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FRACTIONAL CALCULUS IN PROBABILITY

BY

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Abstract. Operators I^α and $D^\alpha (\alpha > 0)$ are defined on i.d.p.m.'s on a Banach space in such a way that they stand for some analogues of fractional integration and differentiation on functions. Further, we apply the theory to give a new characterization of stable measures and Gaussian measures on Banach spaces.

1. Introduction and notation. Throughout the paper* we shall preserve the terminology and notation from [13]. In particular, by X we denote a real separable Banach space. Let $L_0(X)$ be the set of all infinitely divisible probability measures (i.d.p.m.'s) on X and $L_\alpha(X)$ $(\alpha > 0)$ its subsets as defined in [13].

In the sequel we introduce operators I^α and D^α $(\alpha > 0)$ on $L_0(X)$ which satisfy the basic monotonicity and additivity laws and can be considered as fractional calculus on i.d.p.m.'s.

The method of construction of the operators I^α is based on the well-known definition of vector-valued integrals. Namely, they are first defined on simple Poisson measures and then are extended to some larger classes of p.m.'s. In this context, by a *simple Poisson measure* we mean a p.m. μ of the form $\mu = [G]$, where

$$G = \sum_{i=1}^{k} \lambda_i \delta_{x_i} \quad \text{for some } \lambda_i \geqslant 0 \text{ and } x_i \in X \setminus \{0\}.$$

Further, the operators D^α are defined via the decomposability properties of p.m.'s.

As an application of the study we prove that the only solutions of the differential equation $D^\alpha \mu = \mu^\beta * \delta_x$ are stable and Gaussian measures.

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The value of our results is that they open a new direction in the study of decomposability properties of p.m.'s and they are not known even in the one-dimensional case.

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2. Fractional integration on $L_0(X)$. Let J^α $(\alpha > 0)$ be operators on semi-finite measures defined by means of (2.4) in [13]. From Proposition 2.10 in [13] it follows that if $[G]$ is a simple Poisson measure on X, then, for every $\alpha > 0$, $J^\alpha G$ is a Lévy measure on X.

Modifying the well-known definition of integration with respect to a vector-valued measure (cf. [4], p. 239) one can define the following integration:

Given a simple Poisson measure $\mu = [G]$ on X and $\alpha > 0$ we put

$$I^\alpha \mu = [J^\alpha G]. \tag{2.1}$$

A p.m. μ on X is said to be α-*integrable* if there exist a sequence $\{[G_n]\}$ of simple Poisson measures on X and a vector $x \in X$ such that

$$[G_n] * \delta_x \Rightarrow \mu \tag{2.2}$$

and $I^\alpha [G_n]$ converges weakly to some p.m. Define

$$I^\alpha \mu = \lim_{n\to\infty} I^\alpha [G_n] * \delta_x. \tag{2.3}$$

The limit p.m. $I^\alpha \mu$ is called an *integral of μ of fractional order α*. It should be noted, by Lemma 2.1 (below), that if the limit measure $I^\alpha \mu$ exists, then it is uniquely determined by μ.

2.1. Lemma. *Let $\{[G_n]\}$ be a sequence of simple Poisson measures on X such that*

$$[G_n] * \delta_x \Rightarrow \mu = [x, R, G] \quad \text{and} \quad I^\alpha [G_n] * \delta_x \Rightarrow \nu$$

for some $x \in X$ and $\alpha > 0$. Then $J^\alpha G$ is a Lévy measure, $\nu = [x, 2^{-\alpha} R, J^\alpha G]$ and, consequently,

$$I^\alpha [x, R, G] = [x, 2^{-\alpha} R, J^\alpha G]. \tag{2.4}$$

Proof. Let $\nu = [x, R_1, M]$. Since, by assumption, $G_n \Rightarrow G$ and $J^\alpha G_n \Rightarrow M$, we infer from Theorem 2.5 in [13] that

$$M = J^\alpha G. \tag{2.5}$$

Further, since M and G are Lévy measures, it follows, by (2.4) in [13] and by a simple computation, that for every $\delta > 0$ and every $y \in X^*$ we have

$$\int_{B_\delta} \langle x, y\rangle^2 M(dx) < \infty, \quad \int_{B_\delta} \langle x, y\rangle^2 G(dx) < \infty,$$

and

$$(2.6)\qquad \int_{B_\delta} \langle x, y\rangle^2 M(dx)$$

$$= 2^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 G(dx) + \frac{1}{\Gamma(\alpha)} \int_{B_\delta} \int_{\log\|x\|/\delta}^{\infty} e^{-2t} t^{\alpha-1} \langle x, y\rangle^2 dt\, G(dx).$$

Consequently,

$$(2.7)\qquad \lim_{\delta\downarrow 0} \frac{1}{\Gamma(\alpha)} \int_{B_\delta} \int_{\log\|x\|/\delta}^{\infty} e^{-2t} t^{\alpha-1} dt \langle x, y\rangle^2 G(dx) = 0.$$

On the other hand, by Theorem 1.7 from [5] we have

$$(2.8)\qquad \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int_{B_\delta} \langle x, y\rangle^2 G_n(dx) = \langle Ry, y\rangle$$

and

$$(2.9)\qquad \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int_{B_\delta} \langle x, y\rangle^2 J^\alpha G_n(dx) = \langle R_1 y, y\rangle.$$

From (2.9) and (2.7) we get

$$\lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int_{B_\delta} \langle x, y\rangle^2 J^\alpha G_n(dx) = \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} 2^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 G_n(dx) +$$

$$+ \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int_{B_\delta} \int_{\log\|x\|/\delta}^{\infty} e^{-2t} t^{\alpha-1} dt \langle x, y\rangle^2 G_n(dx) = 2^{-\alpha} \langle Ry, y\rangle$$

for every $y \in X^*$, which together with (2.9) implies that $R_1 = 2^{-\alpha} R$ and, by (2.5), equation (2.4) holds. Thus the lemma is proved.

