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ON POLYNOMIAL CHAOS AND INTEGRABILITY

BY

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Abstract. Given a polynomial chaos $\mathscr{P}$, the main purpose of the present paper is to estimate the quantity $\sup\{\|X\|_p;\ \|X\|_2 \leqslant 1,\ X \in \mathscr{P}\}$ as a function of $p \in]2, \infty[$. The results obtained will then be used to decide L_p-convergence of certain random series with partial sums belonging to $\mathscr{P}$.

1. Introduction. Consider a probability space $(\Omega, \mathscr{A}, P)$ and a class $\mathscr{C} = (V_i)_{i \in N}$ of vector subspaces of $L_2(\Omega, \mathscr{A}, P; \boldsymbol{R})$ having the following properties:

(α) $V_i \perp \boldsymbol{R},\ i \in N$;

(β) $\dim V_i = d,\ i \in N$ $(1 \leqslant d < \infty)$;

(γ) the σ-algebras $\sigma(\xi;\ \xi \in V_i)$, $i \in N$, are stochastically independent.

For any Banach space $B = (B, \|\cdot\|)$ and $n \in N_+$, we introduce the algebraic sum

$$\mathscr{P}_n^a(\mathscr{C}; B) = \sum_{\substack{i_1 < \ldots < i_r \\ 0 \leqslant r \leqslant n}} BV_{i_1} \ldots V_{i_r}$$

and let $\mathscr{P}_n(\mathscr{C}; B)$ denote the closure of $\mathscr{P}_n^a(\mathscr{C}; B)$ in probability, i.e., in $L_0(\Omega, \mathscr{A}, P; B)$. The vector space $\mathscr{P}_n(\mathscr{C}; B)$ is called the *n-th B-valued polynomial chaos* generated by $\mathscr{C}$.

In the subsequent sections the quantity

$$M_q(\mathscr{C}) = \sup\{\|\xi\|_q;\ \|\xi\|_2 \leqslant 1,\ \xi \in \bigcup_0^\infty V_i\}, \qquad 2 < q \leqslant \infty,$$

will play an important role. To avoid unnecessary repetitions, we will from now on always assume that

(δ) $2 < p < \infty$ is fixed and $M_p(\mathscr{C}) < \infty$.

The main aim of this paper is to determine sufficient conditions on $\mathscr{C}$ and B to ensure that

$$\mathscr{P}_n(\mathscr{C}; B) \subseteq L_p(\Omega, \mathscr{A}, P; B)$$

and, in that case, to estimate the finite quantity

$$\sup\{\|X\|_p; \|X\|_2 \leqslant 1, X \in \mathscr{P}_n(\mathscr{C}; B)\}.$$

As an example of an application of the L_q-estimates obtained for $\mathscr{P}_n(\mathscr{C}; \boldsymbol{R})$, we will discuss L_p-convergence of certain martingales in $\mathscr{P}_n(\mathscr{C}; C(T))$ arising as partial sums of formally given random series.

L_q-estimates for polynomial chaos have been studied, among others, by Bonami ([1], p. 366), Schreiber [12], Segal ([13], Corollary 1.1), and the author ([2], Theorem 4.1, and [3]). In the linear case, that is for $n = 1$, the list of contributors is, of course, enormously extensive.

The papers [1], [13], and [3] are all based on the idea of hypercontractivity. In the sequel we will show that this, indeed, is a general principle which applies to all polynomial chaos with $B = \boldsymbol{R}$ or even with $B = H$, where H is a Hilbert space. Moreover, for an arbitrary Banach space B the same technique works if $M_\chi(\mathscr{C}) < \infty$, a condition which does not seem possible to improve very much in general.

2. Polynomial chaos and coordinates. The purpose of the present section is to introduce some notation and to give a fragmentary discussion of series expansions of random vectors belonging to a fixed polynomial chaos. The latter subject is further developed in Section 5.

