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# ON THE CENTRAL LIMIT THEOREM IN BANACH SPACE $c_0$

BY

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*Abstract*. In the paper the central limit theorem and the rates of convergence in this theorem in Banach space $c_0$ are considered. Let $\xi_i = (\xi_i^{(1)}, \ldots, \xi_i^{(n)}, \ldots)$, $i = 1, 2, \ldots$, be i.i.d. $c_0$-valued random variables with $E\xi_1 = 0$ and covariance matrix $T$. Let $\mu$ be a zero-mean Gaussian measure on $c_0$ with covariance matrix $T$,

$$F_n(A) = P\{n^{-1/2} \sum_{i=1}^{n} \xi_i \in A\}.$$

The main result of the paper can be formulated as follows: if $|\xi_1^{(j)}| < M_j = (\ln j)^{-1/2} a_j$, $j > j_0$, where $\{a_j\}$ is an arbitrary sequence of positive numbers tending to zero, then $F_n$ converges weakly to $\mu$. Moreover, if instead of $a_j$ we take a slowly increasing sequence $(\ln_k j)^{1/2+\varepsilon}$, where $\ln_k x = \ln \ln_{k-1} x$ and $k \geqslant 2$ is an arbitrary integer, then it is possible to construct $\xi_i$, $i \geqslant 1$, failing the central limit theorem.

If $|\xi_1^{(j)}| < M\sigma_j$, $\sigma_j^2 = E(\xi_1^{(j)})^2 = (\ln j)^{-(1+\delta)}$, $j \geqslant 2$, $\delta > 0$, and $T$ satisfies one additional condition, then we get the estimate

$$\sup_{r \geqslant 0} |F_n(\|x\| < r) - \mu(\|x\| < r)| = O(n^{-1/2+\varepsilon}), \quad \varepsilon > 0.$$

**1. Introduction.** In the paper we consider the central limit theorem (CLT) and the rate of convergence in this theorem in separable Banach space

$$c_0 = \{x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots), \ \lim_{n\to\infty} x^{(n)} = 0, \ \|x\| = \sup_n |x^{(n)}| < \infty\}.$$

Let $\xi$ be a random variable with values in a separable Banach space $B$ (*B-valued r.v.*), with distribution $F$, $E\xi = 0$, and covariance operator $T$. Let $\xi_i$, $i \geqslant 1$,, be i.i.d. $B$-valued r.v.'s with distribution $F$,

$$F_n(A) = P\{n^{-1/2} \sum_{i=1}^{n} \xi_i \in A\}.$$

We say that $\xi$ satisfies *CLT* (shortly, $\xi \in$ CLT) if there exists a Gaussian $B$-valued r.v. $\eta$ with distribution $\mu$, mean zero, and covariance operator $T$ such that $F_n \Rightarrow \mu$ ($\Rightarrow$ denotes weak convergence of probability measures). By $\mathscr{R}(B)$ we denote the class of Gaussian covariances in $B$, i.e., $T \in \mathscr{R}(B)$ if there exists a Gaussian $B$-valued r.v. $\eta$ with the covariance operator $T$.

Our choice of particular Banach space $c_0$ is motivated by the fact that this space plays a rather important role in theory of probability on Banach spaces. There are many statements (see, e.g., [13], [15], [27]) which are proved for Banach spaces not containing subspaces isomorphic to $c_0$ and, as a rule, these statements are false in $c_0$. A similar situation is for CLT in Banach spaces. At present it is well known ([9], [5], [24]) that in Banach spaces of type 2 (and only in such spaces) the condition

$$\mathrm{E}\|\xi\|^2 < \infty \tag{1.1}$$

implies $\xi \in$ CLT; in Banach spaces of cotype 2 (and only in such spaces) the condition

$$T \in \mathscr{R}(B) \tag{1.2}$$

implies $\xi \in$ CLT. (For notions of spaces of some type and cotype we refer the reader to [5] and [9].) On the other hand, in [4] it is proved that in spaces containing $l_\infty^n$ uniformly ($C(0, 1)$ and $c_0$ are examples of such spaces; $C(S)$ denotes the space of continuous functions on metric compact $S$ with sup norm) there exists a symmetric, bounded random variable satisfying (1.2) but not satisfying CLT. The first example of such a kind was constructed in 1969 in [6], where CLT in $C(S)$ was considered. Later there appeared more papers concerning CLT in $C(S)$, but as far as we know none of them was dealt with CLT in $c_0$.

In Section 2 we examine $\mathscr{R}(c_0)$ and CLT in $c_0$. Theorem 2.3 strengthens one of the results from [26] about $\mathscr{R}(c_0)$. Theorem 2.5 states that if coordinates of a $c_0$-valued r.v. $\xi = (\xi^{(1)}, \ldots, \xi^{(n)}, \ldots)$ satisfy the condition

$$|\xi^{(n)}| < M_n, \quad n \geqslant n_0, \tag{1.3}$$

with $M_n = (\ln n)^{-1/2} a_n$, where $\{a_n\}$ is an arbitrary sequence of positive numbers, tending to zero, then $\xi \in$ CLT. Moreover, using the idea of example in [4], we show that it is possible to construct a symmetric $c_0$-valued r.v., satisfying (1.2) with $B = c_0$ and (1.3) with

$$M_n = (\ln n)^{-1/2} (\ln_k n)^{1/2+\varepsilon}$$

but not satisfying CLT. Here $k \geqslant 2$ is an arbitrary integer, $\varepsilon > 0$, and $\ln_k x = \ln \ln_{k-1} x$. Thus, only the case $M_n = (\ln n)^{-1/2}$ remains open.