2.2 COROLLARY. *For every $\alpha > 0$, I^α is a one-to-one operator from $L_0(X)$ into $L_\alpha(X)$ such that if μ_1 and μ_2 are α-integrable, A is a bounded linear operator on X, and $\gamma > 0$, then*

$$(2.10)\qquad I^\alpha (A\mu_1^\gamma * \mu_2) = (AI^\alpha \mu_1)^\gamma * I^\alpha \mu_2.$$

Proof. From the definition of I^α it follows that I^α transforms $L_0(X)$ into $L_\alpha(X)$. Moreover, from Lemma 2.1 and from Theorem 2.9 in [13] we infer that I^α is one-to-one. Formula (2.10) is a simple consequence of (2.5) from [13] and (2.4). Thus the corollary is proved.

2.3 LEMMA. *Let, for $n = 1, 2, \ldots$,*

$$(2.11)\qquad G_n = n(\delta_{(2n)^{-1/2}} + \delta_{-(2n)^{-1/2}}).$$

Then

$$[G_n] \Rightarrow N(0, 1) \tag{2.12}$$

and, for every $\alpha > 0$,

$$I^\alpha [G_n] \Rightarrow N(0, 2^{-\alpha}). \tag{2.13}$$

Proof. By the classical central limit theorem we obtain (2.12). Further, given $\varepsilon > 0$, we have, by (2.7) in [13], the formula

$$J^\alpha G_n(\{x \in R^1 \colon |x| > \varepsilon\}) = \frac{1}{\Gamma(\alpha+1)} \int_{|x|>\varepsilon} \log^\alpha |x| \, \varepsilon^{-1} G_n(dx) = 0 \tag{2.14}$$

for sufficiently large n. On the other hand, for every $r > 0$ we have, by (2.8) in [13], the formula

$$\int_{-r}^{r} x^2 J^\alpha G_n(dx) = 2^{-\alpha} \int_{-r}^{r} x^2 G_n(dx) +$$

$$+ \frac{r^2}{\Gamma(\alpha)} \int_{|x|>r} \int_0^\infty e^{-2t} (t + \log |x| r^{-1})^{\alpha-1} dt G(dx) = 2^{-\alpha}$$

for sufficiently large n, which together with (2.14) implies that $I^\alpha [G_n] \Rightarrow N(0, 2^{-\alpha})$. Thus the lemma is proved.

2.4. Lemma. *Every Gaussian measure* ϱ *on* X *is* α*-integrable* ($\alpha > 0$) *and, for some* $x \in X$,

$$I^\alpha \varrho = \varrho^{2^{-\alpha}} * \delta_x. \tag{2.15}$$

Proof. We may assume that ϱ is a nondegenerate symmetric Gaussian measure on X. Let Z be an X-valued r.v. with distribution ϱ. From the Jain-Kallianpur theorem ([7], Theorem 3) it follows that there exist a sequence $\{x_k\} \subset X \setminus \{0\}$ and a sequence $\{z_k\}$ of i.i.d. real valued r.v.'s with distribution $N(0, 1)$ such that

$$Z = \sum_k x_k z_k, \tag{2.16}$$

where the series is convergent with probability 1. Let ϱ_m ($m = 1, 2, \ldots$) be the distribution of $\sum_{k=1}^{m} x_k z_k$. We consider ϱ_m as a measure on the finite-dimensional space $X_m := \mathrm{lin}(x_1, \ldots, x_m)$. Putting, for $n, m = 1, 2, \ldots$,

$$G_{n,m} = n \sum_{k=1}^{m} (\delta_{x_k/(2n)^{1/2}} + \delta_{-x_k/(2n)^{1/2}}) \tag{2.17}$$

and taking into account Lemma 2.3 we get, for any $\alpha > 0$ and $m = 1, 2, \ldots$,

$$[G_{n,m}] \Rightarrow \varrho_m \tag{2.18}$$

and

$$I^{\alpha}[G_{n,m}] \Rightarrow \varrho_m^{2-\alpha} \tag{2.19}$$

as $n \to \infty$. Now, since $\varrho_m \Rightarrow \varrho$ and $\varrho_m^{2-\alpha} \Rightarrow \varrho^{2-\alpha}$ as $m \to \infty$, by (2.18) and (2.19) we can choose sequences $\{n_k\}$ and $\{m_k\}$ such that $[G_{n_k,m_k}] \Rightarrow \varrho$ and $I^{\alpha}[G_{n_k,m_k}] \Rightarrow \varrho^{2-\alpha}$ as $k \to \infty$. Thus the lemma is proved.

2.5. Lemma. *For every $\alpha > 0$ and for every i.d.p.m. $\mu = [x, R, G]$ on X,*

$$\int_{B_1'} \log^{\alpha} \|x\| \, G(dx) < \infty \tag{2.20}$$

if and only if

$$\int_{B_1'} \log^{\alpha} \|x\| \, \mu(dx) < \infty. \tag{2.21}$$

Proof. Without loss of generality we may assume that $x = 0$ and $R = 0$. Let G_1 and G_2 be restrictions of G to B_1 and B_1', respectively. Then $[G] = [G_1] * [G_2]$ and, by results of Yurinski [15], for every $\alpha > 0$ we obtain

$$\int_{B_1'} \log^{\alpha} \|x\| \, [G_1](dx) < \infty.$$

Therefore, (2.21) holds if and only if

$$\int_{B_1'} \log^{\alpha} \|x\| \, [G_2](dx) < \infty.$$

Thus, we may assume further that G is concentrated on B_1'. Then

$$\int_{B_1'} \log^{\alpha} \|x\| \, \mu(dx) = e^{-G(X)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{B_1'} \log^{\alpha} \|x\| \, G^{*k}(dx) \tag{2.22}$$

and, consequently, (2.21) implies (2.20).

