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LARGE DEVIATIONS FOR WISHART PROCESSES

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Abstract. Let $X^{(\delta)}$ be a Wishart process of dimension δ, with values in the set of positive matrices of size m. We are interested in the large deviations for a family of matrix-valued processes $\{\delta^{-1}X_t^{(\delta)}, t \leqslant 1\}$ as δ tends to infinity. The process $X^{(\delta)}$ is a solution of a stochastic differential equation with a degenerate diffusion coefficient. Our approach is based upon the introduction of exponential martingales. We give some applications to large deviations for functionals of the Wishart processes, for example the set of eigenvalues.

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1. INTRODUCTION

Let Z_t be a Brownian matrix of size $n \times m$ (with i.i.d. real Brownian entries). The $m \times m$ positive matrix $X_t = Z_t' Z_t$ is called a *Wishart process of dimension n*. Here, the superscript $'$ denotes the transpose of the matrix. In [1], Bru began the study of Wishart processes and extended the definition of such processes for non-integer dimension δ, by the solution of an SDE

$$dX_t = \sqrt{X_t}\, dB_t + dB_t'\sqrt{X_t} + \delta I_m\, dt, \quad X_0 = x, \tag{1.1}$$

where B is an $m \times m$ matrix-valued Brownian motion, $x \in \mathcal{S}_m^+$ the set of $m \times m$ real symmetric non-negative matrices.

We recall a part of an existence theorem obtained in [1], Theorem 2:

If $\delta \geqslant m+1$, and $x \in \widetilde{\mathcal{S}}_m^+$ (the set of positive definite symmetric matrices), then (1.1) has a unique strong solution in $\widetilde{\mathcal{S}}_m^+$.

In fact, Bru (see [1], p. 739) proved that we can define a Wishart process of dimension $\delta \geqslant m+1$, starting from a degenerate condition x. This process satisfies $P[X_t \in \widetilde{\mathcal{S}}_m^+] = 1$ for all $t > 0$ and is the unique solution (in law) of (1.1). We also refer to [4] where we undertook a deepest study of this matrix-valued diffusion, related to well-known properties of squared Bessel processes (i.e. Wishart processes of size $m = 1$). Therefore, in the following, we shall allow $x = 0$.

We shall look for a Large Deviation Principle (LDP) for the $\widetilde{\mathcal{S}}_m^+$ valued diffusion with small diffusion coefficient:

$$
\begin{aligned}
dX_t^\epsilon &= \epsilon(\sqrt{X_t^\epsilon}\, dB_t + dB_t'\sqrt{X_t^\epsilon}) + \delta I_m\, dt, \quad t \leqslant T, \\
X_0^\epsilon &= x,
\end{aligned} \tag{1.2}
$$

with $\delta > 0$. By a scale change, X_t^ϵ/ϵ^2 is a solution of (1.1) for the dimension δ/ϵ^2 and the initial condition x/ϵ^2. According to the above existence theorem, for $\delta > 0$ and ϵ small enough ($\epsilon \leqslant \epsilon_0$), (1.2) has a unique solution $X_t^\epsilon \in \widetilde{\mathcal{S}}_m^+$ for $t > 0$.

Note that this problem is equivalent to an LDP for the family of processes $(n^{-1}X_t^{(n\delta)}; t \leqslant T)$, where $X_t^{(n\delta)}$ denotes a Wishart process of dimension $n\delta$, starting from nx as $n \to \infty$.

We stress that the size m is fixed and only the dimension n tends to ∞; this is outside the scope of the random matrix theory where, in general, both parameters m, n tend to ∞ so that m/n converges to a strictly positive constant. When $m = 1$, (1.1) is the equation for the squared Bessel process (BESQ) of dimension δ. In that case, it is well known that the equation (1.1) has a unique strong solution for $x \geqslant 0$, $\delta \geqslant 0$.

In a companion paper [5], we studied large deviations for BESQ and squared Ornstein–Uhlenbeck processes, that is, for the scalar Wishart processes ($m = 1$). Note that the diffusion coefficient in the BESQ equation is not Lipschitz and the Freidlin–Wentzell theory does not apply directly (in the degenerate cases: $x = 0$ or $\delta = 0$). We gave three approaches; the first one was based upon exponential martingales, the second one uses the infinite divisibility of the law of BESQ processes (and thus a Cramer theorem), and the third method is a consequence of continuity of the Itô map for the Bessel equation (not square), a property proved by McKean [9]. We also refer to Feng [6] for the study of an LDP for squares of Ornstein–Uhlenbeck processes.

In the matrix case, due to the restriction on the dimension δ, the laws Q_x^δ of the Wishart processes are no more infinitely divisible. Moreover, we have no analogue of the Bessel equation for the square root of a Wishart process. Thus, we shall focus on the exponential martingale approach to extend the LDP in the matrix case. Since the delicate point is for a degenerate initial condition, we shall assume in the following that the initial condition is $x = 0$.

We denote by $C_0([0,T]; \widetilde{\mathcal{S}}_m^+)$ the space of continuous paths φ_t from $[0,T]$ to $\mathcal{S}_m^+$ such that $\varphi_0 = 0$ and $\varphi_t \in \widetilde{\mathcal{S}}_m^+$ for $t > 0$, endowed with the supremum norm

$$
\|\varphi\| = \sup_{t \in [0,T]} \|\varphi_t\|_{S_m},
$$

where $\|\cdot\|_{S_m}$ denotes any equivalent norm on the space of symmetric matrices of size m. We also define

$$
\begin{aligned}
\mathcal{H}_0 = \{&\varphi \in C_0([0,T]; \widetilde{\mathcal{S}}_m^+); \\
&\varphi \text{ is absolutely continuous with respect to Lebesgue measure}\}.
\end{aligned}
$$

The main result of the paper is

THEOREM 1.1. *Let $\delta > 0$ and $\epsilon \leqslant \epsilon_0$. Then the family P^ϵ of distributions of $(X_t^\epsilon; t \in [0,T])$, a solution of* (1.2), *satisfies an LDP in $C_0([0,T];\widetilde{\mathcal{S}}_m^+)$ with speed ϵ^2 and good rate function*

$$I(\varphi) = \frac{1}{8}\int_0^T \operatorname{Tr}\big(k_\varphi(s)\varphi(s)k_\varphi(s)\big)ds, \qquad \varphi \in \mathcal{H}_0, \tag{1.3}$$

where $k_\varphi(s)$ is the unique symmetric matrix, solution of the equation

$$k_\varphi(s)\varphi(s) + \varphi(s)k_\varphi(s) = 2\big(\dot{\varphi}(s) - \delta I_m\big) \quad \textit{for a.e. } s > 0, \tag{1.4}$$

and $I(\varphi) = \infty$ for $\varphi \notin \mathcal{H}_0$.