In Section 3, estimates of the quantity

$$\Delta_n = \sup_{r \geqslant 0} \left| P\{\|(1/\sqrt{n}) \sum_{i=1}^{n} \xi_i\| < r\} - P\{\|\eta\| < r\} \right|$$

are given. Now, estimates of this quantity can be obtained in Banach spaces with sufficiently smooth norm ([19], [20], [1]) and in spaces $C(S)$ ([8], [21]). In the last case the finite-dimensional approximation is used, and for the space $c_0$ we apply also this approximation. (See [23] and [25] where the finite-dimensional approximation was used in spaces $l_p$, $2 \leqslant p < \infty$, and $l_2$.)

It is worth while to mention papers [28] and [29] where the rates of convergence in limit theorems are given in any Banach space and are expressed by means of so-called ideal metrics. In these papers the estimate of the quantity

$$\sup_{f \in \mathscr{D}} \left|\int_B f(x)\big(F_n(dx) - \mu(dx)\big)\right|$$

is given, where $\mathscr{D}$ is some class of smooth functions on $B$, e.g., some times differentiable in the sense of Fréchet functions on $B$. But in contrast with finite-dimensional spaces, where the class of differentiable functions is very large, in infinite-dimensional spaces the situation is worse, and it is known [2] that in some Banach spaces (e.g., in $C(0, 1)$ and $l_1$) there exists no non-trivial differentiable function with bounded support. Moreover, the behaviour of differentiable functions in such spaces is rather complicated (see [14]), thus in this case quantities of type $\int_B f(x)\big(F_n(dx) - \mu(dx)\big)$ are not very useful.

Now we introduce some more notation. If $x \in c_0$, then

$$\|x\|_m = \sup_{i > m} |x^{(i)}|.$$

We put

$$\hat{\beta}_3 = \sup_i \big(\mathrm{E}\,|\xi^{(i)}|^3 (\mathrm{E}\,|\xi^{(i)}|^2)^{-3/2}\big), \qquad \bar{\beta}_3 = \sup_i \mathrm{E}\,|\xi^{(i)}|^3.$$

The letter $C$ stands for an absolute constant, and $C(\cdot)$ denotes a constant depending on parameters in the parentheses, not the same in different places. If we want the constant to be distinguished, we shall supply it with an index.

**2. CLT and Gaussian measures in $c_0$.** For simplicity we write $\mathscr{R} = \mathscr{R}(c_0)$. The covariance operator of a $c_0$-valued r.v. can be regarded as an infinite matrix and we put $T = \{t_{ij}\}_{i,j=1}^{\infty}$. It will be convenient to set $\sigma_i^2 = t_{ii}$ and, without loss of generality, in the whole paper we assume that $\sigma_i^2 \geqslant \sigma_{i+1}^2$ for all $i$.

Let $\mathscr{R}_1$ stand for the class of covariance matrices $T$ such that for every $\varepsilon > 0$

$$\sum_{i=1}^{\infty} \sigma_i \exp\{-\varepsilon \sigma_i^{-2}\} < \infty. \tag{2.1}$$

Then the result of Vakhania [26] can be formulated in the following way:

THEOREM 2.1 ([26]). $\mathscr{R}_1 \subset \mathscr{R}$ *and if* $t_{ij} = 0$, $i \neq j$, *then* $T \in \mathscr{R} \Leftrightarrow T \in \mathscr{R}_1$.

Thus diagonal Gaussian covariances (in this case, coordinates of a Gaussian $c_0$-valued r.v. $\eta$ are independent) are completely determined by (2.1). The inverse case, where $\eta = (a^{(1)}\zeta, \ldots, a^{(n)}\zeta, \ldots) = a\zeta$ with $a \in c_0$ and $\zeta$ a standard normal $R_1$-valued r.v., shows that $\sigma_n^2$ can tend to zero arbitrarily slowly if there is a strong dependence between coordinates of $\eta$. In [3] the following result is given:

THEOREM 2.2. ([3]). *If* $\sigma_i^2 \downarrow 0$ ($\downarrow$ *denotes monotonic convergence*),

$$r_i = \max_{j \leqslant i} |t_{ii} - t_{ij}| \downarrow 0,$$

*and for every* $\varepsilon > 0$

$$\sum_{i=1}^{\infty} \exp\{-\varepsilon r_i^{-1}\} < \infty, \tag{2.2}$$

*then* $T \in \mathscr{R}$.

It is easy to see that in the case of the diagonal matrix $T$ condition (2.2) coincides with (2.1) but in the case $\eta = a\zeta$, $a \in c_0$, $\zeta$ being a standard normal, (2.1) as a sufficient condition is stronger.

We show that condition (2.1) is necessary and sufficient in all cases where dependence between coordinates of $\eta$ is weak. By means of the same method as in [26] we prove the following result:

THEOREM 2.3 *Let* $t_{ij} = 0$ *if* $|i-j| > m$, *where* $m$ *is any finite number. Then* $T \in \mathscr{R} \Leftrightarrow T \in \mathscr{R}_1$.