Next, suppose that (2.20) holds. It should be noted that, for any $k = 1, 2, \ldots$ and $a_1, \ldots, a_k \geqslant 0$,

$$\max(1, a_1 + \ldots + a_k) \leqslant k \max(1, a_1) \ldots \max(1, a_k). \tag{2.23}$$

Further, for $k = 1, 2, \ldots$,

$$\int_{B_1} \log^{\alpha} \|x\| \, G^{*k}(dx) = \int_X \log^{\alpha} \max(1, \|x\|) \, G^{*k}(dx)$$

$$= \int_X \ldots \int_X \log^{\alpha} \max(1, \|x_1 + \ldots + x_k\|) \, G(dx_1) \ldots G(dx_k)$$

$$\leqslant \int_X \ldots \int_X \Big(\log k + \sum_{i=1}^{k} \log \max(1, \|x_i\|)\Big)^{\alpha} G(dx_1) \ldots G(dx_k) \quad \text{(by (2.23))}$$

$$\leqslant \int_X \ldots \int_X (k+1)^{\alpha} \Big(\log^{\alpha} k + \sum_{i=1}^{k} \log^{\alpha} \max(1, \|x_i\|)\Big) G(dx_1) \ldots G(dx_k)$$

$$= (k+1)^{\alpha} \log^{\alpha} k G^k(X) + k(k+1)^{\alpha} G^{k-1}(X) \int_X \log^{\alpha} \max(1, \|x\|) \, G(dx).$$

Consequently,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int\limits_{B_1'} \log^{\alpha} \|x\| \, G^{*k}(dx) < \infty,$$

which together with (2.22) implies (2.21). Thus the lemma is proved.

2.6. COROLLARY. *For every α-integrable p.m. μ on X, condition* (2.21) *is satisfied.*

Proof. Let G be a Lévy measure corresponding to an α-integrable p.m. μ on X. By Lemma 2.1, $J^{\alpha}G$ is a Lévy measure, and hence G satisfies (2.20). Consequently, by Lemma 2.5, (2.21) holds, which completes the proof of the corollary.

2.7. LEMMA. *Let G be a Lévy measure concentrated on $B_{r,s}$ for some r $(0 < r < s)$. Then $[G]$ is α-integrable for every $\alpha > 0$.*

Proof. Let $\{[G_n]\}$ be a sequence of simple Poisson measures converging to $[G]$, where G_n $(n = 1, 2, \ldots)$ are concentrated on $B_{r,s}$ and $G_n \Rightarrow G$. By Proposition 2.10 in [13], $J^{\alpha}G_n$ and $J^{\alpha}G$ are Lévy measures. We shall prove that $I^{\alpha}[G_n] \Rightarrow I^{\alpha}[G]$.

Accordingly, by Corrolary 2.4 in [13], we obtain

$$J^{\alpha}G_n \Rightarrow J^{\alpha}G. \tag{2.24}$$

Further, since $\int_{B_1} \|x\| \, G(dx) < \infty$, by Proposition 2.1 from [13] we have

$$\int_{B_1} \|x\| \, J^{\alpha}G(dx) < \infty. \tag{2.25}$$

Now, by the assumption that $G_n \Rightarrow G$ and by (2.8) in [13], we get

$$\lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int\limits_{B_\delta} \|x\| \, J^{\alpha}G_n(dx) = \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \int\limits_{B_\delta} \|x\| \, G_n(dx) +$$

$$+ \lim_{\delta\downarrow 0} \overline{\lim_{n\to\infty}} \frac{\delta}{\Gamma(\alpha)} \int\limits_{B_\delta'} \int\limits_{0} e^{-t}(t + \log \|x\| \, \delta^{-1})^{\alpha-1} \, dt G_n(dx)$$

$$= \lim_{\delta\downarrow 0} \int\limits_{B_\delta} \|x\| \, G(dx) + \lim_{\delta\downarrow 0} \frac{\delta}{\Gamma(\alpha)} \int\limits_{B_\delta'} \int\limits_{0}^{\infty} e^{-t}(t + \log \|x\| \, \delta^{-1})^{\alpha-1} \, dt G(dx)$$

$$= \lim_{\delta\downarrow 0} \int\limits_{B_\delta} \|x\| \, J^{\alpha}G(dx) = 0 \quad \text{(by (2.25))}.$$

Hence and by (2.24) it follows from Corollary 1.8 in [5] that $I^\alpha[G_n] \Rightarrow I^\alpha[G]$. Thus, $[G]$ is α-integrable, which completes the proof of the lemma.

The following theorem gives a characterization of α-integrable p.m.'s on X:

2.8. THEOREM. *A p.m. $\mu = [x, R, G]$ on X is α-integrable ($\alpha > 0$) if and only if $J^\alpha G$ is a Lévy measure on X.*

Proof. From Lemma 2.1 it follows that if $\mu = [x, R, G]$ is α-integrable, then $J^\alpha G$ is a Lévy measure.

Conversely, suppose that $J^\alpha G$ is a Lévy measure on X, where G is a Lévy measure corresponding to $\mu = [x, R, G]$. Since, by Lemma 2.4, the Gaussian component $[x, R, 0]$ of μ is α-integrable, it suffices to show that $[G]$ is α-integrable.