REMARK 1.1. *In the real case* $(m = 1)$, *we obtain* (see [5]),

$$I(\varphi) = \frac{1}{8}\int_0^T \frac{\big(\dot{\varphi}(s) - \delta\big)^2}{\varphi(s)} ds.$$

The outline of the paper is the following. In Section 2, we prove an exponential tightness result for the distribution P^ϵ of X^ϵ. In Section 3, we prove Theorem 1.1 using the approach of exponential martingales. In Section 4, we discuss the Cramer's approach, using the additivity of Wishart processes, when we put some restriction on the parameter δ. In Section 5, we give some applications of the contraction principle to obtain an LDP for some functionals of the Wishart process.

2. EXPONENTIAL TIGHTNESS

We follow the same lines as in [5], Section 2. Since the paths of the Wishart process are a.s. α-Hölderian for $\alpha < 1/2$, we prove exponential tightness in the space $C_0^\alpha([0,T],\widetilde{\mathcal{S}}_m^+)$ of α-Hölder continuous functions in $C_0([0,T];\widetilde{\mathcal{S}}_m^+)$ endowed with the norm

$$\|\varphi\|_\alpha = \sup_{0\leqslant s\neq t\leqslant T} \frac{\|\varphi_t - \varphi_s\|}{|t-s|^\alpha},$$

where $\|\cdot\|$ is a norm on $\mathcal{S}_m^+$. Since all the norms are equivalent, we shall choose a suitable norm and we consider in this section $\|M\| = \sum_{1\leqslant i,j\leqslant m} |M_{ij}|$.

PROPOSITION 2.1. *Let $\alpha < 1/2$. The family of distributions P_ϵ of X^ϵ is exponentially tight in $C_0^\alpha([0,T],\widetilde{\mathcal{S}}_m^+)$, in scale ϵ^2, i.e., for $L > 0$ there exists a compact set K_L in C_α such that*

$$\limsup_{\epsilon\to 0} \epsilon^2 \ln P(X^\epsilon \notin K_L) \leqslant -L. \tag{2.1}$$

Proof. Let us fix $\alpha' \in (\alpha, 1/2)$ and $R > 0$. The closed Hölder ball $B_{\alpha'}(0, R)$ is a compact set of $C_0^\alpha([0,T], \widetilde{\mathcal{S}}_m^+)$. Thus it is enough to estimate $P(\|X^\epsilon\|_{\alpha'} \geqslant R)$. For simplicity, we assume $T = 1$. Then

$$\|X^\epsilon\|_{\alpha'} \leqslant \|M^\epsilon\|_{\alpha'} + \delta m,$$

where M^ϵ is a martingale defined by

$$M_t^\epsilon = \epsilon(\sqrt{X_t^\epsilon}\, dB_t + dB_t' \sqrt{X_t^\epsilon}).$$

Bounds for $\|M^\epsilon\|_{\alpha'}$. We shall use the Garsia–Rodemich–Rumsey lemma (see [12], p. 47) with $\Psi(x) = \exp(c\epsilon^{-2}x) - 1$ for some $0 < c < 1/2$ and $p(x) = x^{1/2}$. Then

$$\Psi^{-1}(y) = \frac{\epsilon^2}{c} \log(1 + y).$$

The lemma asserts that if

$$\int_0^1 \int_0^1 \Psi\left(\frac{\|M_t^\epsilon - M_s^\epsilon\|}{p(|t-s|)}\right) ds dt \leqslant K,$$

then

$$\|M_t^\epsilon - M_s^\epsilon\| \leqslant 8 \int_0^{|t-s|} \Psi^{-1}(4K/u^2) dp(u).$$

This yields (see the same computations in [5]):

$$P(\|M^\epsilon\|_{\alpha'} \geqslant R) \leqslant P\left(\int_0^1 \int_0^1 \exp\left(c\epsilon^{-2} \frac{\|M_t^\epsilon - M_s^\epsilon\|}{|t-s|^{1/2}}\right) ds dt \geqslant K + 1\right) \tag{2.2}$$

with

$$K = \frac{1}{4}\left[\exp\left(\left(\frac{c\epsilon^{-2}R}{8} - K_2\right) - 4\right) - 1\right] \quad \text{and} \quad K_2 = 2 \sup_{u \in [0,1]} u^{1/2-\alpha'} \log \frac{1}{u}.$$

Now, by Markov's inequality,

$$P(\|M^\epsilon\|_{\alpha'} \geqslant R) \leqslant \frac{1}{K+1} \int_0^1 \int_0^1 E\left[\exp\left(c\epsilon^{-2} \frac{\|M_t^\epsilon - M_s^\epsilon\|}{|t-s|^{1/2}}\right)\right] ds dt. \tag{2.3}$$

Consequently, for a matrix M,

$$\begin{aligned}
\exp(\lambda\|M\|) = \prod_{i,j} \exp(\lambda|M_{ij}|) &\leqslant \prod_{i,j} [\exp(\lambda M_{ij}) + \exp(-\lambda M_{ij})] \\
&\leqslant m^2 \max[\exp(\lambda M_{ij}) + \exp(-\lambda M_{ij})] \\
&\leqslant m^2 \sum_{i,j} [\exp(\lambda M_{ij}) + \exp(-\lambda M_{ij})].
\end{aligned}$$

Thus,

$$\begin{aligned}
&E[\exp(\lambda\|M^\epsilon_t - M^\epsilon_s\|)] \\
&\leqslant m^2\sum_{i,j}\Big(E\Big[\exp\Big(\lambda\big(M^\epsilon_{i,j}(t)-M^\epsilon_{i,j}(s)\big)\Big)\Big]+E\Big[\exp\Big(-\lambda\big(M^\epsilon_{i,j}(t)-M^\epsilon_{i,j}(s)\big)\Big)\Big]\Big) \\
&\leqslant 2m^4\max_{i,j} E[\exp(2\lambda^2\langle M^\epsilon_{i,j}\rangle^t_s)],
\end{aligned}$$

where in the last inequality we use the exponential inequality for continuous martingales

$$E[\exp(\lambda Z_t)] \leqslant E[\exp(2\lambda^2\langle Z\rangle_t)].$$

Now,

$$\begin{aligned}
\langle M^\epsilon_{i,j}\rangle^t_s &= \epsilon^2 \int_s^t \big(X^\epsilon_{ii}(u) + X^\epsilon_{jj}(u)\big)du \\
&\leqslant \epsilon^2 \int_s^t \mathrm{Tr}(X^\epsilon_u)du.
\end{aligned}$$

Let us set $Y^\epsilon_u := \mathrm{Tr}(X^\epsilon_u)$. Then Y^ϵ_u is a squared Bessel process, a solution of the following SDE:

$$\begin{aligned}
dY^\epsilon_u &= 2\epsilon\sqrt{Y^\epsilon_u}d\beta_u + \delta m\, dt, \\
Y^\epsilon_0 &= 0,
\end{aligned} \tag{2.4}$$

with β a real Brownian motion. Thus, we obtain

$$\begin{aligned}
E\left[\exp\left(c\epsilon^{-2}\frac{\|M^\epsilon_t - M^\epsilon_s\|}{|t-s|^{1/2}}\right)\right] &\leqslant 2m^4\left\{E\left[\exp\left(\frac{2c^2\epsilon^{-2}}{t-s}\int_s^t Y^\epsilon_u du\right)\right]\right\}^{1/2} \\
&\leqslant 2m^4\left\{\frac{1}{t-s}\int_s^t E[\exp(2c^2\epsilon^{-2}Y^\epsilon_u)]\,du\right\}^{1/2}
\end{aligned} \tag{2.5}$$