Proof. Let $\eta = (\eta^{(1)}, \ldots, \eta^{(n)}, \ldots)$ be a Gaussian random vector with covariance matrix $T$ and let $\mu$ be its distribution (considered as a distribution in $R^\infty$). It is known [26] that $\mu(c_0) = 1$ is equivalent to the condition

$$\lim_{n \to \infty} \mu\Big(\bigcup_{k \geqslant n} \{x\colon |x^{(k)}| > \varepsilon\}\Big) = 0 \quad \text{for every } \varepsilon > 0. \tag{2.3}$$

Since $\mathscr{R}_1 \subset \mathscr{R}$, we need only to prove the necessity of condition (2.1). Let us put

$$A_k = A_k(\varepsilon) = \{\sup_{(2k-2)m < j \leqslant (2k-1)m} |\eta^{(j)}| > \varepsilon\}, \qquad \tilde{A}_n = \bigcup_{k \geqslant n} A_k,$$

$$B_k = B_k(\varepsilon) = \{\sup_{(2k-1)m < j \leqslant 2km} |\eta^{(j)}| > \varepsilon\}, \qquad \tilde{B}_n = \bigcup_{k \geqslant n} B_k.$$

From (2.3) it follows that

$$\lim_{n \to \infty} P\Big\{\bigcup_{k \geqslant n} (A_k \cup B_k)\Big\} = \lim_{n \to \infty} P\{\tilde{A}_n \cup \tilde{B}_n\} = 0$$

and

$$\lim_{n\to\infty} P(\tilde{A}_n) = \lim_{n\to\infty} P(\tilde{B}_n) = 0.$$

Since the events $A_k$, $k \geqslant 1$, are independent, using the same argument as in the proof of Theorem 2.1 (cf. [26]) we infer that the condition $\lim P(\tilde{A}_n) = 0$ implies

$$\sum_k P(A_k) < \infty. \tag{2.4}$$

Analogously, we get

$$\sum_k P(B_k) < \infty. \tag{2.5}$$

Now for any $j$ $\big((2k-2)m < j \leqslant (2k-1)m\big)$ we have

$$\begin{aligned} P(A_k) &\geqslant P\{|\eta^{(j)}| > \varepsilon\} \\ &\geqslant \sqrt{\frac{2}{\pi}}\frac{1}{1+\varepsilon\sigma_j^{-1}}\exp\left\{-\frac{\varepsilon^2}{2\sigma_j^2}\right\} \geqslant \frac{1}{\sqrt{2\pi}}\min\left(1, \frac{\sigma_j}{\varepsilon}\right)\exp\left\{-\frac{\varepsilon^2}{2\sigma_j^2}\right\}. \end{aligned} \tag{2.6}$$

In the same way we obtain

$$\begin{aligned} P(B_k) & \\ &\geqslant (2\pi)^{-1/2}\min(1, \sigma_j\varepsilon^{-1})\exp\left\{-\frac{\varepsilon^2}{2\sigma_j^2}\right\}, \quad (2k-1)m < j \leqslant 2km. \end{aligned} \tag{2.7}$$

Now it is sufficient to note that if $a_n \to 0$, $a_n > 0$, then the series

$$\sum_n \exp\left\{-\frac{\varepsilon}{a_n^2}\right\} \quad \text{and} \quad \sum_n a_n \exp\left\{-\frac{\varepsilon}{a_n^2}\right\}$$

converge or diverge simultaneously, since

$$a_n \exp\left\{-\frac{\varepsilon}{a_n^2}\right\} = \exp\left\{-\frac{\varepsilon}{a_n^2}\left(1 - \frac{a_n^2}{\varepsilon}\ln a_n\right)\right\}$$

and $a_n^2 \ln a_n \to 0$. From (2.4)-(2.7) we get (2.1). Thus the theorem is proved.

Now we turn to CLT in $c_0$. We want to find the conditions on a $c_0$-valued r.v. $\xi$ that imply $F_n \Rightarrow \mu$. The main tool in proving the result of this section and the results on the rates of convergence is the inequality of large deviations for sums of idependent bounded real r.v.'s. The idea of this inequality goes back to Khintchine [12] who proved the exponential inequality for the sum of i.i.d. Rademacher r.v.'s. Later in [7] such an inequality was proved for symmetric bounded r.v.'s. In [22] these results were generalized to the multidimensional case and, moreover, the symmetry assumption was removed. We formulate this result:

THEOREM 2.4 ([22]). *Let* $X_i$, $i \geqslant 1$, *be i.i.d.* $R_1$*-valued r.v.'s,* $\mathrm{E}X_1 = 0$, *and* $|X_1| < M$. *Let* $Z_n = \sum_{i=1}^{n} X_i$ *and assume that the following condition is satisfied for* $p > 0$:

(2.8) $$P\{Z_n < 0\} \leqslant 1-p, \quad P\{Z_n > 0\} \leqslant 1-p.$$

*Then for all* $t > 0$

(2.9) $$P\{|Z_n| > t\sqrt{n}\} \leqslant \sqrt{2e^3}\, p^{-1} \exp\{-t^2(16M^2)^{-1}\}.$$

The main result of this section is the following

THEOREM 2.5. *Let* $\xi$ *be a* $c_0$*-valued r.v. with* $\mathrm{E}\xi = 0$, *covariance operator* $T$, *and* $|\xi^{(j)}| < M_j$ *for* $j \geqslant m_0$. *If for any* $\varepsilon > 0$

(2.10) $$\sum_{j \geqslant m_0} \exp\{-\varepsilon M_j^{-2}\} < \infty,$$

*then* $\xi \in CLT$.