Accordingly, for every $m = 1, 2, \ldots$ we put $G_m = G|B_{1/m,m}$. By Lemma 2.7, every $[G_m]$ is α-integrable. Moreover, it can be seen that $[G_m] \Rightarrow [G]$ and $I^\alpha[G_m] \Rightarrow [J^\alpha G]$ as $m \to \infty$. For every $m = 1, 2, \ldots$ let $[G_{n,m}]$ $(n = 1, 2, \ldots)$ be a sequence of simple Poisson measures such that $[G_{m,n}] \Rightarrow [G_m]$ and $I^\alpha[G_{m,n}] \Rightarrow I^\alpha[G_m]$ as $n \to \infty$. Then we can choose sequences $\{m_k\}$ and $\{n_k\}$ of natural numbers such that $[G_{m_k,n_k}] \Rightarrow [G]$ and $I^\alpha[G_{m_k,n_k}] \Rightarrow [J^\alpha G]$ as $k \to \infty$, which shows that $[G]$ is α-integrable. Thus the theorem is proved.

2.9. COROLLARY. *If X is of type p $(0 < p \leqslant 2)$, then $\mu = [x, R, G]$ with*

$$\int_{B_1} \|x\|^p G(dx) < \infty \tag{2.26}$$

is α-integrable ($\alpha > 0$) if and only if

$$\int_X \log^\alpha \max(1, \|x\|)\, \mu(dx) < \infty. \tag{2.27}$$

Proof. Let G be a Lévy measure corresponding to μ and assume that (2.26) holds. Then, by Proposition 2.1 in [13], we have

$$\int_{B_1} \|x\|^p J^\alpha G(dx) < \infty.$$

Moreover, since X is of type p, $J^\alpha G$ is a Lévy measure if and only if (2.20) is satisfied, which, by Lemma 2.5 and Theorem 2.8, implies that μ is α-integrable if and only if (2.27) holds. Thus the corollary is proved.

Since every Hilbert space is of type 2, Corollary 2.9 implies the following

2.10. COROLLARY. *The class of all α-integrable ($\alpha > 0$) p.m.'s on a Hilbert space H coincides with the class of all i.d.p.m.'s μ on H such that*

$$\int_H \log^\alpha \max(1, \|x\|)\, \mu(dx) < \infty.$$

The proof of the following theorem is the same as the proof of Lemma 2.1 and will be omitted.

2.11. THEOREM. *Let $\{\mu_n\}$ be a sequence of α-integrable ($\alpha > 0$) p.m.'s on X such that $\mu_n \Rightarrow \mu$ and $I^\alpha \mu_n \Rightarrow \nu$. Then μ is α-integrable and, moreover, $I^\alpha \mu = \nu$.*

In the sequel, for any $\mu, \nu \in L_0(X)$ we write $\mu \prec \nu$ if there exists $\tau \in L_0(X)$ such that $\mu * \tau = \nu$. Then $\prec$ is a partial ordering in $L_0(X)$. In terms of the relation $\prec$ we get the following analogue of the Dominated Convergence Theorem for ordinary integrals:

2.12. THEOREM. *Let ν be an α-integrable ($\alpha > 0$) p.m. on X and $\{\mu_n\}$ a sequence of measures in $L_0(X)$ such that $\mu_n \Rightarrow \mu$ and $\mu_n \prec \nu$ for every $n = 1, 2, \ldots$ Then μ and μ_n are α-integrable and $I^\alpha \mu_n \Rightarrow I^\alpha \mu$.*

Proof. By assumption, $\mu \prec \nu$, and if $\mu_n = [x_n, R_n, G_n]$, $\mu = [x, R, G]$, and $\nu = [x_0, R_0, G_0]$, then $J^\alpha G_n \leqslant J^\alpha G_0$ and $J^\alpha G \leqslant J^\alpha G_0$ $(n = 1, 2, \ldots)$. Consequently, $J^\alpha G_n$ and $J^\alpha G$ are Lévy measures on X. Thus, by Theorem 2.8, μ_n and μ are α-integrable. Moreover, $[J^\alpha G_n] \prec [J^\alpha G_0]$ $(n = 1, 2, \ldots)$. Hence and by Theorem 2.2 in [10], the sequence $\{[J^\alpha G_n]\}$ is relatively shift compact. Further, by Corollary 1.5 from [5], $\{[J^\alpha G_n]\}$ is relatively compact. Let $\{[J^\alpha G_{n_k}]\}$ be an arbitrary convergent subsequence of $\{[J^\alpha G_n]\}$. Since $[G_{n_k}] \Rightarrow [G]$, by Theorem 2.11 we have $[J^\alpha G_{n_k}] \Rightarrow I^\alpha [G]$. Consequently, $[J^\alpha G_n] \Rightarrow I^\alpha [G]$.

Finally,

$$I^\alpha \mu_n = [x_n, 2^{-\alpha} R_n, 0] * [J^\alpha G_n] \Rightarrow [x, 2^{-\alpha} R, 0] * [J^\alpha G] = I^\alpha \mu,$$

which completes the proof of the theorem.

The following theorems are concerned with the basic monotonicity and additivity laws for I^α.

2.13. THEOREM. *Suppose that μ is an α-integrable ($\alpha > 0$) p.m. on X such that, for some $\beta > 0$, $I^\alpha \mu$ is β-integrable. Then μ is $(\alpha+\beta)$-integrable and*

$$I^{\alpha+\beta} \mu = I^\beta I^\alpha \mu. \tag{2.28}$$

Proof. Let $\mu = [x, R, G]$. Then, by (2.4), we have $I^\alpha \mu = [x, 2^{-\alpha} R, J^\alpha G]$. Further, since $I^\alpha \mu$ is β-integrable, we obtain

$$I^\beta I^\alpha \mu = [x, 2^{-\alpha-\beta} R, J^\beta J^\alpha G],$$

which, by (2.9) in [13], implies that $J^{\alpha+\beta} G$ is a Lévy measure, and hence μ is $(\alpha+\beta)$-integrable. Moreover, (2.28) holds, which completes the proof of the theorem.

