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SMALL DEVIATION OF SUBORDINATED PROCESSES OVER COMPACT SETS

BY

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Abstract. Let $A = \big(A(t)\big)_{t \geqslant 0}$ be a subordinator. Given a compact set $K \subset [0, \infty)$ we prove two-sided estimates for the covering numbers of the random set $\{A(t) : t \in K\}$ which depend on the Laplace exponent Φ of A and on the covering numbers of K. This extends former results in the case $K = [0, 1]$. Using this we find the behavior of the small deviation probabilities for subordinated processes $\big(W_H\big(A(t)\big)\big)_{t \in K}$, where W_H is a fractional Brownian motion with Hurst index $0 < H < 1$. The results are valid in the quenched as well as in the annealed case. In particular, those questions are investigated for Gamma processes. Here some surprising new phenomena appear. As application of the general results we find the behavior of $\log P(\sup_{t \in K} |Z_\alpha(t)| < \varepsilon)$ as $\varepsilon \to 0$ for the α-stable Lévy motion Z_α. For example, if K is a self-similar set with Hausdorff dimension $D > 0$, then this behavior is of order $-\varepsilon^{-\alpha D}$ in complete accordance with the Gaussian case $\alpha = 2$.

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1. INTRODUCTION

Let $A = \big(A(t)\big)_{t \geqslant 0}$ be some subordinator in the sense of [4] and [5], i.e., A is a non-decreasing Lévy process. Paths of subordinators are very irregular and, consequently, the random sets $\{A(t) : t \geqslant 0\}$ are in general "small" subsets of the real line. There exists a very precise description of the size of those sets. To formulate this result we need the following notation.

Given a subset $E \subset \mathbb{R}$ and a number $\delta > 0$ the *covering number* $N(E, \delta)$ is defined by

$$N(E, \delta) := \inf \Big\{ n \geqslant 1 : \exists\, I_1, \ldots, I_n \text{ such that } E \subseteq \bigcup_{j=1}^{n} I_j \Big\},$$

where the $I_j \subset \mathbb{R}$ are intervals of length $|I_j| \leqslant \delta$. Furthermore, if A is a subordinator, its Laplace exponent Φ is the function from $[0,\infty)$ to $[0,\infty)$ defined by the equation

$$\mathbb{E}\, e^{-x\,A(t)} = e^{-t\,\Phi(x)}, \quad 0 \leqslant t, x < \infty.$$

Then the above-mentioned result about the size of the range of a subordinator A is as follows (cf. [13], Corollary 3.2):

THEOREM 1.1. *Suppose that*

$$\liminf_{x\to\infty} \frac{\Phi(x)}{\log x} > 0. \tag{1.1}$$

Then for almost all paths of A it follows that

$$\frac{1}{8}\,\Phi(\delta^{-1}) \leqslant N\big(A([0,1]),\delta\big) \leqslant 40\,\Phi(\delta^{-1}) \tag{1.2}$$

provided that $0<\delta<\delta_0$ *for some* $($*random*$)$ $\delta_0>0$.

Because of $N([0,1],\delta) \approx \delta^{-1}$ the estimates in (1.2) may also be written as

$$N\big(A([0,1]),\delta\big) \approx N\left([0,1], \frac{1}{\Phi(\delta^{-1})}\right) \tag{1.3}$$

for almost all paths of A. This relates directly the size of $[0,1]$ to that of $A([0,1])$. Here and in the sequel we write $f \approx g$ for two functions f and g provided there are $c_1, c_2 > 0$ such that $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$ for all x where f and g are defined.

One may ask now whether or not relation (1.3) depends on the special structure of $[0,1]$ or if it is true even for more general compact sets $K \subset [0,\infty)$. Surprisingly, it turns out that (1.3) is valid in this much more general setting. For that purpose condition (1.1) has to be adapted suitably. We suppose now that there is some $\beta > 0$ such that

$$\liminf_{x\to\infty} \frac{N\big(K, 1/\Phi(x)\big)}{(\log x)^{\beta}} := C_\beta(K,\Phi) > 0. \tag{1.4}$$

Condition (1.4) excludes compact sets K being too small and Laplace exponents Φ which increase too slowly.

One of our main results is the following general version of Theorem 1.1:

THEOREM 1.2. *Let K be a compact subset in* $[0,\infty)$ *and let A be a subordinator with Laplace exponent Φ such that* (1.4) *holds for some* $\beta>0$*. Then for almost all paths of A there is a random δ_0 such that for* $0<\delta<\delta_0$ *it follows that*

$$\frac{1}{14}\,N\left(K, \frac{2}{\Phi(\delta^{-1})}\right) \leqslant N\big(A(K),\delta\big) \leqslant 100\,N\left(K, \frac{1}{2\,\Phi(\delta^{-1})}\right). \tag{1.5}$$

In particular, if there is a $c_0 > 0$ *such that*

$$N(K,\delta) \leqslant c_0 \cdot N(K, 2\,\delta), \tag{1.6}$$

then for almost all paths of A *the condition*

$$c_1 \cdot N\left(K, \frac{1}{\Phi(\delta^{-1})}\right) \leqslant N\big(A(K), \delta\big) \leqslant c_2 \cdot N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)$$

holds whenever $0 < \delta < \delta_0$ *for some random* $\delta_0 > 0$.

The basic idea for the proof of Theorem 1.2 is similar to that of Theorem 1.1. New difficulties appear in the proof of the right-hand estimate of (1.5) because the technique of stopping times has to be modified in the case of arbitrary compact sets K in $[0, \infty)$. The basic new ingredient is to cover K by suitable intervals and to define stopping times on each of these intervals separately (cf. the proof of Proposition 2.1). Also the left-hand estimate in (1.5) requires new techniques. Here two stopping times, depending on the set K, have to be used (cf. the proof of Proposition 2.2). Additional problems appear since we do no longer know that the covering numbers of K behave regularly as for $K = [0, 1]$, i.e., whether or not K satisfies (1.6).

Subordinators play an important role as random time change. More precisely, given a stochastic process $X = \big(X(t)\big)_{t\geqslant 0}$ and a subordinator A which is independent of X, a new process Y is defined by

$$Y(t) := X\big(A(t)\big), \quad 0 \leqslant t < \infty.$$

The investigation of the subordinated process Y can be carried out in two directions: either one investigates Y for each fixed path of A (quenched case) or one may look for Y as a process modeled over $(\Omega, \mathbb{P}) := (\Omega_A \times \Omega_X, \mathbb{P}_A \times \mathbb{P}_X)$ (annealed case). Here we assume that X is defined on $(\Omega_X, \mathbb{P}_X)$ and A on $(\Omega_A, \mathbb{P}_A)$.

Our objective is to investigate subordinated processes when X is a fractional Brownian motion $\big(W_H(t)\big)_{t\geqslant 0}$ and A is an arbitrary subordinator. Recall that for $0 < H < 1$ the process W_H is a centered Gaussian process with a.s. continuous paths satisfying

$$\mathbb{E}\; W_H(t) W_H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \geqslant 0.$$

We investigate the process $\Big(W_H\big(A(t)\big)\Big)_{t\geqslant 0}$ for fixed paths of A and as a process modeled over $(\Omega_A \times \Omega_W, \mathbb{P}_A \times \mathbb{P}_W)$. Here and in the sequel we use the notation $(\Omega_W, \mathbb{P}_W)$ instead of $(\Omega_{W_H}, \mathbb{P}_{W_H})$ and we write $\mathbb{P} = \mathbb{P}_A \times \mathbb{P}_W$.

We shall prove that under some natural regularity assumptions about the compact set K and about A it follows that for almost all paths of A (cf. Theorem 3.3 below for the exact formulation)

$$\log \mathbb{P}_W\big\{\sup_{t\in K} \big|W_H\big(A(t)\big)\big| < \varepsilon\big\} \approx -N\left(K, \frac{1}{\Phi(\varepsilon^{-1/H})}\right)$$

as well as

$$\log \mathbb{P}\Big(\sup_{t\in K} \big| W_H\big(A(t)\big)\big| < \varepsilon \Big) \approx -N\left(K, \frac{1}{\Phi(\varepsilon^{-1/H})}\right)$$

as $\varepsilon \to 0$.