Set

$$\langle \nu \rangle = \mathrm{card}\{i \in N; \nu(i) \neq 0\}, \quad \nu \in \{0, 1, \ldots, d\}^N,$$
$$N_{dn} = \{\nu \in \{0, 1, \ldots, d\}^N; \langle \nu \rangle \leqslant n\},$$

and

$$N_{dn}(m) = \{\nu \in N_{dn}; \nu(i) = 0, i > m\}, \quad m \in N.$$

Moreover, for fixed $i \in N$, let $\xi_{i1}, \ldots, \xi_{id}$ be an orthonormal basis for V_i, let $\xi_{i0} = 1$, and introduce the orthonormal family

$$\xi^\nu = \prod_0^\infty \xi_{i\nu(i)}, \quad \nu \in N_{dn}.$$

Note that every $X \in \mathscr{P}_n^a(\mathscr{C}; B)$ may be written in the form

$$X = \sum_{N_{dn}(m)} \mathrm{E}[X\xi^\nu]\xi^\nu$$

for a suitable $m \in N$.

We will frequently use the notation $L_p(B) = L_p(\Omega, \mathscr{A}, P; B)$.

THEOREM 2.1. *If*

$$\mathscr{P}_n(\mathscr{C}; B) \subseteq L_p(B), \tag{2.1}$$

then, for any $X \in \mathscr{P}_n(\mathscr{C}; B)$,

$$X = \lim_{m\to\infty} \sum_{N_{dn}(m)} \mathrm{E}[X\xi^\nu]\,\xi^\nu \quad \textit{a.s. and in } L_p(B).$$

Theorem 2.1 follows at once from the martingale theory and the next lemma, which will be useful several times below. We postpone the proof of Theorem 2.1 to Section 5, where a more illuminating result is given.

LEMMA 2.1. *The following assertions are equivalent*:

(i) (2.1) *is true*;

(ii) $L_q(B)$ *induces the same topology on* $\mathscr{P}_n(\mathscr{C}; B)$ *for each* q $(0 \leqslant q \leqslant p)$;

(iii) *there exists a constant* $A \in \mathbf{R}_+$ *such that*

$$\|X\|_p \leqslant A\,\|X\|_2, \quad X \in \mathscr{P}_n^a(\mathscr{C}; B).$$

Clearly, the proof of Lemma 2.1 is very simple, so we just state some informative remarks. First note that the canonical translation-invariant $L_0(B)$-metric is given by

$$\|X\|_0 = \mathrm{E}\,[\|X\|/(1+\|X\|)],$$

and hence $L_0(B)$ is a Fréchet space. The implication (i)$\Rightarrow$(ii) now follows from the closed graph theorem. Further, the implication (ii)$\Rightarrow$(iii) is trivial and the remaining implication (iii)$\Rightarrow$(i) is a consequence of the Hölder and Minkowski inequalities ([11], Lemme 1.1).

3. Hilbert space valued polynomial chaos. Throughout this section H stands for an arbitrary Hilbert space.

THEOREM 3.1. $\mathscr{P}_n(H) \subseteq L_p(H)$. *In addition, there exists a finite constant* A, *which depends only on* (d, n), *such that, for any* $X \in \mathscr{P}_n(\mathscr{C}; H)$,

$$\|X\|_p \leqslant A p^{n/2} M(X)\,\|X\|_2, \tag{3.1}$$

where

$$M(X) = \sup\{\|\xi^\nu\|_p;\ \mathrm{E}[X\xi^\nu] \neq 0,\ \nu \in N_{dn}\}.$$

Granting the validity of Theorem 3.1 with H replaced by the scalar field of H it is simple to show (3.1) with n replaced by $n+1$ for any H. However, Theorem 3.1 seems to give some new information on scalar valued polynomial chaos as well.

We recall that a random vector is said to be *symmetric* if its distribution law is invariant under reflection with respect to the origin. Below we use the notation

$$|\nu| = \sum_0^\infty \nu(i), \quad \nu \in N_{dn}.$$

THEOREM 3.2. *Let θ_k be a positive root of the equation*

$$4^k\theta^2 \frac{1-\theta^{2d}}{1-\theta^2} = \frac{1}{p-1} \quad (k = 0, 1).$$