2.14. THEOREM. *Suppose that X is of type p ($0 < p \leqslant 2$), $\mu = [x, R, G]$ is $(\alpha+\beta)$-integrable ($\alpha, \beta > 0$), and (2.26) holds. Then μ is α-integrable and $I^\alpha \mu$ is β-integrable.*

Proof. By assumption, $J^{\alpha+\beta}G$ is a Lévy measure. Hence and by Proposition 2.10 from [13] we have

$$\int_{B_1} \log^{\alpha+\beta}\|x\|\, G(dx) < \infty,$$

which implies (2.20). Again by Proposition 2.10 in [13] and by (2.26), $J^{\alpha}G$ is a Lévy measure. Consequently, by Theorem 2.8, μ is α-integrable and $I^{\alpha}\mu = [x, 2^{-\alpha}R, J^{\alpha}G]$. Further, since $J^{\alpha+\beta}G = J^{\beta}J^{\alpha}G$ and $J^{\alpha+\beta}G$ ia a Lévy measure, $I^{\alpha}\mu$ is β-integrable. Thus the theorem is proved.

Now, by Theorems 2.13 and 2.14 we get the following

2.15. COROLLARY. *A p.m. μ on a Hilbert space is $(\alpha+\beta)$-integrable if and only if it is α-integrable and $I^{\alpha}\mu$ is β-integrable. In any case, (2.28) holds.*

3. Fractional derivative of p.m.'s on X. Since operators I^{α} $(\alpha > 0)$ are one-to-one, we can define differentiations D^{α} $(\alpha > 0)$ as operations converse to I^{α}. Thus, $D^{\alpha} = I^{-\alpha}$. Putting, in addition, $I^{0}\mu = \mu$ $(\mu \in L_0(X))$, we obtain a family I^{α} $(\alpha \in R^1)$ of operators on $L_0(X)$ with the group property

$$I^{\alpha}I^{\beta}\mu = I^{\alpha+\beta}\mu \quad (\alpha, \beta \in R^1) \tag{3.1}$$

whenever $I^{\beta}\mu$ and $I^{\alpha}I^{\beta}\mu$ exist.

Our further aim is to give another approach to the definition of fractional derivatives. Namely, we introduce fractional derivatives via decomposability properties of p.m.'s.

For simplicity of the notation we put

$$A(\mu, \beta) = \mu^{\beta} \quad (\mu \in L_0(X),\ \beta > 0)$$

and

$$\left|\begin{matrix}\alpha\\ k\end{matrix}\right| = \frac{|\alpha(\alpha-1)\ldots(\alpha-k+1)|}{k!} \quad (\alpha > 0,\ k = 1, 2, \ldots). \tag{3.2}$$

3.1. LEMMA. *For any $\alpha > 0$, $c \in (0, 1)$, and $\mu \in L_0(X)$, the series*

$$\mathop{*}_{k=1}^{\infty} T_{c^k} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right)$$

is convergent.

Proof. Since $\sum_{k=1}^{\infty} \left|\begin{matrix}\alpha\\ k\end{matrix}\right| < \infty$, the series $\mathop{*}_{k=1}^{\infty} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right)$ is convergent for every $\mu \in L_0(X)$. Let $\{z_k\}$ be a sequence of X-valued independent, r.v.'s such that z_k is distributed as $A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right)$ $(k = 1, 2, \ldots)$. Then the series $\sum_{k=1}^{\infty} z_k$ is convergent with probability 1. Therefore, for every $c \in (0, 1)$ the series $\sum_{k=1}^{\infty} c^k z_k$

is convergent with probability 1 and, consequently, the series $\mathop{*}\limits_{k=1}^{\infty} T_{c^k} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right)$ is convergent. Thus the lemma is proved.

By virtue of Lemma 3.1 we can define operators T_c^α on the whole $L_0(X)$ by

$$T_c^\alpha \mu = \mathop{*}\limits_{k=1}^{\infty} T_{c^k} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right) \quad (\mu \in L_0(X)), \tag{3.3}$$

where $\alpha > 0$ and $c \in (0, 1)$.

3.2. Lemma. *Let μ be a self-decomposable p.m. on X. Then for any $\alpha, c \in (0, 1)$ there exists $\mu_{\alpha,c} \in L_0(X)$ such that*

$$\mu = T_c^\alpha \mu * \mu_{\alpha,c}. \tag{3.4}$$

Proof. Suppose that $\alpha, c \in (0, 1)$ and $\mu \in L_1(X)$. Then there exists a p.m. $\mu_c \in L_0(X)$ such that

$$\begin{aligned}\mu = T_c \mu * \mu_c &= T_c A\left(\mu, 1 - \left|\begin{matrix}\alpha\\ 1\end{matrix}\right|\right) * T_c A\left(\mu, \left|\begin{matrix}\alpha\\ 1\end{matrix}\right|\right) * \mu_c \\ &= T_{c^2} A\left(\mu, 1 - \left|\begin{matrix}\alpha\\ 1\end{matrix}\right| - \left|\begin{matrix}\alpha\\ 2\end{matrix}\right|\right) * T_{c^2} A\left(\mu, \left|\begin{matrix}\alpha\\ 2\end{matrix}\right|\right) * \\ &\quad * T_c A\left(\mu, \left|\begin{matrix}\alpha\\ 1\end{matrix}\right|\right) * T_c A\left(\mu_c, 1 - \left|\begin{matrix}\alpha\\ 1\end{matrix}\right|\right) * \mu_c.\end{aligned}$$