Let Γ_b be a Gamma distribution with parameter $b > 0$, i.e.,

$$\Gamma_b([0,t]) = \frac{1}{\Gamma(b)} \int_0^t x^{b-1} e^{-x} dx \tag{1.7}$$

for all $t \geqslant 0$. A subordinator A is said to be a *Gamma process* provided that $A(1)$ is Γ_b-distributed for a certain $b > 0$, and hence $\Phi(x) = b \log(1 + x)$. In this case the above-mentioned general small deviation result for $W_H\big(A(t)\big)_{t\in K}$ (cf. Theorem 3.3 below) does not apply. Nevertheless, by refined methods also for Gamma processes A we find the behavior of

$$\log \mathbb{P}\Big(\sup_{t\in K} \big| W_H\big(A(t)\big)\big| < \varepsilon \Big) \tag{1.8}$$

as $\varepsilon \to 0$ in the quenched as well as in the annealed case. For example, if $K = [0, 1]$, then the asymptotic behavior of (1.8) is $-|\log \varepsilon|^2$ in the quenched case and $-|\log \varepsilon|$ in the annealed one. This result is very surprising and, to our knowledge, it is the first example where the quenched and annealed cases behave differently.

For $0 < \alpha < 2$ the α-stable Lévy motion $\big(Z_\alpha(t)\big)_{t\geqslant 0}$ is a Lévy process with symmetric α-stable increments. It is well known (cf. [14]) that Z_α may be represented as

$$Z_\alpha(t) = W\big(A_{\alpha/2}(t)\big), \quad 0 \leqslant t < \infty,$$

where $W = W_{1/2}$ is the Wiener process and $A_{\alpha/2}$ is an $\alpha/2$-stable subordinator independent of W, i.e., its Laplace exponent is given by $\Phi(x) = c\, x^{\alpha/2}$ with a suitable constant $c > 0$. Thus our results apply directly to this case and lead to small deviation results for Z_α indexed by compact sets. For example, the following is an immediate consequence of Theorems 4.2 and 4.3 below and the fact that $N(K, \delta) \approx \delta^{-D}$ for self-similar sets with Hausdorff dimension $D > 0$ (see [6], Theorem 1).

PROPOSITION 1.1. *Let* $K \subset [0, \infty)$ *be a self-similar compact set satisfying the open set condition* (cf. [6], p. 700, for the definition). *If* $D > 0$ *is its Hausdorff dimension*, *then*

$$\log \mathbb{P}\big(\sup_{t\in K} |Z_\alpha(t)| < \varepsilon\big) \approx -\varepsilon^{-\alpha D}$$

as $\varepsilon \to 0$.

In the Gaussian case, i.e. for $\alpha = 2$, this was proved (even for fractional processes) in [12] and [9], while for the N-parameter fractional Brownian motion those questions were investigated in [10] for compact sets in $\mathbb{R}^N$.

The organization of the paper is as follows. In Section 2 we show that with high probability the covering numbers $N\big(A(K), \delta\big)$ are near to $N\big(K, 1/\Phi(\delta^{-1})\big)$. This is the key to prove Theorem 1.2. Section 3 is devoted to the announced small deviation results for processes subordinated to fractional Brownian motions. Basic ingredients are the relation between fractional motions and Riemann–Liouville processes, a conditional Anderson inequality and Talagrand's small deviation result. Finally, in Section 4 we present examples and applications. Of special interest are here Gamma processes. Since their Laplace exponents increase only of logarithmic order, new phenomena appear for processes subordinated by Gamma processes. For example, the quenched case behaves differently as the annealed one. Furthermore, in this section we prove the announced small deviation result for stable Lévy motions indexed by compact "small" sets.

Let us finally mention that throughout the paper c and C (with or without subscript) denote some positive constants which may be different even if they have the same subscript.

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2. COVERING NUMBERS OF THE RANGE OF SUBORDINATORS

Let $A = \big(A(t)\big)_{t \geqslant 0}$ be a subordinator and let $[a, b]$ be some finite interval in $[0, \infty)$. Given $\delta > 0$ we define stopping times $T_k = T_k(\delta)$ as follows: $T_0 := a$ and

$$T_k := \inf\{t > T_{k-1} : A(t) - A(T_{k-1}) > \delta\}. \tag{2.1}$$

Note that by the strong Markov property of A the random variables $\eta_k := T_k - T_{k-1}$ are i.i.d.

LEMMA 2.1. *For $k \geqslant 1$ and $\delta > 0$ the following inclusions hold a.s.*:

$$\{T_{k-1} < b\} \subseteq \big\{N\big(A([a, b], \delta)\big) \geqslant k\big\} \subseteq \{T_{k-1} \leqslant b\}. \tag{2.2}$$

Proof. Suppose first that $T_{k-1} < b$. Then we choose an arbitrary point in $(T_{k-1}, b]$ and denote it by t_{k-1}. Consequently, by construction of the T_j this implies $A(t_{k-1}) - A(T_{k-2}) > \delta$. Since A has a.s. right continuous paths, there is a point $t_{k-2} > T_{k-2}$ such that $A(t_{k-1}) - A(t_{k-2}) > \delta$. This construction is continued until we get some $t_1 > T_1$ with $A(t_2) - A(t_1) > \delta$. Finally, set $t_0 := a$. Thus there exist at least k points $t_0 < \ldots < t_{k-1}$ in $[a, b]$ with $A(t_j) - A(t_{j-1}) > \delta$ for $1 \leqslant j \leqslant k-1$. Of course, this implies $N\big(A([a,b]), \delta\big) \geqslant k$.

To prove the right inclusion suppose that $T_{k-1} > b$. Then this implies

$$A([a,b]) \subseteq \bigcup_{j=1}^{k-1} [A(T_{j-1}), A(T_j - 0)], \tag{2.3}$$

where, as usual, $A(T_j - 0)$ denotes the left-hand limit of A at T_j. By construction of the T_j we have $A(T_j - 0) - A(T_{j-1}) \leqslant \delta$. Hence the inclusion (2.3) implies $N\big(A([a,b]), \delta\big) \leqslant k-1$, which, of course, proves the right-hand side in (2.2). ∎

To state the next result, let us introduce the following notation. Fix $\delta > 0$. We define the random variable η by

$$\eta := \inf\{t > 0\colon A(t) > \delta\}. \tag{2.4}$$

It should be noted that $\eta \overset{d}{=} T_k - T_{k-1}$ whenever $k \geqslant 1$, where $T_0 < T_1 < \ldots$ were defined in (2.1) (for the same $\delta > 0$). Furthermore, recall that Φ denotes the Laplace exponent of the subordinator A.

LEMMA 2.2. *With the previous notation we have*

$$\mathbb{E} \exp\big(-2\Phi(\delta^{-1})\eta\big) \leqslant 7/8 \tag{2.5}$$

and

$$\mathbb{E} \exp\big(\Phi(\delta^{-1})\eta/2\big) \leqslant 1 + e \leqslant e^{4/3}. \tag{2.6}$$

Proof. Estimate (2.5) was obtained in [13] during the proof of Lemma 3.1. Although (2.6) was proved there as well, we repeat the first step since its proof in [13] contains a small incorrectness.

For any $\lambda > 0$ it follows that

$$\mathbb{E}\, e^{\lambda\eta} = \int_0^\infty \mathbb{P}(e^{\lambda\eta} > x)dx = 1 + \lambda \int_0^\infty e^{\lambda s}\, \mathbb{P}(\eta > s)ds. \tag{2.7}$$

Applying this with $\lambda := \Phi(\delta^{-1})/2$, during the proof of Lemma 3.1 in [13], p. 277, it was shown that λ times the integral on the right-hand side of (2.7) is less than or equal to e. Thus (2.6) follows by (2.7). ∎

For later purposes we need the following easy property of Laplace exponents.