(by Jensen's inequality). Consequently, we get

$$P(\|M^\epsilon\|_\alpha \geqslant R) \leqslant \frac{2m^4}{K+1}\{\sup_{u\in[0,1]} E[\exp(2c^2\epsilon^{-2}Y^\epsilon_u)]\}^{1/2}, \tag{2.6}$$

where $K+1 = C\exp(cR\epsilon^{-2}/8)$, and C is a constant. Now,

$$E[\exp(2c^2\epsilon^{-2}Y^\epsilon_u)] = Q^{m\delta\epsilon^{-2}}_0[\exp(2c^2X_u)],$$

where Q^ρ_x denotes the distribution of a squared Bessel process, starting from x, of dimension ρ. The Laplace transform of the BESQ is known (see [11]) and for $c < 1/2$ we obtain

$$Q^{m\delta\epsilon^{-2}}_0[\exp(2c^2X_u)] = (1-4c^2u)^{-(m\delta\epsilon^{-2})/2},$$

which implies

$$P(\|M^\epsilon\|_{\alpha'} \geqslant R) \leqslant C_m A^{m\delta\epsilon^{-2}} \exp(-cR\epsilon^{-2}/8) \tag{2.7}$$

for a positive constant A. Thus,

$$\lim_{R\to+\infty} \limsup_{\epsilon\to 0} \epsilon^2 \ln P(\|M^\epsilon\|_{\alpha'} \geqslant R) = -\infty. \quad \blacksquare$$

3. PROOF OF THEOREM 1.1

From the previous section it follows that the distribution P^ϵ of X^ϵ is exponentially tight in $C_0([0,1]; \widetilde{\mathcal{S}}_m^+)$. Then we need to prove a weak LDP, that is, to prove the upper bound for compact sets. We assume that $T = 1$. According to [3], Exercise 2.1.14 (v) (or [2], Theorem 4.1.11), it is enough to show that

$$-I(\varphi) = \lim_{r\to 0} \limsup_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) = \lim_{r\to 0} \liminf_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big),$$

where $B_r(\varphi)$ denotes the open ball with center $\varphi \in C_0([0,1]; \widetilde{\mathcal{S}}_m^+)$ and radius r, and I is a lower-semicontinuous function.

3.1. The upper bound. First we show the following bound:

$$\lim_{r\to 0} \limsup_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \leqslant -I(\varphi). \tag{3.1}$$

We denote by $\mathcal{M}_m$, respectively $\mathcal{S}_m$, the space of $m \times m$ matrices, respectively symmetric matrices, endowed with the scalar product:

$$\langle A, B\rangle = \mathrm{Tr}(AB').$$

The corresponding norm is denoted by $\|A\|_2$. Set

$$H = \{h \in C([0,1]; \mathcal{S}_m) : \dot{h} \in L^2([0,1]; \mathcal{S}_m)\}.$$

For $h \in H$ let

$$M_t^{\epsilon,h} = \exp\left(\frac{1}{\epsilon^2}\left\{\int_0^t \mathrm{Tr}\left(h(s)(dX_s^\epsilon - \delta I_m\, ds)\right) - \frac{1}{2}\langle Z^\epsilon, Z^\epsilon\rangle_t\right\}\right), \quad t \leqslant 1,$$

where

$$Z_t^\epsilon = \int_0^t \mathrm{Tr}\left(h(s)\sqrt{X_s^\epsilon} dB_s + h(s) dB_s' \sqrt{X_s^\epsilon}\right),$$

$$\langle Z^\epsilon, Z^\epsilon\rangle_t = 4\int_0^t \mathrm{Tr}\left(h(s) X_s^\epsilon\, h(s)\right) ds.$$

$M^{\epsilon,h}$ is a positive, local martingale. In fact, using a Novikov's type criterion (see [11], Exercise VIII.1.40, and [5]), the condition

$$E\Big[\exp\Big(a\,\mathrm{Tr}\,\big(h(s)X_s^\epsilon h(s)\big)\Big)\Big] < c$$

for every $s \geqslant 1$ and two constants a and c ensures that $M^{\epsilon,h}$ is a martingale. Now,

$$\mathrm{Tr}\,\big(h(s)X_s^\epsilon h(s)\big) \leqslant C(h) \sup_{s\in[0,1]} \|X_s^\epsilon\|.$$

With the notation of Section 2, for any $\alpha' < 1/2$ we have

$$\sup_s \|X_s^\epsilon\| \leqslant \|M_s^\epsilon\|_{\alpha'} + \delta m$$

and from (2.7) it follows that $\sup_{s\in[0,1]} \|X_s^\epsilon\|$ has some exponential moments. Therefore, $M^{\epsilon,h}$ is a martingale, and so $E(M_t^{\epsilon,h}) = 1$. Integrating by parts, we obtain

$$M_1^{\epsilon,h} = \exp\left(\frac{1}{\epsilon^2}\Phi(X^\epsilon;h)\right)$$

with

$$\Phi(\varphi;h) = G(\varphi;h) - 2\int_0^1 \mathrm{Tr}\,\big(h(s)\varphi(s)h(s)\big)ds$$

and

$$G(\varphi;h) = \mathrm{Tr}\,\big(h_1(\varphi_1 - \delta I_m)\big) - \int_0^1 \mathrm{Tr}\,\big((\varphi_s - \delta s I_m)\dot{h}_s\big)ds$$

for $\varphi \in C_0([0,1];\widetilde{\mathcal{S}}_m^+)$. Note that if φ is absolutely continuous, then

$$G(\varphi;h) = \int_0^1 \mathrm{Tr}\,\big(h(s)(\dot{\varphi}_s - \delta I_m\, ds)\big).$$

For $\varphi \in C_0([0,1];\widetilde{\mathcal{S}}_m^+)$, $h \in H$, we get

$$\begin{aligned}
P\big(X^\epsilon \in B_r(\varphi)\big) &= P\bigg(X^\epsilon \in B_r(\varphi); \frac{M_1^{\epsilon,h}}{M_1^{\epsilon,h}}\bigg) \\
&\leqslant \exp\left(-\frac{1}{\epsilon^2}\inf_{\psi\in B_r(\varphi)}\Phi(\psi;h)\right)E(M_1^{\epsilon,h}) \\
&\leqslant \exp\left(-\frac{1}{\epsilon^2}\inf_{\psi\in B_r(\varphi)}\Phi(\psi;h)\right),
\end{aligned}$$

which yields

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \leqslant -\inf_{\psi\in B_r(\varphi)}\Phi(\psi;h).$$

For $h \in H$, the map $\varphi \to \Phi(\varphi; h)$ is continuous on $C_0([0,1], \mathcal{S}_m^+)$, so that

$$\lim_{r\to 0} \limsup_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \leqslant -\Phi(\varphi; h).$$