Then, for each $X \in \mathscr{P}_n(\mathscr{C}; H)$,

$$\|X\|_p^2 \leqslant \sum_{N_{dn}} \theta_1^{-2|\nu|} \|\xi^\nu\|_p^2 \|\mathrm{E}[X\xi^\nu]\|^2$$

and, moreover, if any $\xi \in \bigcup_0^\infty V_i$ is symmetric, then, in fact,

$$\|X\|_p^2 \leqslant \sum_{N_{dn}} \theta_0^{-2|\nu|} \|\xi^\nu\|_p^2 \|\mathrm{E}[X\xi^\nu]\|^2.$$

During the past decade there has been an intensive search for hypercontractive maps (i.e. norm $\leqslant 1$) between special L_q-spaces and, as is well known, this work has resulted in a series of important articles (see, e.g., [14], [9], and the papers referred to therein). If $\xi_0 = 1, \xi_1, \xi_2, \ldots$ is an orthonormal basis for $L_2(\boldsymbol{R})$, then Theorem 3.2 shows, in particular, that the Fourier multipliers

$$\left(\frac{1}{\|\xi_j\|_p (4p-3)^{j/2}}\right)_{j\in N}$$

induce a hypercontractive map from $L_2(\boldsymbol{R})$ into $L_p(\boldsymbol{R})$. Note here that essentially no restrictions are made on the distribution of the random vector $(\xi_j)_{j\in N}$. Indeed, the idea of hypercontractivity is a general probabilistic principle.

The proofs of Theorems 3.1 and 3.2 are based on several lemmas.

LEMMA 3.1. *Let $K \in L_2(\Omega \times \Omega, \mathscr{A} \otimes \mathscr{A}, P \otimes P; \boldsymbol{R})$ and set*

$$N_p(K; B) = \sup\left\{\left\|\int K(\cdot, \tau) X(\tau)\, dP(\tau)\right\|_p;\ \|X\|_2 \leqslant 1,\ X \in L_2(\Omega, \mathscr{A}, P; B)\right\}.$$

If $K \geqslant 0$ and $N_p(K; \boldsymbol{R}) \leqslant 1$, then $N_p(K; B) \leqslant 1$.

Lemma 3.1 follows at once from the Jensen inequality and needs no proof. It should be emphasized that Lemma 3.1 does not extend to signed kernels.

LEMMA 3.2. *For any $a, b \in H$,*

$$\tfrac{1}{2}[\|a-b\|^p + \|a+b\|^p] \leqslant [\|a\|^2 + (p-1)\|b\|^2]^{p/2}.$$

We assume familiarity with Lemma 3.2 for $H = \boldsymbol{R}$ (see [1], p. 378, [5], Theorem 3, and [10], p. 75).

Proof. Consider the probability space

$$(\Omega, \mathscr{A}, P) = (\{-1, 1\}, 2^{\{-1,1\}}, (\delta_{-1} + \delta_{+1})/2)$$

and introduce the kernel

$$K(\omega, \tau) = 1+(p-1)^{-1/2}\omega\tau, \qquad \omega, \tau\in\Omega.$$

Since $K\geqslant 0$ and $N_p(K; R)\leqslant 1$, Lemma 3.1 gives $N_p(K; H)\leqslant 1$. However, if $a, b\in H$, then

$$\int K(\omega, \tau)\big(a+(p-1)^{1/2}b\tau\big)dP(\tau) = a+b\omega, \qquad \omega\in\Omega,$$

and Lemma 3.2 follows at once.

LEMMA 3.3. *Let $1, \xi_1, \ldots, \xi_d\in L_p(\Omega, \mathscr{A}; R)$ be an orthonormal sequence in $L_2(R)$, suppose $\lambda = (\lambda_1, \ldots, \lambda_d)\in R^d$, and set*

$$K_\lambda = 1+\sum_1^d \lambda_j(\xi_j\otimes\xi_j).$$

Then

$$N_p(K_\lambda; H)\leqslant 1 \quad \text{if } 4(p-1)\sum_1^d \lambda_j^2\,\|\xi_j\|_p^2\leqslant 1. \tag{3.2}$$

If the random vector $(\xi_j)_1^d$ is symmetric, then

$$N_p(K_\lambda; H)\leqslant 1 \quad \text{if } (p-1)\sum_1^d \lambda_j^2\,\|\xi_j\|_p^2\leqslant 1. \tag{3.3}$$