LEMMA 2.3. *Let Φ be the Laplace exponent of some subordinator A. Then for each $c > 0$ it follows that*

$$\min\{c, 1\}\ \Phi(x) \leqslant \Phi(c\,x) \leqslant \max\{c, 1\}\ \Phi(x), \qquad x \geqslant 0.$$

P r o o f. Suppose first $c \geqslant 1$. Then the left-hand inequality follows by the monotonicity of Φ. To prove the right-hand estimate let us note that by Jensen's inequality

$$e^{-c\,\Phi(x)} = (e^{-\Phi(x)})^c = (\mathbb{E}\, e^{-x\,A(1)})^c \leqslant \mathbb{E}\, e^{-c\,x\,A(1)} = e^{-\Phi(c\,x)}.$$

If $0 < c < 1$, we apply the above estimate with c^{-1} and $c\,x$. This completes the proof. ∎

The next proposition shows that with large probability the covering numbers of $A(K)$ are bounded by suitable covering numbers of K.

PROPOSITION 2.1. *There is a universal $c > 0$ (for example, one may choose $c = 1/16$) such that for each subordinator A on $[0, \infty)$, for each $\delta > 0$ and each compact set $K \subset [0, \infty)$ it follows that*

(2.8)
$$\mathbb{P}\bigg(N\big(A(K), \delta\big) \geqslant 100 N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\bigg) \leqslant \exp\left(-c\,N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right).$$

P r o o f. For given $\varepsilon > 0$ we cover K by $n = n(\varepsilon)$ disjoint intervals $I_j = [a_j, a_j + \varepsilon)$, where $n \leqslant 2N(K, \varepsilon)$. If $\delta > 0$, then for each $j \leqslant n$ we define stopping times T_i^j as in (2.1), i.e., $T_0^j = a_j$ and if $i > 1$, then

$$T_i^j := \inf\{t > T_{i-1}^j : A(t) - A(T_{i-1}^j) > \delta\}.$$

Let

$$m_j := \max\{k \geqslant 0 : T_k^j < a_j + \varepsilon\}.$$

First note that by the strong Markov property of A the random integers $m_1, \ldots, m_n$ are independent and identically distributed. Furthermore, since

$$A(I_j) \subseteq \bigcup_{i=1}^{m_j+1} [A(T_{i-1}^j), A(T_i^j - 0)],$$

the set $A(I_j)$ can be covered by at most $m_j + 1$ intervals of length less than or equal to δ. Consequently, by the choice of the I_j we have

$$N\big(A(K), \delta\big) \leqslant \sum_{j=1}^{n} (1 + m_j) = n + \sum_{j=1}^{n} m_j. \tag{2.9}$$

To proceed we need more information about the random numbers m_j. Thus fix $j \geqslant 1$ for a moment and set $\eta_i := T_i^j - T_{i-1}^j$. Note that the η_i are independent and distributed as η defined by (2.4). From the definition of the m_j we derive that for any $\nu \in \mathbb{N}$

$$\mathbb{P}(m_j \geqslant \nu) = \mathbb{P}\big(\sum_{k=1}^{\nu} \eta_k < \varepsilon\big).$$

The exponential Chebychev inequality implies for any $\lambda > 0$ that

$$\mathbb{P}(m_j \geqslant \nu) \leqslant e^{\lambda\varepsilon}\, \mathbb{E} \exp\big(-\lambda(\eta_1 + \ldots + \eta_\nu)\big) = e^{\lambda\varepsilon}(\mathbb{E}\, e^{-\lambda\eta})^\nu.$$

Set $\lambda := 2\Phi(\delta^{-1})$. Then we are in the situation of (2.5) and conclude

$$\mathbb{P}(m_j \geqslant \nu) \leqslant (7/8)^\nu \exp\big(2\Phi(\delta^{-1})\varepsilon\big) \leqslant \exp\big(2\,\varepsilon\,\Phi(\delta^{-1}) - \nu/8\big). \tag{2.10}$$

Observe that until now all considerations were valid for any pair $\varepsilon > 0$ and $\delta > 0$. Now we choose ε depending on δ by setting $\varepsilon := 1/\Phi(\delta^{-1})$. Then estimate (2.10) leads to

$$\mathbb{P}(m_j \geqslant \nu) \leqslant e^{2-\nu/8}. \tag{2.11}$$

Since m_j is an integer, estimate (2.11) also holds for real numbers $\nu > 0$. By standard methods (2.11) yields

$$\mathbb{E} \exp(\rho m_j) \leqslant 1 + e^2 \frac{8\rho}{1 - 8\rho} < \infty \tag{2.12}$$

provided that $0 < \rho < 1/8$. Next we apply (2.9) for $n = n(\varepsilon)$ with $\varepsilon = \varepsilon(\delta)$ as chosen above. Then we get

$$\mathbb{P}\Big(N\big(A(K), \delta\big) \geqslant 50n\Big) \leqslant \mathbb{P}\Big(\sum_{j=1}^{n} m_j \geqslant 49n\Big);$$

hence an application of the exponential Chebychev inequality leads to

$$\mathbb{P}\Big(N\big(A(K), \delta\big) \geqslant 50n\Big) \leqslant e^{-49\lambda n}\, \mathbb{E} \exp\big(\lambda(m_1 + \ldots + m_n)\big)$$

for any $\lambda > 0$. Setting $\lambda := 1/16$, by (2.12) this implies

$$\mathbb{P}\Big(N\big(A(K), \delta\big) \geqslant 50n\Big) \leqslant e^{-49n/16}\,(1 + e^2)^n \leqslant e^{-n/16}, \tag{2.13}$$

and because of $N(K, \varepsilon) \leqslant n \leqslant 2N(K, \varepsilon)$ and $\varepsilon = 1/\Phi(\delta^{-1})$, inequality (2.8) is a direct consequence of (2.13). ■

Our next aim is to prove that with large probability the covering numbers of $A(K)$ are also bounded from below by suitable covering numbers of K.

PROPOSITION 2.2. *There is a universal constant $c > 0$ such that the following holds. For each subordinator $A = \big(A(t)\big)_{t \geqslant 0}$, each compact set $K \subset [0, \infty)$ and each $\delta > 0$ we have*

$$\mathbb{P}\bigg(N\big(A(K), \delta\big) \leqslant \frac{1}{14}\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\bigg) \leqslant \exp\left(-c\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right).$$

For example, c *may be chosen as* $1/42$.

Proof. Given $\delta > 0$ we define two sequences $T_0, T_1, \ldots$ and $\tau_0, \tau_1, \ldots$ of stopping times as follows. Set $T_0 := 0$ and $\tau_0 := \inf\{t \geqslant T_0 : t \in K\}$. Suppose that for some $k \geqslant 1$ the random times T_{k-1} and τ_{k-1} have been already defined. Then we set

$$T_k := \inf\{s > \tau_{k-1} : A(s) - A(\tau_{k-1}) > \delta\}$$

and

$$\tau_k := \inf\{t \geqslant T_k : t \in K\}$$

provided that there exists at least one element $t \in K$ such that $t \geqslant T_k$. Otherwise put $\tau_k := T_k$. By the strong Markov property of A the random variables $\eta_k := T_k - \tau_{k-1}$ are independent and distributed as η in (2.4). Next we define a random integer $m \geqslant 1$ by

$$m := \inf\{k \geqslant 1 : K \subseteq [0, T_k]\}.$$

Then we get $\tau_0, \ldots, \tau_{m-1} \in K$ and, furthermore,

$$K \subseteq \bigcup_{k=1}^{m} [\tau_{k-1}, T_k]. \tag{2.14}$$

Moreover, since for all $k \geqslant 0$ we have

$$A(\tau_{k+2}) - A(\tau_k) \geqslant 2\delta > \delta,$$

by $\tau_k \in K$ for $0 \leqslant k \leqslant m-1$, it follows that

$$m \leqslant 2\, N\big(A(K), \delta\big). \tag{2.15}$$

Let $\varepsilon > 0$ be given. Then we find $n := N(K, \varepsilon)$ elements $t_1, \ldots, t_n$ in K satisfying $|\,t_i - t_j| \geqslant \varepsilon/2$ whenever $1 \leqslant i, j \leqslant n$ and $i \neq j$. Since the t_i belong to K, from (2.14) we derive

$$\bigcup_{i=1}^{n} [t_i, t_i + \varepsilon/2) \subseteq \bigcup_{k=1}^{m} [\tau_{k-1}, T_k + \varepsilon/2). \tag{2.16}$$