Minimizing in $h \in H$, we obtain

$$\lim_{r\to 0} \limsup_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \leqslant -\sup_{h\in H} \Phi(\varphi; h).$$

PROPOSITION 3.1. *For* $\varphi \in C_0([0,1]; \widetilde{\mathcal{S}}_m^+)$,

$$\sup_{h\in H} \Phi(\varphi; h) = I(\varphi),$$

where $I(\varphi)$ *is defined by* (1.3). *As a consequence*, I *is lower-semicontinuous.*

Proof. Since $\varphi \in C_0([0,1]; \widetilde{\mathcal{S}}_m^+)$, $\int_0^1 \mathrm{Tr}(h_s\varphi_s h_s)ds > 0$ for $h \not\equiv 0$. Replacing h by λh for $\lambda \in \mathbb{R}$, we can see that

$$J(\varphi) := \sup_{h\in H} \Phi(\varphi; h) = \frac{1}{8} \sup_{h\in H} \frac{G^2(\varphi; h)}{\int_0^1 \mathrm{Tr}(h_s\varphi_s h_s)ds}.$$

1. We assume that $J(\varphi) < \infty$. Let us denote by $\|h\|_{L^2(\varphi)}$ the Hilbert norm on $C_0([0,1]; \mathcal{S}_m)$ given by

$$\|h\|^2_{L^2(\varphi)} = \int_0^1 \mathrm{Tr}(h_s\varphi_s h_s)ds.$$

The linear form $G_\varphi \colon h \to G(\varphi; h)$ can be extended to the space $L^2(\varphi)$ and, by the Riesz theorem, there exists a function $k_\varphi \in L^2(\varphi)$ such that

$$G(\varphi; h) = \int_0^1 \mathrm{Tr}\,\big(h_s\varphi_s k_\varphi(s)\big)ds. \tag{3.2}$$

Thus, φ is absolutely continuous and we have

$$\int_0^1 \mathrm{Tr}\,\big(h_s(\dot\varphi_s - \delta I_m)\big)ds = \int_0^1 \mathrm{Tr}\,\big(h_s\varphi_s k_\varphi(s)\big)ds \tag{3.3}$$

for all symmetric matrices $h(s)$. Let k_φ be given by (1.4). We refer to the Appendix for the existence of a unique solution of (1.4) (k_φ is unique in L^0, i.e., as a class of equivalent functions a.e. equal, since $\dot\varphi$ is also defined a.e.). Then, it is easy to see that (3.3) is satisfied for all h symmetric. Moreover, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_0^1 \mathrm{Tr}\,\big(h_s\varphi_s k_\varphi(s)\big)ds &\leqslant \int_0^1 \mathrm{Tr}(h_s\varphi_s h_s)^{1/2}\,\mathrm{Tr}\,\big(k_\varphi(s)\varphi_s k_\varphi(s)\big)^{1/2}ds \\ &\leqslant \Big(\int_0^1 \mathrm{Tr}(h_s\varphi_s h_s)ds\Big)^{1/2}\Big(\int_0^1 \mathrm{Tr}\,\big(k_\varphi(s)\varphi_s k_\varphi(s)\big)ds\Big)^{1/2}\end{aligned} \tag{3.4}$$

with equality for $h = k_\varphi$. Thus,

$$\frac{1}{8} \sup_{h \in L^2(\varphi)} \frac{G^2(\varphi; h)}{\|h\|^2_{L^2(\varphi)}} = I(\varphi).$$

Now, the equality between $I(\varphi)$ and $J(\varphi)$ follows by density of H in $L^2(\varphi)$.

2. We assume now that $I(\varphi) < \infty$. Then φ is absolutely continuous and we define k_φ by (1.4). Consequently, (3.2) holds and from (3.4) we obtain $J(\varphi) < I(\varphi) < \infty$. Thus, $I(\varphi) = J(\varphi)$ in all cases. Since I is a supremum of continuous functions, I is lower-semicontinuous. ■

3.2. The lower bound. In order to obtain the lower bound, we first prove

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \geqslant -I(\varphi)$$

for all $r > 0$ and for φ in a subclass $\mathcal{K}$ of $C_0([0,1]; \widetilde{\mathcal{S}}^+_m)$. Then we shall show that this subclass is rich enough.

Let $\mathcal{K}$ be the set of functions φ such that $I(\varphi) < \infty$ and such that k_φ defined by (1.4) belongs to H. For $\varphi \in \mathcal{K}$, set $h_\varphi = \frac{1}{4} k_\varphi$. As in the previous subsection, we introduce the new probability measure

$$\hat{P} := M_1^{\epsilon, h_\varphi} P,$$

where P is the Wiener measure on $C([0,1]; \mathcal{M}_{m,m})$. Under $\hat{P}$, we get

$$B_t = \hat{B}_t + \frac{2}{\epsilon} \int_0^t \big(\sqrt{X^\epsilon_s}\, h_\varphi(s)\big)\, ds,$$

where $\hat{B}$ is a Brownian matrix on $\hat{P}$. Thus, under $\hat{P}$, X^ϵ solves the SDE

$$dX^\epsilon_t = \epsilon(\sqrt{X^\epsilon_t} d\hat{B}_t + d\hat{B}'_t \sqrt{X^\epsilon_t}) + \Big(2\big(X^\epsilon_t h_\varphi(t) + h_\varphi(t) X^\epsilon_t\big) + \delta I_m\Big) dt.$$

Under $\hat{P}$, $X^\epsilon_t \underset{\epsilon \to 0}{\longrightarrow} \Psi_t$ a.s. solution of

$$d\Psi_t = \big[2\big(\Psi_t h_\varphi(t) + h_\varphi(t)\Psi_t\big) + \delta I_m\big] dt,$$

i.e.,

$$\dot{\Psi}_t - \delta I_m = 2\big(\Psi_t h_\varphi(t) + h_\varphi(t)\Psi_t\big) = \frac{1}{2}\big(\Psi_t k_\varphi(t) + k_\varphi(t)\Psi_t\big).$$

Since k_φ is continuous, this equation has φ as a unique solution; thus

$$X^\epsilon_t \underset{\epsilon \to 0}{\longrightarrow} \varphi_t \ \hat{P} \text{ a.s.}$$

and $\lim_{\epsilon\to 0} \hat{P}\big(X^\epsilon \in B_r(\varphi)\big) = 1$ for every $r > 0$. Now,

$$P\big(X^\epsilon \in B_r(\varphi)\big) = \hat{P}\left(X^\epsilon \in B_r(\varphi)\frac{1}{M_1^{\epsilon,h_\varphi}}\right)$$
$$\geqslant \exp\left(-\frac{1}{\epsilon^2}\sup_{\psi\in B_r(\varphi)} F(\psi;h_\varphi)\right)\hat{P}\big(X^\epsilon \in B_r(\varphi)\big),$$

which yields

$$\liminf_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \geqslant -\sup_{\psi\in B_r(\varphi)} F(\psi;h_\varphi)$$

and, by continuity of $F(\cdot,h)$,

$$\lim_{r\to 0}\liminf_{\epsilon\to 0} \epsilon^2 \ln P\big(X^\epsilon \in B_r(\varphi)\big) \geqslant F(\varphi;h_\varphi) = I(\varphi).$$

We now prove the following

PROPOSITION 3.2. *For any $\varphi \in C_0([0,1];\widetilde{\mathcal{S}}_m^+)$ such that $I(\varphi) < \infty$, there exists a sequence φ_n of elements of $\mathcal{K}$ such that $\varphi_n \to \varphi$ in $C_0([0,1];\widetilde{\mathcal{S}}_m^+)$ and $I(\varphi_n) \to I(\varphi)$.*

Proof. We follow the same lines as in the proof of the corresponding result for the scalar case in [5].