Proof. We first prove (3.2). To this end let $(\xi'_j)_1^d$ be a stochastically independent copy of the random vector $(\xi_j)_1^d$ and set $\eta_j = \xi_j-\xi'_j$, $j = 1, \ldots, d$, and

$$K'_\lambda(\omega, \tau) = 1+\sum_1^d \lambda_j\eta_j(\omega)\xi_j(\tau), \qquad \omega, \tau\in\Omega.$$

Furthermore, suppose $X\in L_2(\Omega, \mathscr{A}, P; H)$ is arbitrary and define

$$a = \mathrm{E}[X], \qquad Y = X-a, \qquad Z_\lambda = \int K'_\lambda(\cdot, \tau)\, Y(\tau)\,dP(\tau).$$

Note that $Z_\lambda = \sum_1^d \lambda_j\eta_j\mathrm{E}[Y\xi_j]$, and hence

$$\|Z_\lambda\|_p\leqslant \Big(4\sum_1^d \lambda_j^2\,\|\xi_j\|_p^2\Big)^{1/2}\|Y\|_2 \tag{3.4}$$

by the Cauchy-Schwarz and Parseval inequalities.

Now we use Lemma 3.2 to get

$$\tfrac{1}{2}[\|a-Z_\lambda\|^p+\|a+Z_\lambda\|^p]\leqslant [\|a\|^2+(p-1)\|Z_\lambda\|^2]^{p/2}.$$

Taking the expectation and remembering the symmetry of Z_λ, we obtain

$$\|a+Z_\lambda\|_p^p\leqslant \big\|\|a\|^2+(p-1)\|Z_\lambda\|^2\big\|_{p/2}^{p/2},$$

and the Minkowski inequality yields immediately

$$\|a+Z_\lambda\|_p^2 \leqslant \|a\|^2+(p-1)\|Z_\lambda\|_p^2.$$

Combining this and (3.4) we have

$$\|a+Z_\lambda\|_p \leqslant \|X\|_2, \quad 4(p-1)\sum_1^d \lambda_j^2\|\xi_j\|_p^2 \leqslant 1.$$

Finally, for any $\lambda\in R^d$, we get

$$\mathrm{E}[a+Z_\lambda \mid \xi_1, \ldots, \xi_d] = \int K_\lambda(\cdot, \tau)X(\tau)\,dP(\tau),$$

and the Jensen inequality for conditional expectation gives

$$\left\|\int K_\lambda(\cdot, \tau)X(\tau)\,dP(\tau)\right\|_p \leqslant \|a+Z_\lambda\|_p.$$

Summing up, we have thus proved (3.2).

If the random vector $(\xi_j)_1^d$ is symmetric, the above line of proof applies directly to the kernel K_λ and (3.3) follows in a similar way.

This completes the proof of Lemma 3.3.

LEMMA 3.4. *Let $(\Omega_i, \mathscr{A}_i, P_i)$, $i = 0, \ldots, m$, be probability spaces and assume*

$$K_i\in L_2(\Omega_i\times\Omega_i, \mathscr{A}_i\otimes\mathscr{A}_i, P_i\otimes P_i), \quad i = 0, \ldots, m.$$

If $N_p(K_i; B)\leqslant 1$, $i = 0, \ldots, m$, then $N_p(K_0\otimes\ldots\otimes K_m; B)\leqslant 1$.

Lemma 3.4 is a direct consequence of the Minkowski inequality as in the special case $B = C$ (see [1], p. 375, and [13], Lemma 1.4). The argument will not be repeated here.

Proof of Theorem 3.1. To prove (3.1) we assume that

$$X = \sum_{N_{dn}(m)} a_\nu \xi^\nu$$

for appropriate $m\in N$ and $a_\nu\in H$, $\nu\in N_{dn}(m)$, depending on Lemma 2.1. Moreover, since the σ-algebras $\mathscr{A}_i = \sigma(\xi; \xi\in V_i)$, $i = 0, \ldots, m$, are stochastically independent and X is $\left(\bigotimes_0^m \mathscr{A}_i, \mathscr{B}(H)\right)$-measurable, there is no loss of generality in assuming the underlying probability space to be of the form

$$(\Omega, \mathscr{A}, P) = \left(\mathop{\mathrm{X}}_0^m \Omega_i, \bigotimes_0^m \mathscr{A}_i, \bigotimes_0^m P_i\right).$$