By the choice of the t_i the left-hand intervals in (2.16) are disjoint. Thus comparing the lengths of the sets in (2.16) leads to

$$\frac{n\varepsilon}{2} \leqslant \sum_{k=1}^{m} (T_k - \tau_{k-1}) + \frac{m\varepsilon}{2}$$

or, equivalently, to

$$\frac{\varepsilon}{2}\,[n-m] \leqslant \sum_{k=1}^{m}(T_k-\tau_{k-1}). \tag{2.17}$$

Suppose now that $N\big(A(K),\delta\big)\leqslant\nu$ for some integer $\nu\geqslant 1$. By (2.15) this implies $m\leqslant 2\nu$; hence by (2.17) it follows that

$$\frac{\varepsilon}{2}\,[n-2\nu] \leqslant \sum_{k=1}^{2\nu}(T_k-\tau_{k-1}) = \sum_{k=1}^{2\nu}\eta_k,$$

where, as above, $\eta_k = T_k-\tau_{k-1}$. Consequently, from this we derive

$$\mathbb{P}\Big(N\big(A(K),\delta\big)\leqslant\nu\Big) \leqslant \mathbb{P}\big(\sum_{k=1}^{2\nu}\eta_k \geqslant a\big) \tag{2.18}$$

with

$$a := \frac{\varepsilon}{2}\,[N(K,\varepsilon)-2\nu], \tag{2.19}$$

recalling that $n=N(K,\varepsilon)$. For each $\lambda>0$ the exponential Chebyshev inequality implies

$$\mathbb{P}\big(\sum_{k=1}^{2\nu}\eta_k \geqslant a\big) \leqslant e^{-\lambda a}\,\mathbb{E}\,\exp\big(\lambda\sum_{k=1}^{2\nu}\eta_k\big) = e^{-\lambda a}\,(\mathbb{E}\,e^{\lambda\eta})^{2\nu}.$$

Next set $\lambda := \Phi(\delta^{-1})/2$ and apply (2.6). Consequently,

$$\mathbb{P}\big(\sum_{k=1}^{2\nu}\eta_k \geqslant a\big) \leqslant \exp\big(-a\,\Phi(\delta^{-1})/2 + 8\nu/3\big);$$

thus by (2.18) and (2.19) the preceding estimate yields

$$\mathbb{P}\Big(N\big(A(K),\delta\big)\leqslant\nu\Big) \leqslant \exp\Big(-\frac{\varepsilon}{4}\,[N(K,\varepsilon)-2\nu]\,\Phi(\delta^{-1}) + 8\nu/3\Big). \tag{2.20}$$

Since $N\big(A(K),\delta\big)$ is an integer, it is not difficult to see that (2.20) not only holds for $\nu\in\mathbb{N}$, but also if $\nu>0$ is any real number. Hence, we may choose

$$\varepsilon := \frac{1}{\Phi(\delta^{-1})} \quad\text{and}\quad \nu := N(K,\varepsilon)/14,$$

which finally leads to

$$\mathbb{P}\bigg(N\big(A(K),\delta\big)\leqslant\frac{1}{14}\,N\left(K,\frac{1}{\Phi(\delta^{-1})}\right)\bigg) \leqslant \exp\left(-c\,N\left(K,\frac{1}{\Phi(\delta^{-1})}\right)\right),$$

as asserted. Easy calculations imply that c may be taken as $1/42$. ∎

Now we are in position to prove Theorem 1.2.

P r o o f o f T h e o r e m 1.2. Set $\delta_j := 2^{-j}$. Whenever $c > 0$, by (1.4) it follows that

$$\sum_{j=1}^{\infty} \exp\left(- c\, N\left(K, \frac{1}{\Phi(\delta_j^{-1})}\right)\right) \leqslant \sum_{j=1}^{\infty} \exp(-c'\, j^{\beta}) < \infty$$

because of $\beta > 0$. Hence, in view of Propositions 2.1 and 2.2 the Borel–Cantelli lemma applies and for almost all paths of A there is a $j_0 \geqslant 1$ such that

$$\frac{1}{14}\, N\left(K, \frac{1}{\Phi(\delta_j^{-1})}\right) \leqslant N\big(A(K), \delta_j\big) \leqslant 100\, N\left(K, \frac{1}{\Phi(\delta_j^{-1})}\right) \tag{2.21}$$

provided that $j \geqslant j_0$. Fix such a path of A and take a $\delta > 0$ with $\delta_{j+1} \leqslant \delta \leqslant \delta_j$ for some $j \geqslant j_0$. Using the right-hand estimate of (2.21) with δ_{j+1} gives

$$\begin{aligned} N\big(A(K), \delta\big) &\leqslant N\big(A(K), \delta_{j+1}\big) \leqslant 100\, N\left(K, \frac{1}{\Phi(\delta_{j+1}^{-1})}\right) \\ &\leqslant 100\, N\left(K, \frac{1}{\Phi(2/\delta)}\right) \leqslant 100\, N\left(K, \frac{1}{2\,\Phi(\delta^{-1})}\right), \end{aligned} \tag{2.22}$$

where we used $\delta/2 \leqslant \delta_{j+1}$ and Lemma 2.3.

Similarly, the left-hand side of (2.21) implies

$$N\big(A(K), \delta\big) \geqslant N\big(A(K), \delta_j\big) \geqslant \frac{1}{14}\, N\left(K, \frac{1}{\Phi(\delta_j^{-1})}\right) \geqslant \frac{1}{14}\, N\left(K, \frac{2}{\Phi(\delta^{-1})}\right)$$

as asserted. This completes the proof. ■

EXAMPLE 2.1. First we want to show by an example that even if $K = [0, 1]$, Theorem 1.2 applies to cases not covered by Theorem 1.1. Let A be the subordinator with Laplace exponent

$$\Phi(x) = \log(1 + x)^{\alpha}$$

for some $\alpha \in (0, 1)$. Note that Φ is indeed a Laplace exponent because it is the composition of the two completely monotone functions $\log(1 + x)$ and x^{α} with $0 < \alpha < 1$. The corresponding subordinator is a Gamma process with α-stable subordination. Let K be some self-similar set satisfying the open set condition and with Hausdorff dimension $0 < D \leqslant 1$. Then a result of Lalley (cf. [6]) asserts that $N(K, \delta) \approx \delta^{-D}$; hence condition (1.4) holds with $\beta := \alpha\, D$. Thus, Theorem 1.2 applies and shows that almost surely

$$N\big(A(K), \delta\big) \approx |\log \delta|^{\alpha D}.$$

Observe that even for $K = [0, 1]$, hence $D = 1$, condition (1.1) is violated.

EXAMPLE 2.2. We want to investigate now the superposition of two independent subordinators. The first example has been already of this type. But here we investigate the general situation. Let A_1, A_2 be two such subordinators with Laplace exponents Φ_1 and Φ_2, respectively. Define A_3 by

$$A_3(t) := A_2\big(A_1(t)\big), \qquad t \geqslant 0.$$

Of course, A_3 is a subordinator as well and its Laplace exponent equals

$$\Phi_3(x) = \Phi_1\big(\Phi_2(x)\big), \qquad x \geqslant 0.$$

We want to show now that a direct application of Theorem 1.2 to A_3 and $K = [0,1]$ leads exactly to the same estimates for $N\big(A_3([0,1]), \delta\big)$ (with slightly better constants) as an iterative application.