Proof. We must show that $F_n \Rightarrow \mu$, where $F_n$ is the distribution of $n^{-1/2} \sum_{i=1}^{n} \xi_i$, $\xi_i$ being independent copies of $\xi$. Obviously, $T \in \mathscr{R}_1 \subset \mathscr{R}$ because $\sigma_j^2 \leqslant M_j^2$ if $|\xi^{(j)}| < M_j$. Since the finite-dimensional distributions of $F_n$ converge weakly to the corresponding finite-dimensional distributions of $\mu$, we need only to show the tightness of the family $\{F_n\}$. Using the form of compact sets in $c_0$ (cf., e.g., [16] or [18]) we see that it is sufficient to show that for any $\varepsilon > 0$ and $\delta > 0$ there exist $n_0 = n_0(\varepsilon, \delta)$ and $k_0 = k_0(\varepsilon, \delta)$ such that for all $n > n_0$

(2.11) $$P\{\|S_n\|_{k_0} > \delta\} < \varepsilon.$$

We first assume that a $c_0$-valued r.v. $\xi$ is symmetric. Since

$$P\{\|S_n\|_k > \delta\} \leqslant \sum_{j>k} P\{|S_n^{(j)}| > \delta\},$$

for $S_n^{(j)}$ we apply Theorem 2.4 with $p = 1/2$ and we get the estimate

(2.12) $$P\{|S_n^{(j)}| > \delta\} \leqslant C \exp\{-\delta^2(16M_j^2)^{-1}\}.$$

From (2.12) and (2.10) we deduce (2.11).

It remains to remove the assumption of symmetry. For this purpose we formulate the following result:

LEMMA 2.1. *Let* $Z_n$, $n \geqslant 1$, *be a sequence of* $c_0$*-valued r.v.'s satisfying the* *following condition: for every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists* $k_0 = k_0(\varepsilon, \delta)$ *such that for all* $n \geqslant 1$ *and* $j > k_0$

$$P\{|Z_n^{(j)}| > \delta\} < \varepsilon. \tag{2.13}$$

*Let* $G_n$ *and* $\tilde{G}_n$ *denote the distributions of* $Z_n$ *and* $-Z_n$, *respectively. Then the tightness of the sequence* $\{G_n * \tilde{G}_n\}$ *implies the tightness of* $\{G_n\}$.

The proof of Lemma 2.1 goes along the lines of the proof of Lemma 2 from [11] and is omitted.

Now, if we put $\tilde{\xi}_i = \xi - \xi_i'$, where $\xi_i'$ is an independent copy of $\xi_i$, then from the first part of the proof we infer that the sequence of measures induced by

$$\tilde{S}_n = n^{-1/2} \sum_{i=1}^{n} \tilde{\xi}_i$$

is tight (recall that $|\tilde{\xi}_i^{(j)}| < 2M_j$). Further, by Chebyschev's inequality, for any $\varepsilon > 0$ and $\delta > 0$ we have

$$P\{|S_n^{(j)}| > \delta\} \leqslant \delta^{-2} \mathrm{E}|S_n^{(j)}|^2 = \delta^{-1} \mathrm{E}(\xi_1^{(j)})^2 \leqslant \delta^{-2} M_j^2 \leqslant \varepsilon$$

for all $j > k_0 = \{\inf m\colon\ M_n^2 < \varepsilon\delta^2 \text{ for all } n > m\}$. Therefore, (2.13) is satisfied and from Lemma 2.1 we get (2.11) in a general case. Thus the theorem is proved.

Remark. The result of Theorem 2.5 can be obtained by using operators of type 2. (see a joint paper with A. Račkauskas and V. Sakalauskas *).

Now we construct a $c_0$-valued r.v. mentioned in the Introduction.

PROPOSITION 2.1. *There exists a symmetric* $c_0$*-valued r.v.* $\xi$ *having independent coordinates, covariance operator* $T \in \mathscr{R}_1$, *satisfying* (1.3) *with*

$$M_n = (\ln n)^{-1/2} (\ln_k n)^{1/2+\varepsilon}, \tag{2.14}$$

*where* $k \geqslant 2$ *is an arbitrary integer and* $\varepsilon > 0$, *but not satisfying CLT.*

Proof. We follow the construction from [4], but our evaluation is more precise. Let $\xi = (\xi^{(1)}, \ldots, \xi^{(n)}, \ldots)$ with independent coordinates $\xi^{(n)}$ and let

$$P\{\xi^{(n)} = a_n\} = P\{\xi^{(n)} = -a_n\} = p_n, \qquad P\{\xi^{(n)} = 0\} = 1 - 2p_n,$$

where $a_n = (\ln n)^{-1/2} (\ln_{k+1} n)^{1/2+\varepsilon}$, $p_n = (\ln_k n)^{-1}$, $k \geqslant 1$ is some fixed number, $\varepsilon > 0$, and $n > n_0(k)$ in order that all expressions be correctly defined. For the first coordinates $\xi^{(n)}$, $n \leqslant n_0(k)$, we can take $p_n$ and $a_n$ arbitrary. For example, we can take $a_n = 1$ and $p_n = 1/2$, and this does not affect our evaluations, since we deal with limit behaviour of sums

$$S_n^{(j)} = n^{-1/2} \sum_{i=1}^{n} \xi_i^{(j)}$$

* This paper appeared in Litovsk. Matem. Sb., 23, 1 (1983), p. 163-174 (in Russian).