Hence and by a simple induction we get, for $n = 1, 2, \ldots$,

$$\begin{aligned}\mu = T_{c^n} A\left(\mu, 1 - \sum_{k=1}^{n} \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right) * \mathop{*}\limits_{k=1}^{n} T_{c^k} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right) * \\ * \mathop{*}\limits_{k=0}^{n-1} T_{c^k} A\left(\mu_c, 1 - \sum_{k=0}^{n-1} \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right).\end{aligned} \tag{3.5}$$

Since, by Lemma 3.1, $\mathop{*}\limits_{k=1}^{n} T_{c^k} A\left(\mu, \left|\begin{matrix}\alpha\\ k\end{matrix}\right|\right)$ is convergent to $T_c^\alpha \mu$, we infer from (3.5) that there exists $\mu_{\alpha,c} \in L_0(X)$ such that (3.4) holds, which completes the proof of the lemma.

From Lemma 3.2 we obtain immediately the following

3.3. Corollary. *For any $c \in (0, 1)$, $\alpha > 0$, and $\mu \in L_n(X)$, where n is the smallest integer greater than α, there exists a finite sequence $\mu_{1,c}, \ldots, \mu_{n-1,c}, \mu_{\alpha,c}$ such that*

$$\begin{aligned}\mu = T_c \mu * \mu_{1,c}, \quad \mu_{1,c} = T_c \mu_{1,c} * \mu_{2,c}, \ldots, \\ \mu_{n-2,c} = T_c \mu_{n-2,c} * \mu_{n-1,c}, \quad \mu_{n-1,c} = T_c^{\alpha-n+1} \mu_{n-1,c} * \mu_{\alpha,c}.\end{aligned} \tag{3.6}$$

Suppose that $\alpha > 0$ and $n = [\alpha]+1$. A p.m. μ on X is said to be *α-differentiable* if $\mu \in L_n(X)$ and there exists a weak limit, say $D^{(\alpha)}\mu$,

(3.7) $$D^{(\alpha)}\mu = \lim_{t \downarrow 0} \mu_{\alpha,c}^{t^{-\alpha}},$$

where for $c = e^{-t} \in (0, 1)$ the measure $\mu_{\alpha,c}$ is defined by (3.6). In particular, for $\alpha = 1$, 1-differentiable p.m.'s are called *differentiable*. The limit measure $D^{(\alpha)}\mu$ in (3.7) is said to be a *derivative of μ of fractional order α*.

3.4. THEOREM. *For every α-differentiable p.m. μ on X, $D^{(\alpha)}\mu$ is α-integrable,*

(3.8) $$I^{\alpha} D^{(\alpha)}\mu = \mu$$

and, consequently,

(3.9) $$D^{(\alpha)}\mu = D^{\alpha}\mu.$$

Proof. Given $\alpha > 0$, $t = -\log c > 0$, and a Lévy measure M, we put

(3.10) $$\Delta_t^{\alpha} M(E) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} M(E)$$

for every Borel subset E of X such that $0 \notin \bar{E}$, where

$$\binom{\alpha}{0} = 1$$

and, for $k = 1, 2, \ldots$,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}.$$

By an easy computation it follows that for every $c = e^{-t} \in (0, 1)$ the measure $\mu_{\alpha,c}$ given by (3.6) is of the form

(3.11) $$\mu_{\alpha,c} = [(1-e^{-t})^{\alpha} x, (1-e^{-2t})^{\alpha} R, \Delta_t^{\alpha} M].$$

In particular, $\Delta_t^{\alpha} M$ is a Lévy measure if $\mu = [x, R, M]$ is $[\alpha]+1$ times self-decomposable. Further, suppose that $D^{(\alpha)}\mu = [x_1, R_1, G]$. Then, by the definition of $D^{(\alpha)}\mu$ and by Theorem 1.7 from [5] we obtain

(3.12) $$\lim_{t \downarrow 0} t^{-\alpha}(1-e^{-t})^{\alpha} x = x = x_1,$$

(3.13) $$\lim_{t \downarrow 0} t^{-\alpha} \Delta_t^{\alpha} M = G$$

and, for every $y \in X^*$,

(3.14) $$\langle R_1 y, y\rangle = \lim_{\delta \downarrow 0} \overline{\lim_{t \downarrow 0}}\, t^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 \Delta_t^{\alpha} M(dx) + \lim_{t \downarrow 0} t^{-\alpha}(1-e^{-2t})^{\alpha} \langle Ry, y\rangle.$$

On the other hand, since $\mu \in L_n(X)$ and $\alpha \leqslant n$, we infer that $\mu \in L_\alpha(X)$ and, by (3.11) and by Theorem 2.9 in [13], we get

$$M = J^\alpha G. \tag{3.15}$$

We shall prove that, for every $y \in X^*$,

$$\lim_{\delta\downarrow 0} \lim_{t\downarrow 0} t^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 \Delta_t^\alpha M(dx) = 0. \tag{3.16}$$

In fact, let m_t^α $(t > 0)$ be the signed measures defined by (2.14) and (2.16) in [13]. Then, by (2.4) in [13], (3.15), and by some computation, we get

$$t^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 \Delta_t^\alpha M(dx) = \int_X \int_0^\infty 1_{B_\delta}(e^{-u}x) e^{-2u} \langle x, y\rangle^2 m_t^\alpha(du) G(dx) \tag{3.17}$$

for every $\delta > 0$, which implies

$$\begin{aligned} t^{-\alpha} \int_{B_\delta} \langle x, y\rangle^2 \Delta_t^\alpha M(dx) \\ = \int_{B_\delta} \int_0^\infty e^{-2u} m_t^\alpha(du) \langle x, y\rangle^2 G(dx) + \int_{B'_\delta} \int_{\log\|x\|/\delta}^\infty e^{-2u} m_t^\alpha(du) \langle x, y\rangle^2 G(dx). \end{aligned} \tag{3.18}$$

By Lemma 2.6 from [13], m_t^α have a common finite variation, say K, and $m_t^\alpha \downarrow \delta_0$ as $t \downarrow 0$. Hence for any $x \in X$ and $y \in X^*$ we get

$$\Big|\int_0^\infty e^{-2u} m_t^\alpha(du)\Big| \leqslant K/2$$

and

$$\langle x, y\rangle^2 \Big| \int_{\log\|x\|/\delta}^\infty e^{-2u} m_t^\alpha(du)\Big| \leqslant K\delta^2 \|y\|^2,$$

which, by (3.18) and by the Dominated Convergence Theorem, implies (3.16).