Accordingly, for each fixed $i\in\{0, \ldots, m\}$, the random vector $(\xi_{ij})_{j=1}^d$ depends only on the i-th coordinate ω_i of $\omega\in\Omega$. Introducing $\theta = (4d(p-1))^{-1/2}$ and

$$K_i(\omega_i, \tau_i) = 1+\theta\sum_{j=1}^d [\xi_{ij}(\omega_i)\xi_{ij}(\tau_i)/\|\xi_{ij}\|_p], \quad \omega_i, \tau_i\in\Omega_i,$$

for $i = 0, \ldots, m$, we have

$$\int (K_0 \otimes \ldots \otimes K_m)(\cdot, \tau) X(\tau) dP(\tau) = \sum_{N_{dn}(m)} [\theta^{\langle v \rangle} a_v \xi^v / \|\xi^v\|_p].$$

Now, applying Lemmas 3.3 and 3.4, we obtain

$$\Big\| \sum_{N_{dn}(m)} [\theta^{\langle v \rangle} a_v \xi^v / \|\xi^v\|_p] \Big\|_p \leqslant \|X\|_2,$$

and hence

$$\|X\|_p^2 \leqslant \sum_{N_{dn}(m)} \theta^{-2\langle v \rangle} \|\xi^v\|_p^2 \|a_v\|^2,$$

which gives us immediately (3.1). This completes the proof of Theorem 3.1.

Proof of Theorem. 3.2. Setting

$$K_i(\omega_i, \tau_i) = 1 + \sum_{j=1}^{d} [\theta_1^j \xi_{ij}(\omega_i) \xi_{ij}(\tau_i) / \|\xi_{ij}\|_p], \quad \omega_i, \tau_i \in \Omega_i, \tag{3.5}$$

for $i = 0, \ldots, m$, and arguing as in the proof of Theorem 3.1, we prove at once the first part of Theorem 3.2. The remaining part of Theorem 3.2 may be proved in a similar way if we replace θ_1 in (3.5) by θ_0.

4. Banach space valued polynomial chaos. As one might expect, there is no hope of a pure extension of Theorem 3.1 to Banach space valued polynomial chaos. Indeed, the supremum of a real-valued stochastic process may have very bad integrability properties under fairly restrictive conditions.

THEOREM 4.1. *Suppose $M_\infty(\mathscr{C}) < \infty$. Then $\mathscr{P}_n(\mathscr{C}; B) \subseteq L_p(B)$. More precisely, there exists a finite constant A, which depends only on (d, n), such that*

$$\|X\|_p \leqslant A p^{n/2} M_\infty^{2n}(\mathscr{C}) \|X\|_2, \quad X \in \mathscr{P}_n(\mathscr{C}; B). \tag{4.1}$$

In addition, there exists a constant $\alpha > 0$, depending only on $(d, n, M_\infty(\mathscr{C}))$, such that

$$\sup\{E[\exp(\alpha \|X\|^{2/n})]; \|X\|_2 \leqslant 1, X \in \mathscr{P}_n(\mathscr{C}; B)\} \leqslant 2. \tag{4.2}$$

In particular, $[\exp(\|X\|^{2/n})] \in L_1(R)$ for each $X \in \mathscr{P}_n(\mathscr{C}; B)$.

Proof. Suppose

$$X = \sum_{N_{dn}(m)} a_v \xi^v,$$

where $m \in N$ and $a_v \in B$, $v \in N_{dn}(m)$. Arguing as in the proof of Theorem 3.1, we set, for any fixed $i \in \{0, \ldots, m\}$,

$$K_i^\theta(\omega_i, \tau_i) = 1 + \theta \sum_{j=1}^{d} \xi_{ij}(\omega_i) \xi_{ij}(\tau_i), \quad \omega_i, \tau_i \in \Omega_i, \ \theta \in R.$$