Assuming

$$\liminf_{x\to\infty} \frac{\Phi_3(x)}{(\log x)^{\beta_3}} > 0 \tag{2.23}$$

for some $\beta_3 > 0$, from Theorem 1.2 (with $K = [0,1]$) we derive

$$c_1\, \Phi_3(\delta^{-1}) \leqslant N\big(A_3([0,1]), \delta\big) \leqslant c_2\, \Phi_3(\delta^{-1}). \tag{2.24}$$

On the other hand, if

$$\liminf_{x\to\infty} \frac{\Phi_1(x)}{(\log x)^{\beta_1}} > 0 \tag{2.25}$$

for some $\beta_1 > 0$, then Theorem 1.2 applies to A_1 and $[0,1]$ and implies that a.s.

$$c_1\, \Phi_1(\delta^{-1}) \leqslant N\big(A_1([0,1]), \delta\big) \leqslant c_2\, \Phi_1(\delta^{-1}). \tag{2.26}$$

In order to apply Theorem 1.2 to A_2 and to $K = \overline{A_1([0,1])}$ condition (1.4) has to be satisfied for some $\beta_2 > 0$. In view of (2.26) this holds if and only if (2.23) is valid with $\beta_2 = \beta_3$. Thus, under the assumption (2.23) the direct and the iterative application of Theorem 1.2 lead to the same estimates for $N\big(A_3([0,1]), \delta\big)$ as stated in (2.24). Observe that by $\sup_{x\geqslant 1} \Phi_2(x)/x < \infty$ condition (2.23) implies (2.25) with $\beta_1 = \beta_3$.

3. SMALL DEVIATIONS OF SUBORDINATED PROCESSES

3.1. Upper estimates

LEMMA 3.1. *If* $0 < H < 1$, *then there is a constant* $c_H > 0$ *such that for all* $\varepsilon > 0$ *and all real numbers* $0 = t_0 < t_1 < \ldots < t_n$ *we have*

$$\mathbb{P}\big(\sup_{1\leqslant j\leqslant n} |W_H(t_j)| < \varepsilon\big) \leqslant \prod_{j=1}^{n} \mathbb{P}\big(|\xi| < c_H\,(t_j - t_{j-1})^{-H}\,\varepsilon\big)$$

with ξ *distributed according to* $\mathcal{N}(0,1)$.

P r o o f. For $H > 0$ let R_H be the Riemann–Liouville process defined by

$$R_H(t) := \int_0^t (t-x)^{H-1/2} dW(x), \quad 0 \leqslant t < \infty.$$

If $0 < H < 1$, then with

$$\kappa_H := \left(\frac{1}{2H} + \int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2}\right)^2 dx \right)^{1/2}$$

we have (cf. [8])

$$W_H = \kappa_H^{-1} R_H + \Delta_H,$$

where Δ_H is a centered Gaussian process independent of R_H. Hence, by Anderson's inequality (cf. [1]), it follows that

$$\begin{aligned}\mathbb{P}\big(\sup_{1\leqslant j\leqslant n} |W_H(t_j)| < \varepsilon\big) &\leqslant \mathbb{P}\big(\sup_{1\leqslant j\leqslant n} |R_H(t_j)| < \kappa_H\,\varepsilon\big) \\ &= \mathbb{P}\big(\sup_{1\leqslant j\leqslant n} \big|\int_0^{t_j} (t_j - x)^{H-1/2} dW(x)\big| < \kappa_H\,\varepsilon\big).\end{aligned}$$

Next we apply the conditional Anderson inequality as in [11] or [8] and see that the last probability is less than

$$\mathbb{P}\big(\sup_{1\leqslant j\leqslant n} \big|\int_{t_{j-1}}^{t_j} (t_j - x)^{H-1/2} dW(x)\big| < \kappa_H\,\varepsilon\big). \tag{3.1}$$

Set

$$\sigma_j := \int_{t_{j-1}}^{t_j} (t_j - x)^{H-1/2} dW(x).$$

Then the σ_j are independent centered normal random variables with variance

$$\mathbb{E}\,|\sigma_j|^2 = \int_{t_{j-1}}^{t_j} (t_j - x)^{2H-1} dx = \frac{1}{2H}\,(t_j - t_{j-1})^{2H}.$$

Consequently, (3.1) coincides with

$$\prod_{j=1}^n \mathbb{P}\big(|\xi| < \kappa_H\,(2H)^{1/2}\,(t_j - t_{j-1})^{-H}\,\varepsilon\big)$$

proving our assertion with $c_H := \kappa_H\,(2H)^{1/2}$. ■

As a consequence of Lemma 3.1 we obtain the following useful estimate:

PROPOSITION 3.1. *There is a constant $c_0 > 0$ only depending on H such that for all compact sets $K \subset [0, \infty)$ and for all $\varepsilon > 0$ we have*

$$\mathbb{P}\big(\sup_{t \in K} |W_H(t)| < \varepsilon\big) \leqslant \exp\big(-N(K, c_0\,\varepsilon^{1/H}) + 1\big). \tag{3.2}$$

Proof. Let $\delta > 0$ be given and set $n := N(K, \delta)$. Then there are points $\tilde{t}_j \in K$ with $0 \leqslant \tilde{t}_1 < \tilde{t}_2 < \ldots < \tilde{t}_n$ and with $\tilde{t}_j - \tilde{t}_{j-1} \geqslant \delta/2$ for $2 \leqslant j \leqslant n$. Set $t_0 := 0$, $t_1 := \tilde{t}_2$ until $t_{n-1} := \tilde{t}_n$. In this way we obtain n points $0 = t_0 < t_1 < \ldots < t_{n-1}$ with $t_1, \ldots, t_{n-1} \in K$ satisfying $t_j - t_{j-1} \geqslant \delta/2$ for $1 \leqslant j \leqslant n-1$. We apply now Lemma 3.1 to $t_1, \ldots, t_{n-1}$ and conclude

$$\begin{aligned}\mathbb{P}\big(&\sup_{t \in K} |W_H(t)| < \varepsilon\big)\\ &\leqslant \mathbb{P}\big(\sup_{1 \leqslant j \leqslant n-1} |W_H(t_j)| < \varepsilon\big)\\ &\leqslant \prod_{j=1}^{n-1} \mathbb{P}\big(|\xi| < c_H\,(t_j - t_{j-1})^{-H}\,\varepsilon\big) \leqslant \mathbb{P}(|\xi| < c_H\,2^H\,\delta^{-H}\,\varepsilon)^{n-1}.\end{aligned} \tag{3.3}$$

Next we choose a constant $c_1 > 0$ for which

$$\mathbb{P}(|\xi| < c_1) \leqslant e^{-1}$$

and with this c_1 we define $\delta > 0$ by $\delta := 2\,c_H^{1/H}\,c_1^{-1/H}\,\varepsilon^{1/H}$. Consequently, by (3.3) this implies

$$\begin{aligned}\mathbb{P}\big(\sup_{t \in K} |W_H(t)| < \varepsilon\big) &\leqslant \mathbb{P}(|\xi| < c_1)^{n-1}\\ &\leqslant \exp\big(-(n-1)\big) = \exp\big(-N(K, c_0\,\varepsilon^{1/H}) + 1\big)\end{aligned}$$

with $c_0 := 2\,c_H^{1/H}\,c_1^{-1/H}$. This completes the proof. ■

REMARK 3.1. Estimate (3.2) is very useful in the case $N(K, \delta) \geqslant c\,\delta^{-\alpha}$ for some $\alpha > 0$. But if, for example, $N(K, \delta) \approx |\log \delta|^\beta$, then (3.2) turns out to be too weak for our purposes. Here we need the following modified version of Proposition 3.1.