Now, by (3.14) and (3.16) we get

$$R_1 = 2^\alpha R, \tag{3.19}$$

which together with (3.12) and (3.13) implies that $D^{(\alpha)}\mu$ is α-integrable and (3.8) and (3.9) hold. Thus the theorem is proved.

In the sequel we shall give some sufficient conditions for the existence of $D^{(\alpha)}\mu$. Namely, we get the following

3.5. THEOREM. *Suppose that X is of type p $(0 < p \leqslant 2)$, $\alpha > 0$, and $\mu = [x, R, M] \in L_n(X)$, where n is the smallest integer greater than α and*

$$\int_{B_1} \|x\|^p M(dx) < \infty. \tag{3.20}$$

Then μ is α-differentiable.

Proof. Without loss of generality we may assume that $x = 0$ and $R = 0$. Given $c = e^{-t}$, $t > 0$, define $\mu_{\alpha,c}$ by (3.6). Then, by (3.11), we get

$$\mu_{\alpha,c} = [\Delta_t^\alpha M]. \tag{3.21}$$

Further, since $\mu \in L_n(X)$ and $\alpha \leqslant n$, there exists $G \in M(X)$ such that

$$M = J^\alpha G, \tag{3.22}$$

which, by assumption and by Proposition 2.10 from [13] implies that G is a Lévy measure. Moreover, by (3.22) and by Theorem 2.9 in [13] we obtain

$$t^{-\alpha}\Delta_t^\alpha M \Rightarrow G \quad \text{as } t \downarrow 0. \tag{3.23}$$

Our further aim is to prove that

$$\lim_{\delta\downarrow 0} \overline{\lim_{t\downarrow 0}} \int_{B_\delta} \|x\|^p t^{-\alpha} \Delta_t^\alpha M(dx) = 0, \tag{3.24}$$

which, by Corollary 2.8 from [5] and by (3.23) should imply that

$$\lim_{t\downarrow 0} \mu_{\alpha,c}^{t^{-\alpha}} = [G],$$

and then the theorem should be proved.

Accordingly, for any $t = -\log c > 0$ and $\delta > 0$ we have, by (2.4) in [13] and (3.13), the formulas

$$t^{-\alpha} \int_{B_\delta} \|x\|^p \Delta_t^\alpha M(dx) = t^{-\alpha} \int_{B_\delta} \|x\|^p \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} J^\alpha G(dx)$$

$$= \int_{B_\delta} \int_0^\infty e^{-pu} m_t^\alpha(du) \|x\|^p G(dx) + \int_{B_\delta} \int_{\log\|x\|/\delta}^{\infty} e^{-pu} \|x\|^p m_t^\alpha(du) G(dx),$$

where m_t^α is defined by (2.14) and (2.16) in [13].

Since, by Lemma 2.6 in [13], the signed measures m_t^α $(t > 0)$ have a common finite variation, say K, and $m_t^\alpha \Rightarrow \delta_0$ as $t \downarrow 0$, we get

$$\left|\int_0^\infty e^{-pu} m_t^\alpha(du)\right| \leqslant K/p$$

and

$$\|x\|^p \left|\int_{\log\|x\|/\delta}^{\infty} e^{-pu} m_t^\alpha(du)\right| \leqslant \delta^p K$$

which, by the above formulas and by the Dominated Convergence Theorem, implies (3.24). Thus the theorem is proved.

From Theorem 3.5 we get the following corollaries:

3.6. Corollary. *Suppose that X is of type p $(0 < p \leqslant 2)$. Then every self-decomposable p.m. $\mu = [x, R, M]$ on X satisfying (3.20) is differentiable.*

3.7. COROLLARY. *Every n times* ($n = 1, 2, \ldots$) *self-decomposable p.m. on a Hilbert space is α-differentiable* ($0 < \alpha \leqslant n$).

4. A characterization of stable measures on X. From Lemma 2.4 it follows that every Gaussian measure $\mu = [x, R, 0]$ on X is α-integrable ($\alpha > 0$) and, by (3.11), there exists a limit

$$\lim_{t\downarrow 0} \mu_{\alpha,c}^{t^{-\alpha}} = [x, 2^{\alpha}R, 0], \tag{4.1}$$

where, for $c = e^{-t}$ ($t > 0$), $\mu_{\alpha,c}$ is defined by (3.6).

The same is true for stable measures on X. Namely, we get the following

4.1. THEOREM. *Let* $\mu = [M]$ *be a stable p.m. on* X *with index* p ($0 < p < 2$). *Then, for every* $\alpha > 0$, μ *is* α*-integrable,*

$$I^{\alpha}\mu = \mu^{p^{-\alpha}}, \tag{4.2}$$

and there exists a limit (*in the weak sense*)

$$\lim_{t\downarrow 0} \mu_{\alpha,c}^{t^{-\alpha}} = \mu^{p^{\alpha}}, \tag{4.3.}$$

where, for $c = e^{-t}$ ($t > 0$), $\mu_{\alpha,c}$ *is defined by* (3.6).