(a) First, let us show that the condition $I(\varphi) < \infty$ implies that

$$\lim_{t\to 0}\frac{\varphi_t}{t} = \delta I_m.$$

From the scalar case we know that

$$\lim_{t\to 0}\frac{\mathrm{Tr}(\varphi_t)}{t} = \delta m. \tag{3.5}$$

Indeed, $\mathrm{Tr}(X_t^\varepsilon)$ satisfies an LDP (see (2.4)) with rate function given by

$$J(g) = \frac{1}{8}\int_0^1 \frac{\big(\dot{g}(s)-\delta m\big)^2}{g(s)}ds,$$

and $J(g) < \infty$ implies that $\lim_{t\to 0} g(t)/t = \delta m$ (see [5], [6]). From the upper bound, the condition $I(\varphi) < \infty$ implies $J\big(\mathrm{Tr}(\varphi)\big) < \infty$, and thus the condition (3.5) is satisfied.

Let us put $\|A\|_1 = \mathrm{Tr}(|A|)$ and $\|A\|_2 = \big(\mathrm{Tr}(|A|^2)\big)^{1/2}$ for a matrix A. Then

$$\begin{aligned}
\|\varphi_t - \delta t I_m\|_1 &= \Big\| \int_0^t (\dot{\varphi}_s - \delta I_m) ds \Big\|_1 = \frac{1}{2} \Big\| \int_0^t \big(\varphi_s k_\varphi(s) + k_\varphi(s)\varphi_s\big) ds \Big\|_1 \\
&\leqslant \frac{1}{2} \Big(\int_0^t \|\varphi_s k_\varphi(s)\|_1 ds + \int_0^t \|k_\varphi(s)\varphi_s\|_1 ds \Big) \\
&\leqslant \int_0^t \|\sqrt{\varphi_s}\|_2 \, \|\sqrt{\varphi_s} k_\varphi(s)\|_2 ds \\
&= \int_0^t \big(\mathrm{Tr}(\varphi_s)\big)^{1/2} \Big(\mathrm{Tr}\,\big(k_\varphi(s)\varphi_s k_\varphi(s)\big)\Big)^{1/2} ds \\
&\leqslant \Big(\int_0^t \mathrm{Tr}(\varphi_s) ds\Big)^{1/2} \Big(\int_0^t \mathrm{Tr}\,\big(k_\varphi(s)\varphi_s k_\varphi(s)\big) ds\Big)^{1/2}.
\end{aligned}$$

Thus,

$$\Big\| \frac{\varphi_t}{t} - \delta I_m \Big\|_1 \leqslant \left(\frac{1}{t} \int_0^t \frac{\mathrm{Tr}(\varphi_s)}{s} ds \right)^{1/2} \Big(\int_0^t \mathrm{Tr}\,\big(k_\varphi(s)\varphi_s k_\varphi(s)\big) ds\Big)^{1/2}.$$

According to (3.5), the first term on the right-hand side is bounded and the second tends to 0 as t tends to 0 since $I(\varphi) < \infty$.

(b) As a second step, we approximate φ by ψ such that $k_\psi \in L^2([0,1]; \mathcal{S}_m)$. Let us set

$$\psi_r(t) = \begin{cases} \delta t I_m, & t \leqslant r/2, \\ (\delta r/2) I_m + (t - r/2) a_r, & r/2 \leqslant t \leqslant r, \\ \varphi(t), & t \geqslant r, \end{cases}$$

where the matrix a_r is chosen such that ψ is continuous in r. Let k_ψ be the solution of (1.4) associated with ψ. Since $k_\psi(s) = 0$ for $s \in [0, r/2]$, and $\psi(s)$ is invertible for $s > 0$, we have $k_\psi \in L^2([0,1]; \mathcal{S}_m)$. Obviously, $\psi_r \underset{r\to 0}{\longrightarrow} \varphi$ in $C_0([0,1]; \widetilde{\mathcal{S}}_m^+)$.

It remains to prove the convergence of $I(\psi_r)$ to $I(\varphi)$, or that

$$\int_{r/2}^r \mathrm{Tr}\,\big(k_\psi(s)\psi(s)k_\psi(s)\big) ds \underset{r\to 0}{\longrightarrow} 0.$$

We have

$$\begin{aligned}
\int_{r/2}^r \mathrm{Tr}\,\big(k_\psi(s)\psi(s)k_\psi(s)\big) ds &= \int_{r/2}^r \mathrm{Tr}\,\Big(k_\psi(s)\big(\dot{\psi}(s) - \delta I_m\big)\Big) ds \\
&= \int_{r/2}^r \mathrm{Tr}\,\big(k_\psi(s)(a_r - \delta I_m)\big) ds.
\end{aligned}$$

Note that a_r and $k_{\psi_r}(s)$ for $s \in [r/2, r]$ are diagonalisable in the same basis with respective eigenvalues $(a_i^{(r)})_i$ and $k_i(s)$, where

$$k_i(s) = \frac{a_i^{(r)} - \delta}{\delta r/2 + (s - r/2)a_i^{(r)}},$$

and that, according to step (a), $\lim_{r \to 0} a_i^{(r)} = \delta$. Thus, for r small enough,

$$\int_{r/2}^{r} \operatorname{Tr}\big(k_\psi(s)\psi(s)k_\psi(s)\big)ds = \int_{r/2}^{r} \sum_i \frac{(a_i^{(r)} - \delta)^2}{\delta r/2 + (s - r/2)a_i^{(r)}} ds \leqslant \frac{1}{\delta} \sum_{i=1}^{m} (a_i^{(r)} - \delta)^2,$$

and the last quantity tends to 0 as r tends to 0.

(c) By (b), we must find an approximating sequence $\varphi^{(n)}$ of φ in $\mathcal{K}$ for φ satisfying $k_\varphi \in L^2$.