for large values of $j$ and $n$, where $\xi_i$ are independent copies of $\xi$. It is easy to see that

$$\sigma_j^2 = a_j^2 p_j = (\ln j \ln_k j)^{-1} (\ln_{k+1} j)^{1+2\varepsilon},$$

which implies $T \in \mathscr{R}_1$. Thus it remains to show that $\xi$ does not satisfy CLT and for this reason we show that the sequence $S_n$ is not bounded in probability. Consider the events

$$A_{nj} = \bigcap_{i=1}^{n} \{\xi_i^{(j)} = a_j\} \quad \text{and} \quad A_n = \bigcup_{j \leqslant N_n} A_{nj}$$

and choose $N_n = \exp\{Cn \ln_k n\}$, $C > 1$. We have

$$P(A_{nj}) = (\ln_k j)^{-n}, \quad P(A_n) = 1 - P\big(\bigcap_{j \leqslant N_n} A_{nj}^c\big) = 1 - \prod_{j \leqslant N_n} \big(1 - P(A_{nj})\big).$$

Now we show that

$$P(A_n) \to 1 \quad \text{as } n \to \infty. \tag{2.15}$$

For this purpose we use the estimate

(2.16)
$$\prod_{j \leqslant N_n} \big(1 - P(A_{nj})\big) \leqslant \big(1 - P(A_{nN_n})\big)^{N_n} = \big(1 - (\ln_{k-1}(Cn \ln_k n))^{-n}\big)^{\exp\{Cn \ln_k n\}}$$

Since it is easy to verify that for every $C > 1$

$$(\ln_{k-1}(Cn \ln_k n))^{-n} \exp\{Cn \ln_k n\} \to \infty \quad \text{as } n \to \infty, \tag{2.17}$$

we derive (2.15) from (2.16) and (2.17).

Now let $\omega \in A_n$. Then

$$\sup_{j \leqslant N_n} \Big| n^{-1/2} \sum_{i=1}^{n} \xi_i^{(j)}(\omega) \Big| \geqslant n^{1/2} a_{N_n}.$$

But we have

$$b_n = n^{1/2} a_{N_n} = \frac{(\ln_k (Cn \ln_k n))^{1/2+\varepsilon}}{C(\ln_k n)^{1/2}} \to \infty \quad \text{as } n \to \infty$$

for every $\varepsilon > 0$. Therefore, we infer that for sufficiently large $n$ we have $\|S_n\| > b_n$ with probability close to 1 and $\xi$ does not satisfy CLT.

Remark. It is easy to verify that in this way we can prove the proposition with $M_n = (\ln n)^{-1/2} \big(d(n)\big)^{1/2+\varepsilon}$, where $d(n)$ is a monotonically increasing sequence, $\lim_{n \to \infty} d(n) = \infty$, and $d(n)$ satisfies the following condition: there exist $C > 1$ and $b(n) \to \infty$ such that for all $n > n_0$

$$Cd(n) - \ln\big(d(\exp\{Cnd(n)\})\big) > n^{-1} b(n).$$

For simplicity, in Proposition 2.1 we have taken $d(n) = \ln_k n$.

**3. The rates of convergence in CLT in $c_0$.** To formulate the results we need some more notation. Let $\xi$ be a $c_0$-valued r.v. with distribution $F$, $E\xi = 0$, and covariance matrix $T$. Let $\xi_i$, $i \geqslant 1$, be i.i.d. $c_0$-valued r.v.'s with the same distribution $F$. By $T_{m,\sigma}$ we denote a covariance matrix of the random vector $(\xi^{(1)}\sigma_1^{-1}, \ldots, \xi^{(m)}\sigma_m^{-1})$ and let $\lambda_k^{(m)}$, $k = 1, 2, \ldots, m$, stand for the eigenvalues of $T_{m,\sigma}$. Moreover, without loss of generality we assume that $\lambda_k^{(m)} \geqslant \lambda_{k+1}^{(m)}$ for all $k$ and $m$. In this section we assume that

(3.1) $$\sigma_i^2 = (\ln i)^{-(1+\delta)}, \quad \delta > 0,\ i \geqslant 2,\ \sigma_1^2 = 1.$$

Of course, we can change the quantities $\sigma_i^2$, $i \leqslant i_0$, and since we deal with the estimates of the form $\Delta_n = O(f(n))$, such a change affects only the constants in the last relation.

We say that $\xi$ satisfies *condition* $(B, d_m)$, where $d_m \downarrow 0$, if

(3.2) $$\sup_n P\{\|S_n\|_m > d_m\} < \tfrac{1}{4},$$

and $\xi$ satisfies *condition* $A(k_0, n_0, p)$, where $p > 0$, if for all $j > k_0$ and $n > n_0$

$$P\{S_n^{(j)} < 0\} \leqslant 1-p, \quad P\{S_n^{(j)} > 0\} \leqslant 1-p.$$

The main result of this section can be formulated as follows:

THEOREM 3.1. *Assume that for all $j \geqslant 1$ and $m \geqslant 1$*

(3.3) $$|\xi^{(j)}| < M\sigma_j,$$

(3.4) $$\lambda_m^{(m)} > (\ln m)^{-(\delta_1-\delta)}, \quad \delta_1 > \delta.$$

*Then for sufficiently large $n$*

(3.5) $$\Delta_n = O(n^{-1/2+\nu}),$$

*where $\nu > 0$ and the constant in* (3.5) *tends to infinity as $\nu \to 0$.*

Remark. If (3.3) is replaced by the condition

(3.6) $$|\xi^{(j)}| < M(\ln j)^{-(1+\delta-\delta_2)/2}, \quad \delta_2 < \delta,$$

then estimation (3.5) remains valid, but the constant changes and tends to infinity if $\delta_2 \to \delta$.