Proof. It is well known [3] that $\mu = [M]$ is a stable p.m. on X with index p ($0 < p < 2$) if and only if there exists a finite measure m on the unit sphere S of X such that

$$M(E) = \int_S \int_0^{\infty} 1_E(ux) \frac{du}{u^{p+1}} m(dx) \quad (E \subset X). \tag{4.4}$$

Further, by (2.4) in [13] and by some computation we get

$$J^{\alpha}M(E) = \frac{1}{\Gamma(\alpha)} \int_S \int_0^{\infty} \int_0^{\infty} 1_E(e^{-v}ux) v^{\alpha-1} dv \frac{du}{u^{p+1}} m(dx) = p^{-\alpha}M(E) \quad (E \subset X). \tag{4.5}$$

Consequently, by Theorem 2.8, μ is α-integrable and (4.2) holds.

Now, from (4.4) it follows that, for every $a > 0$,

$$T_a \mu = \mu^{a^p}. \tag{4.6}$$

Therefore, for any $\alpha > 0$ and $t = -\log c > 0$ the measure $\mu_{\alpha,c}$ defined by (3.6) is of the form

$$\mu_{\alpha,c} = A\left(\mu, \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-pkt}\right) = A(\mu, (1-e^{-pt})^{\alpha}), \tag{4.7}$$

which implies that (4.3) holds. Thus the theorem is proved.

The following theorem together with Theorem 4.1 gives a full description of stable measures on X.

4.2 THEOREM. *Suppose that* $\mu \in L_0(X)$ *and, for some* $\beta > 0$, $\alpha \in R^1 \setminus \{0\}$, *and* $z \in X$, *we have*

$$I^\alpha \mu = \mu^\beta * \delta_z. \tag{4.8}$$

Then μ *is a stable p.m. on* X *with index* p $(0 < p \leqslant 2)$, *where*

$$p = \beta^{-1/\alpha} \tag{4.9}$$

Proof. Since operators I^α $(\alpha \in R^1)$ are one-to-one, we may assume that (4.8) holds for some $\alpha > 0$. Let $\mu = [x, R, M]$ be nondegenerate. By (2.4), equation (4.8) is equivalent to the following:

$$x = \beta x + z, \tag{4.10}$$

$$2^{-\alpha} R = \beta R, \tag{4.11}$$

and

$$J^\alpha M = \beta M. \tag{4.12}$$

Consider equation (4.12). Applying successively operators J^α we get, by (2.9) in [13], the formula

$$J^{n\alpha} M = \beta^n M \quad (n = 1, 2, \ldots), \tag{4.13}$$

which, by the definition of classes $L_\gamma(X)$ $(\gamma > 0)$, implies that $\mu \in L_{n\alpha}(X)$ for every $n = 1, 2, \ldots$ Hence μ is completely self-decomposable. Recall ([12], formula (6.8)) that μ is completely self-decomposable if and only if its Lévy measure M is of the form

(4.14)
$$M(E) = \int_B \int_0^\infty 1_E(sx) \frac{ds}{s^{2\|x\|+1}} \left[\int_0^\infty \Phi(tx) \frac{dt}{t^{2\|x\|+1}} \right]^{-1} m(dx) \quad (E \subset X),$$

where B is the open unit ball in X, m a finite measure on B vanishing at 0, and Φ a weight function on X in the Urbanik sense [14]. Moreover, for a fixed weight function Φ the representation (4.14) is unique.

From (2.4) in [13] and (4.14) it follows that for every Borel subset E of X

$$J^\alpha M(E) = \frac{1}{\Gamma(\alpha)} \int_B \int_0^\infty \int_0^\infty 1_E(e^{-t} sx)\, t^{\alpha-1} h(x)\, dt \frac{ds}{s^{2\|x\|+1}} m(dx) \tag{4.15}$$

$$= \int_B \int_0^\infty 1_E(ux) \frac{du}{u^{2\|x\|+1}} (2\|x\|)^{-\alpha} h(x)\, m(dx),$$

where, for $x \in B$,

(4.16)
$$h(x)^{-1} = \int_0^\infty \Phi(tx) \frac{dt}{t^{2\|x\|+1}}.$$

Therefore, by (4.12), (4.15), and by the uniqueness of the representation (4.14), we get the equation

(4.17)
$$(2\|x\|)^{-\alpha} m(dx) = \beta m(dx).$$

It should be noted that (4.17) holds if and only if either $m = 0$ or $m \neq 0$ and m is concentrated on some sphere S_r of X with radius r $(0 < r < 1)$. In the latter case we get

(4.18)
$$(2r)^{-\alpha} = \beta.$$

Hence and by (4.14), for every Borel subset E of X we obtain

(4.19)
$$M(E) = \int_{S_r} \int_0^\infty 1_E(sx) \frac{ds}{s^{2r+1}} h(x) m(dx).$$

Proceeding successively, we infer from (4.10)-(4.12) and (4.18) that either $R \neq 0$, $\beta = 2^{-\alpha}$, and μ is a Gaussian measure, or $R = 0$, $M \neq 0$, M is of the form (4.19), and μ is a stable measure with index $p = 2r$. Thus, in any case, μ is a stable measure with index p $(0 < p \leqslant 2)$, where p is given by (4.9). Thus the theorem is proved.

From Theorem 4.2 we obtain immediately the following

4.3. Corollary. *A p.m. μ on X is Gaussian if and only if, for some $\alpha \in R^1 \setminus \{0\}$ and $z \in X$,*

$$I^\alpha \mu = \mu^{2^{-\alpha}} * \delta_z.$$

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