PROPOSITION 3.2. *For all compact sets $K \subset [0, \infty)$ it follows that*

$$\log \mathbb{P}\big(\sup_{t \in K} |W_H(t)| < \varepsilon\big) \leqslant -c[N(K, \varepsilon^{1/(1+H)}) - 1]\,|\log \varepsilon|$$

provided that $0 < \varepsilon < \varepsilon_H$. Here $c > 0$ and $\varepsilon_H > 0$ only depend on H.

P r o o f. We start with estimate (3.3) where we now set $\delta := \varepsilon^{1/(1+H)}$. Then we get

$$\delta^{-H}\,\varepsilon = \delta = \varepsilon^{1/(1+H)};$$

hence (3.3) yields (recall that in (3.3) we have $n = N(K,\delta)$)

$$\log \mathbb{P}\big(\sup_{t\in K}|W_H(t)| < \varepsilon\big) \leqslant [N(K,\delta)-1]\,\log \mathbb{P}(|\xi| < c_H\,2^H\,\delta). \tag{3.4}$$

Observe that $\mathbb{P}(|\xi| \leqslant x) \leqslant x$ whenever $0 < x < \infty$. Consequently, (3.4) is less than

$$[N(K,\delta)-1]\{\log(c_H\,2^H) - |\log\delta|\} \leqslant -c\,[N(K,\varepsilon^{1/(1+H)})-1]\,|\log\varepsilon|$$

provided that $\varepsilon < \varepsilon_H$ for a suitable $\varepsilon_H > 0$. This completes the proof. ■

REMARK 3.2. Suppose that $N(K,\delta) \approx |\log\delta|^{\beta}$ for a certain $\beta > 0$. Then Proposition 3.2 implies

$$\log \mathbb{P}\big(\sup_{t\in K}|W_H(t)| < \varepsilon\big) \leqslant -c\,|\log\varepsilon|^{\beta+1},$$

while Proposition 3.1 only gives $|\log\varepsilon|^{\beta}$. On the other hand, if $N(K,\delta) \approx \delta^{-\alpha}$ for some $\alpha > 0$, then Proposition 3.1 leads to the right order while Proposition 3.2 does not. Thus one may ask for a general estimate of the small deviation probability of W_H, which applies in all suitable cases of compact sets.

Before proceeding further let us recall that the subordinator A is defined on the probability space $(\Omega_A,\mathbb{P}_A)$ while the fractional Brownian motion W_H is defined on $(\Omega_W,\mathbb{P}_W)$ and that $\mathbb{P} := \mathbb{P}_A\times\mathbb{P}_W$.

THEOREM 3.1. (a) *Suppose* (1.4) *holds for* K *and* Φ. *Then for almost all paths of the subordinator* A *there is a* (*random*) ε_0 *such that for* $0 < \varepsilon < \varepsilon_0$ *it follows that*

$$\log \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant -c_1\,N\left(K,\frac{c_2}{\Phi(\varepsilon^{-1/H})}\right),$$

where c_1 *and* c_2 *only depend on* H *and on* $C_\beta(K,\Phi)$ *in* (1.4).

(b) *For each* $\varepsilon > 0$ *it follows that*

$$\log \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant -c_3\,N\left(K,\frac{1}{\Phi(c_0^{-1}\varepsilon^{-1/H})}\right) + 2, \tag{3.5}$$

where $c_3 > 0$ *is universal. For example*, c_3 *may be chosen as* $1/42$ *and* c_0 *is the constant appearing in* (3.2).

P r o o f. Let us start with the proof of (a). Then assume that (1.4) holds true. Thus Theorem 1.2 applies to almost all paths of A. Taken such a path there is a δ_0 (depending on the chosen path) such that (1.5) holds for all $0 < \delta < \delta_0$. For $\varepsilon > 0$ we apply Proposition 3.1 to the set $\overline{A(K)}$ and obtain

$$\begin{aligned}(3.6)\quad \log \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) &= \log \mathbb{P}_W\big(\sup_{s\in\overline{A(K)}} |W_H(s)| < \varepsilon\big)\\ &\leqslant -N\big(\overline{A(K)},\, c_0\,\varepsilon^{1/H}\big) + 1 \leqslant -\frac{1}{14}\,N\left(K, \frac{2}{\Phi(c_0^{-1}\varepsilon^{-1/H})}\right) + 1\end{aligned}$$

provided that $c_0\,\varepsilon^{1/H} < \delta_0$. Here $c_0 > 0$ is the constant in Proposition 3.1. Note that (1.4) implies $N(K,\delta) \to \infty$ as $\delta \to 0$. From this and Lemma 2.3 we derive the existence of constants $c_1, c_2 > 0$ (depending on H and on $C_\beta(K,\Phi)$) such that (3.6) can be estimated by

$$-c_1\,N\left(K, \frac{c_2}{\Phi(\varepsilon^{-1/H})}\right).$$

This completes the proof of part (a).

To verify (b) we start with (3.2) and apply it to each fixed path of A. Then we get

$$(3.7)\qquad \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant \exp\Big(-N\big(A(K), c_0\,\varepsilon^{1/H}\big) + 1\Big)$$

for all paths of A. We integrate now (3.7) over all paths, i.e., over Ω_A with respect to $\mathbb{P}_A$, and obtain (recall that $\mathbb{P} = \mathbb{P}_A \times \mathbb{P}_W$)

$$(3.8)\qquad \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant e \int_{\Omega_A} \exp\Big(-N\big(A(K), c_0\,\varepsilon^{1/H}\big)\Big)d\mathbb{P}_A.$$

To shorten the notation set $\delta := c_0\,\varepsilon^{1/H}$. Then the right-hand integral in (3.8) equals

$$\int_0^1 \mathbb{P}_A\Big(\exp\big\{-N\big(A(K),\delta\big)\big\} \geqslant t\Big)dt = \int_0^\infty e^{-s}\,\mathbb{P}_A\Big(N\big(A(K),\delta\big) \leqslant s\Big)ds.$$

Set $s_0 := \frac{1}{14}\,N\big(K, 1/\Phi(\delta^{-1})\big)$ and split the last integral as

$$\int_0^{s_0} e^{-s}\,\mathbb{P}_A\Big(N\big(A(K),\delta\big) \leqslant s\Big)ds + \int_{s_0}^\infty e^{-s}\,\mathbb{P}_A\Big(N\big(A(K),\delta\big) \leqslant s\Big)ds.$$

By the choice of s_0 and by Proposition 2.2 the first integral can be estimated by

$$\mathbb{P}_A\Big(N\big(A(K),\delta\big) \leqslant s_0\Big) \leqslant \exp\left(-\frac{1}{42}\,N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right).$$

The second integral is less than

$$\int_{s_0}^{\infty} e^{-s} ds = \exp(-s_0) = \exp\left(-\frac{1}{14}\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right).$$

Summing up, we obtain

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in K} \big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \\
&\leqslant e\left[\exp\left(-\frac{1}{42}\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right) + \exp\left(-\frac{1}{14}\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right)\right)\right] \\
&\leqslant \exp\left(-c_3\, N\left(K, \frac{1}{\Phi(\delta^{-1})}\right) + 2\right),
\end{aligned}$$

where we may take $c_3 = 1/42$. Finally, to complete the proof recall that $\delta := c_0\, \varepsilon^{1/H}$. ■

REMARK 3.3. Note that some additional term has to appear on the right-hand side of (3.5). For example, take $K = \{0\}$. Then the left-hand side equals 0 for all $\varepsilon > 0$ while, without an additional term, the right-hand side would be $-c_3$. Recall that we do not assume (1.4) for the validity of part (b) in Theorem 3.1; thus the case $K = \{0\}$ is not excluded.

3.2. Lower estimates. Our next aim is to prove lower estimates for the small deviation probability of subordinated processes over compact sets. We treat here the regular case which rests on Talagrand's small deviation estimate for Gaussian processes (cf. [7] or [15]).