THEOREM 3.2. *If $\xi$ satisfies condition $(B, d_m)$ with $d_m = (\ln m)^{-\varkappa}$, $0 < \varkappa < \delta/2$, $\bar{\beta}_3$ or $\hat{\beta}_3$ is finite, and*

(3.7) $$\lambda_m^{(m)} > m^{-1/6+\gamma},$$

*then*

(3.8) $$\Delta_n = O((\ln n)^{-2\varkappa+\gamma_1}),$$

*where $\gamma_1 > 0$ is arbitrarily small, but the constant in* (3.8) *tends to infinity if $\gamma_1 \to 0$.*

Before proceeding to the proofs of the theorems we formulate some lemmas which contain the main steps of the proofs. We begin with the basic inequality in the estimation of $\Delta_n$.

LEMMA 3.1. *For any $m \geqslant 1$ and $\varepsilon > 0$ we have*

$$(3.9)\qquad \Delta_n \leqslant C(\Delta_{n,m} + P\{\|S_n\|_m > \varepsilon\} + P\{\|\eta\|_m > \varepsilon\} + P\{\|\eta\| < \varepsilon\}),$$

*where*

$$\Delta_{n,m} = \sup_x \Delta_{n,m}(x), \qquad \Delta_{n,m}(x) = \left|P\{\max_{j\leqslant m} |S_n^{(j)}| < x\} - P\{\max_{j\leqslant m} |\eta^{(j)}| < x\}\right|.$$

Proof. We have

$$(3.10)\qquad \begin{aligned} \Delta_n &\leqslant \max\{\sup_{0\leqslant x\leqslant \varepsilon} \Delta_n(x), \sup_{x>\varepsilon} \Delta_n(x)\} \\ &\leqslant \max\{P\{\|S_n\| < \varepsilon\} + P\{\|\eta\| < \varepsilon\}, \sup_{x>\varepsilon} \Delta_n(x)\} \\ &\leqslant \max\{2P\{\|\eta\| < \varepsilon\} + \Delta_n(\varepsilon), \sup_{x>\varepsilon} \Delta_n(x)\} \\ &\leqslant \sup_{x\geqslant\varepsilon} \Delta_n(x) + 2P\{\|\eta\| < \varepsilon\}, \end{aligned}$$

$$(3.11)\qquad \begin{aligned} \Delta_n(x) &= |P\{\|S_n\| < x\} - P\{\|\eta\| < x\}| \\ &\leqslant |P\{\|S_n\| < x\} - P\{\max_{j\leqslant m} |S_n^{(j)}| < x\}| + \\ &\quad + |P\{\max_{j\leqslant m} |S_n^{(j)}| < x\} - P\{\max_{j\leqslant m} |\eta^{(j)}| < x\}| + \\ &\quad + |P\{\|\eta\| < x\} - P\{\max_{j\leqslant m} |\eta^{(j)}| < x\}|. \end{aligned}$$

It is easy to see that

$$(3.12)\qquad P\{\max_{j\leqslant m} |S_n^{(j)}| < x\} - P\{\|S_n\| < x\} \leqslant P\{\|S_n\|_m > x\},$$

$$(3.13)\qquad P\{\max_{j\leqslant m} |\eta^{(j)}| < x\} - P\{\|\eta\| < x\} \leqslant P\{\|\eta\|_m > x\}.$$

Inequalities (3.10)-(3.13) yield (3.9), so the lemma is proved.

For the estimation of $\Delta_{n,m}$ we shall use the following result from [17]:

LEMMA 3.2. ([17]). *Let $X_i$, $i \geqslant 1$, be i.i.d. $R_k$-valued r.v.'s with $\mathrm{E}X_1 = 0$, unit covariance matrix, and*

$$\beta_3 = \mathrm{E}\Big(\sum_{i=1}^{k} (X_1^{(i)})^2\Big)^{3/2} < \infty.$$

*Then*

$$\sup_{A\in\mathfrak{M}} \Big|P\{n^{-1/2} \sum_{i=1}^{n} X_i \in A\} - \varphi(A)\Big| \leqslant Ck\beta_3 n^{-1/2},$$

*where $\mathfrak{M}$ is the class of all convex Borel sets in $R_k$ and $\varphi$ a standard $k$-dimensional normal distribution.*

By means of this lemma and the estimate

$$E\Big(\sum_{i=1}^{k}(X_1^{(i)})^2\Big)^{3/2} \leqslant k^{3/2}\max_{i\leqslant k} E|X_1^{(i)}|^3,$$

after simple considerations we get the following result:

LEMMA 3.3. *Let $\lambda_m^{(m)} > 0$ for all $m \geqslant 1$. Then*

$$\Delta_{n,m} \leqslant Cm^{5/2}(\lambda_m^{(m)})^{-3/2}\,\tilde{\beta}_3 n^{-1/2}, \tag{3.14}$$

$$\Delta_{n,m} \leqslant Cm^{5/2}(\sigma_m^2\lambda_m^{(m)})^{-3/2}\,\bar{\beta}_3 n^{-1/2}. \tag{3.15}$$

LEMMA 3.4. *Let $\eta$ be a Gaussian $c_0$-valued r.v. satisfying* (3.1). *If*

$$\varepsilon^2(\ln m)^{\delta} > 2, \tag{3.16}$$

*then*

$$P\{\|\eta\|_m > \varepsilon\} \leqslant C\left[\left(1+\varepsilon(\ln m)^{(1+\delta)/2}\right)\left(\frac{\varepsilon^2}{2}(\ln m)^{\delta}-1\right)m^{\varepsilon^2(\ln m)^{\delta}/2-1}\right]^{-1}. \tag{3.17}$$