Let $k^{(n)}$ be a sequence of smooth functions with values in $\mathcal{S}_m$ such that $k^{(n)}$ converges to k_φ in $L^2([0,1], \mathcal{S}_m)$. Let $\varphi^{(n)}$ be the unique solution of the equation

$$\begin{aligned} \dot{\varphi}_t^{(n)} - \delta I_m &= k_t^{(n)}\varphi_t^{(n)} + \varphi_t^{(n)}k_t^{(n)}, \\ \varphi_0^{(n)} &= 0. \end{aligned}$$

Since

$$\|\varphi_t^{(n)}\| \leqslant \int_0^t \|\dot{\varphi}_s^{(n)}\| ds \leqslant 2\int_0^t \|\varphi_s^{(n)}\|\,\|k_s^{(n)}\| ds + \delta,$$

the Gronwall inequality shows that

$$\sup_n \sup_{t \in [0,1]} \|\varphi_t^{(n)}\| < \infty,$$

where we have chosen the operator norm on the set of matrices in the previous inequality. Another application of Gronwall's inequality entails that

$$\sup_{t \in [0,1]} \|\varphi_t - \varphi_t^{(n)}\| \underset{n \to \infty}{\longrightarrow} 0.$$

Now, the convergence of $I(\varphi^{(n)})$ to $I(\varphi)$ follows from the convergence in L^2 of $k^{(n)}$ to k_φ and the convergence in $L^\infty([0,1])$ of $\varphi^{(n)}$ to φ. ■

4. THE CRAMER THEOREM

Let Q_x^δ denote the distribution on $C(\mathbb{R}, \mathcal{S}_m^+)$ of the Wishart process of dimension $\delta \geqslant m + 1$, starting from $x \in \mathcal{S}_m^+$. We recall the following additivity property (see [1]):

$$Q_x^\delta \oplus Q_y^{\delta'} = Q_{x+y}^{\delta+\delta'}.$$

Let $\delta \geqslant m+1$ and take $\epsilon = 1/\sqrt{n}$. Then X^ϵ, a solution of (1.2), is distributed as $n^{-1}\sum_{i=1}^n X_i$, where X_i are independent copies of Q_x^δ. From Cramer's theorem ([2], Chapter 6), we obtain

THEOREM 4.1. *Assume that $\delta \geqslant m+1$. The family P^ϵ of distributions of $(X_t^\epsilon; t\in[0,T])$, solution of* (1.2), *satisfies an LDP in $C_0([0,T];\widetilde{\mathcal{S}}_m^+)$ with speed ϵ^2 and good rate function*

$$\Lambda^*(\varphi) = \sup_{\mu\in\mathcal{M}([0,T],\mathcal{S}_m)} \Big(\int_0^T \mathrm{Tr}(\varphi_t d\mu_t) \ - \Lambda(\mu)\Big), \tag{4.1}$$

where

$$\Lambda(\mu) = \ln\Big[Q_x^\delta\Big(\exp\big(\int_0^T \mathrm{Tr}(X_s d\mu_s)\big)\Big)\Big]. \tag{4.2}$$

The Laplace transform of the Q_x^δ distribution can be computed explicitly in terms of the Ricatti equation, extending to the matrix case the well-known result for the squared Bessel processes (see [10] and [11], Chapter XI).

LEMMA 4.1. *Let μ be a positive $\mathcal{S}_m^+$-valued measure on $[0,T]$. Then*

$$\begin{aligned} Q_x^\delta\bigg(\exp\bigg(-\frac{1}{2}\int_0^T \mathrm{Tr}(X_s d\mu_s)\bigg)\bigg) \\ = \exp\bigg(\frac{1}{2}\mathrm{Tr}\,\big(F_\mu(0)x\big)\bigg)\exp\bigg(\frac{\delta}{2}\int_0^T \mathrm{Tr}\,\big(F_\mu(s)\big)ds\bigg), \end{aligned}\tag{4.3}$$

where $F_\mu(s)$ is the $\mathcal{S}_m$-valued, right continuous solution of the Riccati equation

$$\dot F + F^2 = \mu, \qquad F(T) = 0. \tag{4.4}$$

P r o o f. From Itô's formula we obtain

$$\begin{aligned} F_\mu(t)X_t &= F_\mu(0)x + \int_0^t F_\mu(s)dX_s + \int_0^t dF_\mu(s)X_s \\ &= F_\mu(0)x + \int_0^t F_\mu(s)dX_s + \int_0^t d\mu(s)X_s - \int_0^t F_\mu^2(s)X_s\,ds. \end{aligned}$$

Consider the exponential local martingale

$$Z_t = \exp\bigg(\frac{1}{2}\int_0^t \mathrm{Tr}\,\big(F_\mu(s)dM_s\big) - \frac{1}{2}\int_0^t \mathrm{Tr}\,\big(F_\mu(s)X_sF_\mu(s)\big)ds\bigg),$$

where $M_s = X_s - \delta I_m\, s$. Then

$$Z_t = \exp\Big(\frac{1}{2}\Big(\mathrm{Tr}\,\big(F_\mu(t)X_t\big) - \mathrm{Tr}\,\big(F_\mu(0)x\big) - \delta\int_0^t \mathrm{Tr}\,\big(F_\mu(s)\big)ds - \int_0^t \mathrm{Tr}\,\big(X_s d\mu(s)\big)\Big)\Big).$$

Now, X_t is positive and $F_\mu(t)$ is negative (see Appendix A.2). Thus, $\mathrm{Tr}\,\big(X_t F_\mu(t)\big) \leqslant 0$ and Z_t is a bounded martingale. The lemma follows from the equality $E(Z_0) = E(Z_T)$. ■

REMARK 4.1. 1. *The condition* $F(T) = 0$ *in* (4.4) *is equivalent to* $F(T-) = -\mu(\{T\})$.

2. *Taking* $d\mu_s = 2\Theta\delta_1(ds)$, *where* Θ *is a symmetric positive matrix, we find that* $F_\mu(t) = -2\Theta\big(I_m + 2(1-t)\Theta\big)^{-1}$, $t < 1$, *from which we obtain* (see [1])

$$Q_x^\delta\Big(\exp\big(-\mathrm{Tr}(X_1\Theta)\big)\Big) = \det(I_m + 2\Theta)^{-\delta/2}\exp\Big(-\mathrm{Tr}\,\big(x(I_m+2\Theta)^{-1}\Theta\big)\Big). \tag{4.5}$$

For $m = 1$, this example is given in [5], Subsection 8.3.

Let us try to determine the correspondence between φ and μ in (4.1). If μ is a negative measure, then from (4.3) we get

$$\int_0^T \mathrm{Tr}(\varphi_t d\mu_t) - \Lambda(\mu) = \int_0^T \mathrm{Tr}(\varphi_t d\mu_t) - \frac{1}{2}\mathrm{Tr}\,\big(F_{-2\mu}(0)x\big) - \frac{\delta}{2}\int_0^T \mathrm{Tr}\,\big(F_{-2\mu}(s)\big)ds.$$

Since $d\mu(t) = -\frac{1}{2}(\dot F_t + F_t^2)$, an integration by parts gives

$$\int_0^T \mathrm{Tr}(\varphi_t d\mu_t) - \Lambda(\mu) = \frac{1}{2}\int_0^T \mathrm{Tr}\,\big(F_{-2\mu}(s)(\dot\varphi_s - \delta I_m)\big)ds - \frac{1}{2}\int_0^T \mathrm{Tr}\,\big(F_{-2\mu}^2(s)\varphi_s\big)ds. \tag{4.6}$$

The optimal function $F(s)$ giving the supremum in (4.6) solves the equation

$$\dot\varphi_s - \delta I_m = \varphi_s F(s) + F(s)\varphi_s,$$

that is, $F(s) = k_\varphi(s)/2$, where k_φ is the solution of (1.4), and for this F the right-hand side of (4.6) is exactly $I(\varphi)$.