Let ψ be a non-decreasing continuous function from $(0,\infty)$ to $(0,\infty)$ satisfying the condition

$$C_1\, \psi(x) \leqslant \psi(2x) \leqslant C_2\, \psi(x), \qquad x \geqslant x_0, \tag{3.9}$$

for a certain $x_0 > 0$ and with constants $1 < C_1 \leqslant C_2 < \infty$. Observe that the left-hand estimate in (3.9) excludes functions ψ behaving like a power of $\log x$ as $x \to \infty$.

THEOREM 3.2. *Suppose that the compact set K and the Laplace exponent Φ of A satisfy* (1.4) *and*

$$N\left(K, \frac{1}{\Phi(x)}\right) \leqslant \psi(x) \tag{3.10}$$

for some ψ with property (3.9). *Then this implies the following*:

(a) *For almost all paths of A it follows that*

$$\log \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \geqslant -c\,\psi(\varepsilon^{-1/H}) \tag{3.11}$$

for $\varepsilon < \varepsilon_0$ (random).

(b) *The inequality*

$$\log \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \geqslant -c\,\psi(\varepsilon^{-1/H}) \tag{3.12}$$

holds provided $0 < \varepsilon < \varepsilon_0$ for some fixed $\varepsilon_0 > 0$.

The constant $c > 0$ only depends on $H \in (0,1)$ and on C_1 and C_2 in (3.9).

Proof. First note that we may assume $0 \in K$. Otherwise we add zero and increase the covering numbers of K by at most 1. Take now a path of A such that (2.22) holds for small $\delta > 0$. Consequently, by (3.10) we get

$$N\big(A(K),\delta\big) \leqslant 100\,N\left(K, \frac{1}{\Phi(2/\delta)}\right) \leqslant 100\,\psi(2/\delta). \tag{3.13}$$

Next recall that the Dudley distance $d_H(t,s) := \big(\mathbb{E}\,|W_H(t) - W_H(s)|^2\big)^{1/2}$ equals $|t-s|^H$; hence, defining the covering numbers $N(E,\delta,d_H)$ by using the radius of the covering intervals (our definition is taken with respect to the diameter), by (3.13) it follows that

$$N\big(A(K),\delta,d_H\big) = N\big(A(K),(2\delta)^{1/H}\big) \leqslant 100\psi(2^{1-1/H}\delta^{-1/H}) \leqslant 100\psi(\delta^{-1/H})$$

provided $\delta > 0$ is small enough. Set $\tilde\psi(\delta) := 100\,\psi(\delta^{-1/H})$. In view of (3.9) this function $\tilde\psi$ satisfies the assumptions of Talagrand's theorem in [7], p. 257, and we obtain

$$\log \mathbb{P}_W\big(\sup_{s_1,s_2\in A(K)} |W_H(s_1) - W_H(s_2)| < \varepsilon\big) \geqslant -\tilde\psi(\varepsilon) = -c\,\psi(\varepsilon^{-1/H})$$

for small $\varepsilon > 0$. Since we assumed $0 \in K$, hence also $0 \in A(K)$, inequality (3.11) follows directly from the last estimate.

To verify (3.12) observe that (3.11) may be rewritten as

$$\liminf_{\varepsilon\to 0} \exp\big(c\,\psi(\varepsilon^{-1/H})\big)\cdot \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \geqslant 1$$

almost surely. Consequently, integrating the last estimate with respect to $\mathbb{P}_A$ we obtain

$$\int_{\Omega_A} \Big[\liminf_{\varepsilon\to 0} \exp\big(c\,\psi(\varepsilon^{-1/H})\big)\cdot \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big)\Big] d\mathbb{P}_A \geqslant 1.$$

Hence by Fatou's lemma we arrive at

$$\liminf_{\varepsilon\to 0} \exp\big(c\,\psi(\varepsilon^{-1/H})\big) \int_{\Omega_A} \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) d\mathbb{P}_A \geqslant 1.$$

This completes the proof of (3.12) since

$$\int_{\Omega_A} \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) d\mathbb{P}_A = \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big). \quad \blacksquare$$

Combining Theorems 3.1 and 3.2 gives the following

THEOREM 3.3. *Suppose that the compact set K and the subordinator A with Laplace exponent Φ satisfy*

$$N\left(K, \frac{1}{\Phi(x)}\right) \approx \psi(x) \tag{3.14}$$

for some continuous non-decreasing function ψ with property (3.9). *Then this implies for almost all paths of A that*

$$\log \mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \approx -\psi(\varepsilon^{-1/H})$$

and, *moreover*, *that*

$$\log \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \approx -\psi(\varepsilon^{-1/H}).$$

Proof. We only mention that (3.9) and (3.14) imply (1.4). Hence the assertion follows directly from Theorems 3.1 and 3.2. $\blacksquare$

4. EXAMPLES AND APPLICATIONS

4.1. Gamma processes. In this subsection we suppose that A is a Gamma process, i.e., A is a subordinator such that for a certain $b > 0$ the random variable $A(1)$ is Γ_b-distributed as defined in (1.7). Then the Laplace exponent of A equals

$$\Phi(x) = b\,\log(1+x), \qquad 0 \leqslant x < \infty. \tag{4.1}$$

Furthermore, in this subsection we restrict ourselves to compact sets K in $[0,\infty)$ with

$$N(K,\delta) \approx \delta^{-a} \tag{4.2}$$

for a certain $a \in (0,1]$. This simplifies the considerations and covers the most interesting cases as, for example, self-similar sets K. Clearly, (4.1) and (4.2) imply

$$N\left(K, \frac{1}{\Phi(x)}\right) \approx (\log x)^a. \tag{4.3}$$

Thus property (1.4) is satisfied and Theorem 1.2 implies that for almost all paths of A we have

$$N\big(A(K),\delta\big) \approx |\log\delta|^a \tag{4.4}$$

as $\delta \to 0$.

Our next objective is to investigate the behavior of the expression

$$\log\mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \tag{4.5}$$

in that situation. It should be mentioned that even for $K = [0,1]$ the behavior of (4.5) does not follow from Theorem 2.1 in [13]. The crucial point is that the function $\psi(x) := (\log x)^a$ does not satisfy the condition (3.9) with some $C_1 > 1$, so Talagrand's small deviation result does not apply in that case. This was erroneously not mentioned in the proof of (4.2) in [13], p. 279.

THEOREM 4.1. *Let A be some Gamma process with $A(1)$ distributed according to Γ_b for some $b > 0$ and let K be a compact set in $[0,\infty)$ with $N(K,\delta) \approx \delta^{-a}$ for some $a \in (0,1]$. Then for all $H \in (0,1)$ and almost all paths of A we have*

$$\log\mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \approx -|\log\varepsilon|^{a+1}.$$

Moreover, in the annealed case we have the following general result:

Let $K \subset [0,\infty)$ be a compact set with $K \neq \{0\}$. Then

$$\log\mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \approx -|\log\varepsilon|. \tag{4.6}$$

P r o o f. First note that neither Theorem 3.1 nor Theorem 3.2 can be used here directly. As already mentioned, condition (3.9) does not hold in our situation. Thus the latter theorem does not apply. Moreover, Theorem 3.1 does not lead to the right order.

U p p e r b o u n d s i n t h e q u e n c h e d c a s e. Fix a path of A satisfying (1.5) for $\delta < \delta_0$ and apply Proposition 3.2 to $\overline{A(K)}$. Using (4.4) it follows that

$$\begin{aligned}\log\mathbb{P}_W\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) &\leqslant -c_1\big[N\big(A(K),\varepsilon^{1/(1+H)}\big) - 1\big]|\log\varepsilon| \\ &\leqslant -c_2[\log(\varepsilon^{-1/(1+H)})^a - 1]|\log\varepsilon| \\ &\leqslant -c_3\,|\log\varepsilon|^{a+1}\end{aligned}$$

provided that $\varepsilon < \varepsilon_0$. This gives the desired upper estimate in the quenched case.