Proof. The proof consists of the following two elementary inequalities:

$$P\{\|\eta\|_m > \varepsilon\} \leqslant \sum_{j\geqslant m} P\{|\eta^{(j)}| > \varepsilon\},$$

$$\sqrt{\frac{2}{\pi}}\int_t^{\infty}\exp\left\{-\frac{u^2}{2}\right\}du \leqslant \frac{4}{3}\sqrt{\frac{2}{\pi}}\frac{1}{1+t}\exp\left\{-\frac{t^2}{2}\right\} \leqslant \frac{4}{3}\sqrt{\frac{2}{\pi}}\exp\left\{-\frac{t^2}{2}\right\}.$$

LEMMA 3.5. *Let the eigenvalues $\lambda_m^{(m)}$ satisfy condition* (3.4). *Then for any $\varkappa > 0$ and $\varepsilon\in(0, \varepsilon_0)$, where*

$$\varepsilon_0 = \varepsilon_0(\varkappa, \delta_1) = \max\left\{\varepsilon\colon \ln\frac{1}{\varepsilon} \leqslant \frac{1}{\varkappa}\left(\sqrt{\frac{\pi}{2}}\frac{1}{\varepsilon}\right)^{1/(1+\delta_1)}\right\},$$

*we have*

$$P\{\|\eta\| < \varepsilon\} \leqslant C\exp\left\{-\frac{1}{2}\varepsilon^{-\varkappa}\ln\left(\sqrt{\frac{\pi}{2}}\frac{1}{\varepsilon}\right)\right\}. \tag{3.18}$$

*In the case of the diagonal matrix $T$ (the case $\delta_1 = \delta$) instead of* (3.18) *we can use the following inequality valid for all $\varepsilon$ $(0 < \varepsilon < 1)$:*

$$P\{\|\eta\| < \varepsilon\} \leqslant \exp\left\{-\frac{2}{\varepsilon}\sqrt{\frac{2}{\pi e}}\exp\{\varepsilon^{-2/(1+\delta)}\}\right\}. \tag{3.19}$$

Proof. Let us start with the case of the diagonal matrix $T$. In [10] the following estimate is given:

$$P\{\|\eta\| < \varepsilon\} \leqslant \exp\left\{-\frac{4}{3}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{\sigma_n}{\sigma_n+\varepsilon}\exp\left\{-\frac{\varepsilon^2}{2\sigma_n^2}\right\}\right\}.$$

Substituting to this estimate the values $\sigma_n$ from (3.1) we get the series

$$S(\varepsilon) = \frac{4}{3}\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\varepsilon}\exp\left\{-\frac{\varepsilon^2}{2}\right\}+ \right.$$

$$\left. + \sum_{n=2}^{\infty} \left(1+\varepsilon(\ln n)^{(1+\delta)/2}\right)^{-1} \exp\left\{-\frac{1}{2}\varepsilon^2 (\ln n)^{1+\delta}\right\}\right].$$

Let $n_0 = \exp\{\varepsilon^{-2/(1+\delta)}\}$. Then for $n \leqslant n_0$ we have

$$\exp\left\{-\frac{1}{2}\varepsilon^2(\ln n)^{1+\delta}\right\} \geqslant e^{-1/2} \quad \text{and} \quad 1+\varepsilon(\ln n)^{(1+\delta)/2} < 2.$$

Hence we obtain

$$S(\varepsilon) \geqslant \frac{2}{3}\sqrt{\frac{2}{\pi e}}\, n_0 = \frac{2}{3}\sqrt{\frac{2}{\pi e}} \exp\{\varepsilon^{-2/(1+\delta)}\},$$

which implies (3.19).

In the general case we use the estimate

$$P\{\|\eta\| < \varepsilon\} \leqslant P\{\sup_{j \leqslant k} |\eta^{(j)}| < \varepsilon\} \leqslant (2\varepsilon)^k \left[(2\pi)^{k/2} (\det T_{k,\sigma})^{1/2} \prod_{j=1}^{k} \sigma_j\right]^{-1}$$

$$= \exp\left\{-k \ln\left(\frac{1}{\varepsilon}\sqrt{\frac{\pi}{2}}\right) - \sum_{j=1}^{k} \ln\left((\lambda_j^{(k)})^{1/2}\sigma_j\right)\right\}$$

$$\leqslant C \exp\left\{-k \ln\left(\frac{1}{\varepsilon}\sqrt{\frac{\pi}{2}}\right) + \frac{1+\delta_1}{2}\sum_{j=3}^{k} \ln\ln j\right\}.$$

Now we choose $k = [\varepsilon^{-\varkappa}]+1$ and using a rather rough estimate

$$\sum_{j=3}^{k} \ln\ln j \leqslant k \ln\ln k$$

after simple calculations for all $\varepsilon$ $(0 < \varepsilon < \varepsilon_0)$ we derive (3.18). Thus the lemma is proved.

The estimates of the term $P\{\|S_n\|_m > \varepsilon\}$ are given in the following two lemmas:

LEMMA 3.6. *If condition* (3.3) *is satisfied and*

$$\frac{\varepsilon^2}{16M^2}(\ln m)^\delta > 1, \tag{3.20}$$

*then for* $n > n_0$

$$P\{\|S_n\|_m > \varepsilon\} \leqslant C\left(\frac{\varepsilon^2}{16M^2}(\ln m)^\delta - 1\right)^{-1} m^{-[\varepsilon^2(\ln m)^\delta/16M^2 - 1]}. \tag{3.21}$$

Proof. By (3.3) we can assume that $\xi$ satisfies condition $A(1, n_0, 1/4)$ with sufficiently large $n_0$. Then we can apply Theorem 2.4, and from (2.9) we get

$$P\{\|S_n\|_m > \varepsilon\} \leqslant \sum_{j>m} P\{|S_n^{(j)}| > \varepsilon\} \leqslant C \sum_{j=m}^{\infty} \exp\left\{-\frac{\varepsilon^2}{16M^2\sigma_j^2}\right\}.$$

Now, using (3.1) we obtain (3.21).