5. SOME APPLICATIONS

From the contraction principle we can obtain an LDP for some continuous functionals of the Wishart process X^ϵ.

5.1. The eigenvalues process. Let $\Big(\lambda^\epsilon(t) = \big(\lambda_1^\epsilon(t), \ldots, \lambda_m^\epsilon(t)\big); t \in [0, T]\Big)$ denote the process of eigenvalues of the process X^ϵ, ordered decreasingly.

PROPOSITION 5.1. *The process λ^ϵ satisfies an LDP in $C_0([0,T], \mathbb{R}_+^m)$, in scale ϵ^2, with rate function*

$$J(x) = \frac{1}{8} \sum_{i=1}^{m} \int_0^T \frac{\big(\dot{x}_i(t) - \delta\big)^2}{x_i(t)} dt. \tag{5.1}$$

REMARK 5.1. $\big(\lambda^\epsilon(t)\big)_t$ *is a solution of the SDE* (see [1])

$$d\lambda_i^\epsilon(t) = 2\epsilon \sqrt{\lambda_i^\epsilon(t)} d\beta_i(t) + \left\{ \delta + \epsilon^2 \sum_{k \neq i} \frac{\lambda_i^\epsilon(t) + \lambda_k^\epsilon(t)}{\lambda_i^\epsilon(t) - \lambda_k^\epsilon(t)} \right\} dt$$

from which we can guess the form of the rate function J in (5.1) *since the drift b_ϵ in the above equation satisfies $b_\epsilon(\lambda) \underset{\epsilon \to 0}{\longrightarrow} \delta$. Nevertheless, since the drift $b_\epsilon(\lambda)$ explodes on the hyperplanes $\{\lambda_i = \lambda_j\}$ and the diffusion coefficient is degenerate, the classical results* (see [7], Theorem V.3.1) *do not apply.*

P r o o f o f P r o p o s i t i o n 5.1. According to the contraction principle,

$$J(x) = \inf\{I(\varphi);\ \text{e.v.}(\varphi) = x\}.$$

Write $\varphi_t = P_t^{-1} \Lambda_t P_t$, where Λ_t is the diagonal matrix of eigenvalues of φ_t, and P_t is an orthogonal matrix. Then

$$\dot{\varphi}_t = P_t^{-1} \dot{\Lambda}_t P_t + \dot{P_t}^{-1} \Lambda_t P_t + P_t^{-1} \Lambda_t \dot{P}_t.$$

We denote by $\tilde{k}_t$ the matrix $P_t k_\varphi(t) P_t^{-1}$, where k_φ solves (1.4). Then

$$\text{Tr}\left(k_\varphi(t) \varphi(t) k_\varphi(t)\right) = \text{Tr}\left(\tilde{k}_t \Lambda(t) \tilde{k}_t\right)$$

and

$$\tilde{k}_{ij}(t) \lambda_i(t) + \tilde{k}_{ij}(t) \lambda_j(t) = 2\big(\dot{\lambda}_i(t) - \delta\big) \delta_{ij} + R_{ij}(t),$$

where the matrix R is defined by

$$R(t) = P_t \dot{P_t}^{-1} \Lambda_t + \Lambda_t \dot{P}_t P_t^{-1}.$$

Now, it is easy to verify that $R_{ii}(t) = 0$. Thus

$$\text{Tr}\left(\tilde{k}_t \Lambda(t) \tilde{k}_t\right) = \sum_i \frac{\big(\dot{\lambda}_i(t) - \delta\big)^2}{\lambda_i(t)} + \sum_{i \neq j} \frac{R_{ij}^2(t) \lambda_j(t)}{\lambda_i(t) + \lambda_j(t)},$$

and the infimum of the above quantity is obtained for $R \equiv 0$, corresponding to P_t independent of t. For this choice, $I(\varphi) = J(\lambda)$, where λ is the set of eigenvalues of φ. ∎

5.2. An LDP for the random vector X_1^ϵ

PROPOSITION 5.2. *The random vector X_1^ϵ satisfies an LDP, in scale ϵ^2, with rate function*

$$K(M) = \frac{1}{2}\operatorname{Tr}(M) - \frac{\delta}{2}\ln\big(\det(M)\big) - \frac{m\delta}{2} + \frac{m\delta}{2}\ln(\delta), \qquad M \in \mathcal{S}_m^+. \tag{5.2}$$

REMARK 5.2. *For $m = 1$,*

$$K(a) = \frac{1}{2}[(a-\delta) - \ln(a/\delta)], \qquad a > 0,$$

which corresponds (for $\delta = 1$) to the rate function obtained in the study of an LDP for a $\chi_2(n)$ distribution as $n \to \infty$.

P r o o f (sketch).

(i) Since the application $\varphi \to \varphi(1)$ is continuous, we must minimize $I(\varphi)$ under the constraint $\varphi(1) = M$. The optimal path φ solves the Euler–Lagrange equation (see [8], Chapter 7), given in terms of k_φ by

$$2\dot{k}_\varphi(s) + k_\varphi^2(s) = 0, \qquad s \in (0,1).$$

This leads to $k_\varphi^{-1}(t) = (t/2)I_m + C$ and $\varphi(t) = \delta t I_m + t^2 A$ with matrix A determined by $\varphi(1) = M$. Note that this is the same path as in Remark 4.1. Now, it is easy to verify that for $\varphi(t) = \delta t I_m + t^2(M - \delta I_m)$ we have $I(\varphi) = K(M)$, where K is given by (5.2).

(ii) Of course, we can compute K directly, using the Laplace transform (4.5) (with $x = 0$), and then

$$K(M) = \sup_\Theta \left\{ \operatorname{Tr}(\Theta M) + \frac{\delta}{2}\ln\big(\det(I_m - 2\Theta)\big)\right\}.$$

The optimal Θ_0 is given by $M = \delta(I_m - 2\Theta_0)^{-1}$. ∎

5.3. An LDP for the largest eigenvalue. Let us denote by $\lambda_{\max}^\epsilon = \lambda_1^\epsilon$ the largest eigenvalue of the Wishart process X^ϵ.

PROPOSITION 5.3. *The process $\{\lambda_{\max}^\epsilon(t), t \in [0,T]\}$ satisfies an LDP in the space $C_0\big([0,T];\mathbb{R}_+\big)$, in scale ϵ^2, with rate function given by*

$$I_{\max}(f) = \inf\{J(x), x = (f, x_2, \dots, x_m), x_i(t) \leqslant f(t) \text{ for } i = 2, \dots, m\},$$

where J is given by (5.1). *For f belonging to a class of functions $\mathcal{F}$ to be defined in the proof,*

$$I_{\max}(f) = \frac{1}{8}\left[\int_0^T \frac{(\dot{f}_t - \delta)^2}{f_t}dt + (m-1)\int_0^T \frac{(\dot{\underline{f}}_t - \delta)^2}{\underline{f}_t}\right], \tag{5.3}$$

where $\underline{f}(t) = \delta t + \inf_{s \leqslant t}\big(f(s) - \delta s\big)$.