L o w e r b o u n d s i n t h e q u e n c h e d c a s e. Because of (4.3) there is a $c_0 > 0$ such that for $x \geqslant x_0$ the function $\psi(x) := c_0\,(\log x)^a$ fulfils the inequality

$$N\left(K, \frac{1}{\Phi(x)}\right) \leqslant \psi(x).$$

Then ψ satisfies the right-hand estimate in (3.9). Take a path of A for which (1.5) holds. As in the proof of Theorem 3.2, we get

$$N\big(A(K),\delta,d_H\big) \leqslant 100\,\psi(\delta^{-1/H}) \leqslant c\,|\log\delta|^a$$

for small $\delta > 0$. Thus we are exactly in the situation of Comment 2 in [3] and obtain

$$\log\mathbb{P}\big(\sup_{s\in A(K)} |W_H(s)| < \varepsilon\big) \geqslant -c'\,|\log\varepsilon|^{a+1},$$

as asserted. This completes the proof of this case.

We turn now to the proof of the lower estimate in (4.6). To this end it suffices to treat the case $K=[0,1]$. Recall that W_H is self-similar and A is a Lévy process. Because of $A(K)\subseteq[0,A(1)]$, by $\mathbb{P}=\mathbb{P}_A\times\mathbb{P}_W$ the H-self-similarity of W_H implies

$$\begin{aligned}
\text{(4.7)}\quad \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big|<\varepsilon\Big) &\geqslant \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big|<\varepsilon\,,\ A(1)\leqslant\varepsilon^{1/H}\Big)\\
&\geqslant \mathbb{P}\big(\sup_{0\leqslant s\leqslant\varepsilon^{1/H}}|W_H(s)|<\varepsilon\,,\ A(1)\leqslant\varepsilon^{1/H}\big)\\
&= \mathbb{P}\big(\sup_{0\leqslant s\leqslant 1}|W_H(s)|<1\,,\ A(1)\leqslant\varepsilon^{1/H}\big)\\
&= \mathbb{P}_W\big(\sup_{0\leqslant s\leqslant 1}|W_H(s)|<1\big)\,\mathbb{P}_A\big(A(1)\leqslant\varepsilon^{1/H}\big).
\end{aligned}$$

If $\varepsilon\leqslant 1$, then we get

$$\mathbb{P}\big(A(1)\leqslant\varepsilon^{1/H}\big) = \frac{1}{\Gamma(b)}\int_0^{\varepsilon^{1/H}} x^{b-1}\,e^{-x}\,dx \geqslant e^{-1}\frac{\varepsilon^{b/H}}{\Gamma(b+1)}$$

and from (4.7) we derive

$$\mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big|<\varepsilon\Big) \geqslant c_{H,b}\,\varepsilon^{b/H}$$

with

$$c_{H,b} := e^{-1}\mathbb{P}\big(\sup_{0\leqslant s\leqslant 1}|W_H(s)|<1\big)\,\Gamma(b+1)^{-1}.$$

This completes the proof of the lower estimate.

In order to prove the upper estimate in (4.6) we may assume $K=\{1\}$. Recall that by assumption at least one element in K is different from zero. This implies

$$\text{(4.8)}\qquad \mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big|<\varepsilon\Big) = \mathbb{P}(\zeta^H|\xi|<\varepsilon),$$

where ζ is Γ_b-distributed and independent of ξ, which is as before $\mathcal{N}(0,1)$. The right-hand side of (4.8) equals

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \mathbb{P}(\zeta^H < \varepsilon/x)\, \exp(-x^2/2)dx.$$

Take $\delta > 0$ and split the last integral into two integrals: $\int_0^\delta$ and $\int_\delta^\infty$. The first integral can be estimated by δ. To estimate the second integral we use

$$\mathbb{P}\big(\zeta < (\varepsilon/x)^{1/H}\big) \leqslant \mathbb{P}\big(\zeta < (\varepsilon/\delta)^{1/H}\big) \leqslant \frac{\varepsilon^{b/H}}{\Gamma(b+1)\,\delta^{b/H}}.$$

Summing up, we arrive at

$$\mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant \delta + \frac{\varepsilon^{b/H}}{\Gamma(b+1)\,\delta^{b/H}}.$$

Finally, setting $\delta := \varepsilon^{b/(b+H)}$ gives

$$\mathbb{P}\Big(\sup_{t\in K}\big|W_H\big(A(t)\big)\big| < \varepsilon\Big) \leqslant c\,\varepsilon^{b/(b+H)}$$

for some $c > 0$. This completes the proof of the upper estimate. ■

4.2. Stable Lévy motion. A subordinator A_γ is said to be *γ-stable*, $0<\gamma<1$, provided its Laplace exponent equals $\Phi(x) = c_0\, x^\gamma$ for some $c_0 > 0$. Thus, if $K \subset [0,\infty)$ is a compact set with

$$\liminf_{\delta\to 0} \frac{N(K,\delta)}{|\log\delta|^\beta} > 0$$

for some $\beta > 0$, then the condition (1.4) is satisfied and Theorem 1.2 applies. Hence, in that case for almost all paths of A_γ we have

$$\frac{1}{14}N\left(K, \frac{2\,\delta^\gamma}{c_0}\right) \leqslant N\big(A_\gamma(K),\delta\big) \leqslant 100\,N\left(K, \frac{\delta^\gamma}{2\,c_0}\right)$$

for $0 < \delta < \delta_0$. Here $c_0 > 0$ is the constant in the Laplace exponent of A_γ.

For $0 < \alpha < 2$ let $Z_\alpha = \big(Z_\alpha(t)\big)_{t\geqslant 0}$ be an α-stable Lévy motion. We assume that Z_α is normalized, i.e., $\mathbb{E}\,\exp\big(i\,Z_\alpha(1)\,x\big) = \exp(-|x|^\alpha)$ for $x \in \mathbb{R}$. As well known (cf. [14]) the process Z_α may be represented as

$$Z_\alpha(t) = W\big(A_{\alpha/2}(t)\big), \qquad 0 \leqslant t < \infty, \tag{4.9}$$

where, as before, W is a Wiener process independent of the $\alpha/2$-stable subordinator $A_{\alpha/2}$. Thus our results apply in that case and lead to the following

THEOREM 4.2. *For $0 < \alpha < 2$ let Z_α be an α-stable Lévy motion and let $K \subset [0, \infty)$ be an arbitrary compact set. Then it follows that*

$$\log \mathbb{P}\big(\sup_{t \in K} |Z_\alpha(t)| < \varepsilon\big) \leqslant -c_1\, N(K, c_2\, \varepsilon^{-\alpha}) + 2,$$

where $c_1 > 0$ is a universal constant and $c_2 > 0$ only depends on α.

Proof. By the representation (4.9) this is a direct consequence of Theorem 3.1 (b). Recall that $\Phi(x) = c_0\, x^{\alpha/2}$ and that $H = 1/2$. ■

The corresponding lower estimates for the small deviation probability of Z_α over compact sets follow either from Theorem 3.2 or from recent results in [2] and [3]. For example, Theorem 3.2 implies the following

THEOREM 4.3. *Let $K \subset [0, \infty)$ be a compact set such that*

$$N(K, \delta) \leqslant \varphi(\delta)$$

for some non-increasing continuous function φ satisfying

$$C_1\, \varphi(\delta) \leqslant \varphi(2\,\delta) \leqslant C_2\, \varphi(\delta) \tag{4.10}$$

with some $1 < C_1 \leqslant C_2 < \infty$. Then this implies

$$\log \mathbb{P}\big(\sup_{t \in K} |Z_\alpha(t)| < \varepsilon\big) \geqslant -c\, \varphi(\varepsilon^\alpha). \tag{4.11}$$

REMARK 4.1. The restriction $C_1 > 1$ in (4.10) can be eliminated. But then in (4.11) the function φ has to be replaced by a modified function $\bar{\varphi}$. We refer to [3] for more details.

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