Remark. If instead of (3.3) we have (3.6), then in (3.20) and (3.21) we must replace $\delta$ by $\delta - \delta_2$.

LEMMA 3.7. *If $\xi$ satisfies condition $(B, d_m)$, then for any $p$ $(1 \leqslant p < 2)$ we have*

$$P\{\|S_n\|_m > \varepsilon\} \leqslant C(p)\varepsilon^{-p} d_m^p. \tag{3.22}$$

Proof. First, in the same way as in the proof of Proposition 2.1 from [24] we can show that for all $u > 2d_m$

$$P\{\|S_n\|_m > u\} \leqslant C d_m^2 u^{-2}. \tag{3.23}$$

Now, for $1 \leqslant p < 2$, from (3.23) we get

$$\mathrm{E}\|S_n\|_m^p = \int_0^\infty u^{p-1} P\{\|S_n\|_m > u\}\, du$$

$$= \left(\int_0^{2d_m} + \int_{2d_m}^\infty\right) u^{p-1} P\{\|S_n\|_m > u\}\, du \leqslant C(p) d_m^p,$$

which together with the inequality $P\{\|S_n\|_m > \varepsilon\} \leqslant \varepsilon^{-p} \mathrm{E}\|S_n\|_m^p$ implies (3.22).

We can now proceed to the proofs of the theorems.

Proof of Theorem 3.1. We put $m = [n^\alpha]$ in (3.9), where $\alpha$ is some parameter which will be chosen later. Then from (3.4) and (3.14) we get

$$\Delta_{n,m} = O\left(n^{-1/2+5\alpha/2} (\ln n)^{3(\delta_1 - \delta)}\right). \tag{3.24}$$

Without loss of generality we can assume that $M > 1$ in (3.3). Now set

$$\varepsilon = \frac{4M(\gamma+1)}{(\alpha \ln n)^{\delta/2}},$$

where $\gamma > 0$ will also be chosen later. It is easy to check that conditions (3.16) and (3.20) are satisfied. Therefore, by (3.17) and (3.21) we have

$$P\{\|\eta\|_m > \varepsilon\} \leqslant C(\gamma, M) n^{-\alpha\gamma}, \tag{3.25}$$

$$P\{\|S_n\|_m > \varepsilon\} \leqslant C\gamma^{-1} n^{-\alpha\gamma}. \tag{3.26}$$

Now we estimate the term $P\{\|\eta\| < \varepsilon\}$. Setting $\varkappa = 2/\delta$ in (3.18) for $n > n_0$, where $n_0$ depends on $M$, $\alpha$, $\gamma$,and $\delta$, we have

(3.27)

$$P\{\|\eta\| < \varepsilon\} \leqslant C \exp\left\{-\frac{1}{2}\frac{\alpha \ln n}{(4M(\gamma+1)^{1/2})^{2/\delta}} \ln\left(\sqrt{\frac{\pi}{2}}\frac{(\alpha \ln)^{\delta/2}}{4M(\gamma+1)^{1/2}}\right)\right\}$$

$$\leqslant \exp\left\{-\frac{1}{2} C_1(M, \gamma, \alpha)\ln\left(C_2(M, \alpha, \gamma, \delta)(\ln n)^{\delta/2}\right)\ln n\right\} \leqslant n^{-1/2}.$$

Thus all terms in (3.9) are estimated. Now, let us choose a small number $\nu > 0$. Then we find $\tilde{n}_0$ and $\alpha$ such that for $n > \tilde{n}_0$

$$n^{5\alpha/2}(\ln n)^{3(\delta_1-\delta)} < n^{\nu}.$$

Then we set $\gamma = 1/2\alpha$ and from (3.24)-(3.27) we infer that (3.5) holds for $n > \max(n_0, \tilde{n}_0)$. Thus the theorem is proved.

Proof of Theorem 3.2. We start again with (3.9) and put $m = [n^{\alpha}]$, where $0 < \alpha < 1/5$. Then from (3.14) or (3.15) it follows that $\Delta_{n,m}$ tends to zero as some negative power of $n$. Therefore

$$\Delta_{n,m} = o\left((\ln n)^{-2\varkappa}\right).$$

Now setting $\varepsilon = (\ln n)^{-\gamma_2}$,$0 < \gamma_2 < \delta/2$, from (3.17) and (3.18) we infer that $P\{\|\eta\| < \varepsilon\}$ and $P\{\|\eta\|_m > \varepsilon\}$ tend to zero as some negative powers of $n$. Finally, for the estimate of $P\{\|S_n\|_m > \varepsilon\}$ we use lemma 3.7 with $p = 2-\gamma_3$. From (3.22) we get

$$P\{\|S_n\|_m > \varepsilon\} \leqslant C(\gamma_3)(\ln n)^{-2\varkappa+\gamma_1},$$

where $C(\gamma_3) = C\gamma_3^{-1}$, $\gamma_1 = \gamma_3\varkappa+(2-\gamma_3)\gamma_2$, and this quantity can be made small if $\gamma_3$ and $\gamma_2$ are chosen to be small. Thus the proof is complete.

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