P r o o f. According to the contraction principle, $I_{\max}$ is given by the minimum of the rate function

$$J(x) = \frac{1}{8} \sum_{i=1}^{m} \int_0^T \frac{\big(\dot{x}_i(t) - \delta\big)^2}{x_i(t)} dt$$

under the constraint $\{x_i(t) \leqslant f(t), i = 2, \dots, m\}$ with $x_1 = f$ fixed. Let us set

$$F(y) = \frac{1}{8} \int_0^T \frac{\big(\dot{y}(t) - \delta\big)^2}{y(t)} dt;$$

F is a convex function on $C_0\big([0,T); \mathbb{R}_+\big)$ and let us introduce the convex function $G_f(y) = y - f \in C\big([0,T); \mathbb{R}\big)$. The problem is to minimize $F(y)$ under the constraint $G_f(y) \leqslant 0$. We associate with f the measure μ_f from the Ricatti equation

$$2\mu_f = \dot{H} + H^2 \text{ on } (0,T), \qquad H(T) = -2\mu_f(T)$$

with $H_t = \big((\dot{f}(t) - \delta)\big)/2f(t)$. Then we define the measure

$$d\tilde{\mu}_f(t) = d\mu_f(t) 1_{(\underline{f}(t) = f(t))}.$$

Let $\mathcal{F} = \{f; d\tilde{\mu}_f$ is a positive measure on $[0,T]\}$. For $f \in \mathcal{F}$, let us show that the Lagrangian

$$L(y, \mu) = F(y) + \langle G_f(y), \mu \rangle$$

has a saddle point at $(\underline{f}, \tilde{\mu}_f)$, i.e.,

$$L(\underline{f}, \mu) \leqslant L(\underline{f}, \tilde{\mu}_f) \leqslant L(y, \tilde{\mu}_f) \tag{5.4}$$

for all $y \in C_0\big([0,T); \mathbb{R}_+\big)$ and all positive measures μ.

The first inequality follows from

$$\langle G_f(\underline{f}), \mu \rangle \leqslant 0 = \langle G_f(\underline{f}), \tilde{\mu}_f \rangle$$

since $\operatorname{supp}(\tilde{\mu}_f) \subset \{t, f(t) = \underline{f}(t)\}$.

For the second inequality, we must show that $\underline{f}$ minimize $F(y) + \langle G_f(y), \tilde{\mu}_f \rangle$. The optimal path of this problem of minimization solves the Euler–Lagrange equation (see [5])

$$\frac{d}{dt}\left(\frac{\partial g}{\partial b}(y, \dot{y})\right) = \frac{\partial g}{\partial a}(y, \dot{y}) + \tilde{\mu}_f \text{ on } (0,T), \left(\frac{\partial g}{\partial b}(y, \dot{y})\right)_{t=T} = -\tilde{\mu}_f(T) \tag{5.5}$$

with $g(a,b) = (b - \delta)^2/8a$. The auxiliary function $H_t = \big(\dot{y}(t) - \delta\big)/2y(t)$ associated with the optimal path y satisfies the Ricatti equation

$$2\tilde{\mu}_f = \dot{H} + H^2, \qquad H(T) = -2\tilde{\mu}_f(T).$$

By the choice of $\tilde{\mu}_f$, it is easy to see that $\underline{f}$ solves the Euler–Lagrange equation (5.5) (or the associated Ricatti equation). According to Luenberger [8] (Theorem 2, Section 8.4), the existence of this saddle point implies that $\underline{f}$ minimize $F(y)$ under the constraint $G_f(y) \leqslant 0$. ∎

For a fixed time, we have the following result:

PROPOSITION 5.4. *The random vector* $\lambda^{\epsilon}_{\max}(1)$ *satisfies an LDP in* $\mathbb{R}_+$ *with rate function given by*

$$K_{\max}(a) = \frac{a}{2} - \frac{\delta}{2}\ln(a) - \frac{\delta}{2} + \frac{\delta}{2}\ln(\delta) \quad \textit{if } a > \delta, \tag{5.6}$$

$$K_{\max}(a) = m\left(\frac{a}{2} - \frac{\delta}{2}\ln(a) - \frac{\delta}{2} + \frac{\delta}{2}\ln(\delta)\right) \quad \textit{if } a \leqslant \delta. \tag{5.7}$$

The proof is immediate from (5.2). We minimize $K(M)$ under the constraint $\|M\| = a$, where $\|\cdot\|$ denotes the operator norm.

6. APPENDIX

A.1. On the equation $AX + XA = B$. Let A and B be two symmetric matrices, where A is strictly positive. We are looking for a symmetric matrix X, a solution of the equation (see (1.4))

$$AX + XA = B. \tag{6.1}$$

Since A is symmetric, let P and D be orthogonal and positive diagonal matrices such that $A = P^{-1}DP$. Then, by (6.1), the symmetric matrix $\tilde{X} = PXP^{-1}$ satisfies

$$D\tilde{X} + \tilde{X}D = PBP^{-1} := \tilde{B},$$

that is,

$$d_i\tilde{X}_{ij} + \tilde{X}_{ij}d_j = \tilde{B}_{ij},$$

and consequently $\tilde{X}_{ij} = \tilde{B}_{ij}/(d_i + d_j)$. Thus, X is uniquely determined.

A.2. On the Riccati equation. We consider the Ricatti equation (see (4.4))

$$\dot{F} + F^2 = \mu, \quad F(T) = 0, \tag{6.2}$$

or

$$F(t) = C + \mu(]0,t]) - \int_0^t F^2(s)ds,$$

where C is chosen such that $F(T) = 0$. We diagonalize $F(t)$: $F_t = P_t^{-1} D_t P_t$ with D_t the matrix of eigenvalues of F_t and P_t orthogonal. Then the Ricatti equation can be written as

$$\dot{D}(t) + D^2(t) = P(t)\mu_t P^{-1}(t) + R_t,$$

where R is a matrix whose diagonal entries are zeroes. Set $\nu = P\mu P^{-1}$; then ν is a positive $\mathcal{S}_m^+$-valued measure and the eigenvalues of F satisfy the scalar Riccati equation:

$$\dot{d}_i(t) + d_i^2(t) = \nu_{ii}(t), \qquad d_i(T) = 0,$$

where ν_{ii} is a positive measure on $[0, T]$. We know (see [11], Chapter XI) that $d_i(t) \leqslant 0$ (d_i is related to the decreasing solution of the Sturm–Liouville equation $\phi_i'' = \phi_i \nu_{ii}$). It follows that the matrix $F(t)$ is symmetric negative.

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