By Lemma 3.3, $N_p(K_i^\theta; R) \leqslant 1$ if $|\theta| \leqslant (4d(p-1) M_\infty^2(\mathscr{C}))^{-1/2}$, and, obviously, $K_i^\theta \geqslant 0$ if $|\theta| \leqslant (dM_\infty^2(\mathscr{C}))^{-1}$. Now we use Lemmas 3.1 and 3.4 and conclude that

$$\Big\| \sum_{N_{dn}(m)} \theta^{\langle v \rangle} a_v \xi^v \Big\|_p \leqslant \|X\|_2, \quad |\theta| \leqslant \big(2d(p-1)^{1/2} M_\infty^2(\mathscr{C})\big)^{-1}.$$

However, if $b_0, \ldots, b_n$ belong to a Banach space $(C, |||\cdot|||)$ and

$$\Big|\Big|\Big|\sum_0^n \theta^k b_k\Big|\Big|\Big| \leqslant 1, \quad -\varepsilon \leqslant \theta \leqslant \varepsilon,$$

for a fixed ε $(0 < \varepsilon < 1)$, then, by the Hahn-Banach theorem, there exists a finite constant A_n, depending only on n, such that

$$\Big|\Big|\Big|\sum_0^n b_k\Big|\Big|\Big| \leqslant A_n \varepsilon^{-n}.$$

Since $L_p(B)$ is a Banach space, X necessarily satisfies (4.1).

Suppose now that $X \in \mathscr{P}_n(\mathscr{C}; B)$ and $\|X\|_2 \leqslant 1$. Clearly, for any $\alpha > 0$,

$$\mathrm{E}\left[\exp(\alpha\|X\|^{2/n})\right] = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \|X\|_{(2k)/n}^{(2k)/n}$$

and, consequently, by (4.1) the left-hand side does not exceed

$$\sum_{k=0}^{n} \frac{\alpha^k}{k!} + \sum_{k=n+1}^{\infty} \left[\frac{2\alpha A^{2/n} M_\infty^4(\mathscr{C})}{n}\right]^k \frac{k^k}{k!}.$$

This proves (4.2).

To settle the last part of Theorem 4.1, suppose $X \in \mathscr{P}_n(\mathscr{C}; B)$ and choose $Y \in \mathscr{P}_n^a(\mathscr{C}; B)$ so that $\|X - Y\|_2 \leqslant \alpha^{n/2}/2^{1+n/2}$, where $\alpha > 0$ is as in (4.2). Since

$$\|X\|^{2/n} \leqslant 2^{2/n}\left[\|X - Y\|^{2/n} + \|Y\|^{2/n}\right],$$

we now have $\mathrm{E}\left[\exp(\|X\|^{2/n})\right] < \infty$ by the Cauchy-Schwarz inequality, which completes the proof of Theorem 4.1.

The assumptions on $\mathscr{C}$ in Theorems 3.1 and 3.2 are very different. To throw some light on the underlying reason for this, we will construct a class $\mathscr{C} = (V_i)_{i \in N}$ of one-dimensional subspaces of $L_2(\boldsymbol{R})$ having the following properties:

(i) (α)-(δ) in Section 1 are fulfilled;

(ii) any random vector belonging to $\bigcup_0^\infty V_i$ is symmetric;

(iii) $\bigcup_0^\infty V_i \subseteq L_\infty(\boldsymbol{R})$;

(iv) $M_q(\mathscr{C}) < \infty$ for all $q < \infty$;

(v) $\mathscr{P}_1(\mathscr{C}; c_0) \setminus \left[\bigcup_{q>0} L_q(c_0)\right] \neq \emptyset$, where c_0 denotes the Banach space $(c_0(\boldsymbol{N}), \|\cdot\|_\infty)$.

The construction below is a modification of an example given by Jain and Marcus ([8], Example 4.3). It depends on the next lemma which follows from the convexity of the exponential function:

LEMMA 4.1 ([8], p. 5). *Let ζ_i, $i \in N$, be non-negative stochastically independent random variables and suppose* $(\sup_{i \in N} \zeta_i) \in L_1(R)$. *Then*

$$\sum_0^\infty \int_{z_0}^\infty P[\zeta_i \geqslant z]\, dz < \infty$$

for an appropriate z_0 $(0 < z_0 < \infty)$.

To simplify the notation, set $r_k = 2^{k(k+1)/2}$, $s_k = k^{1/2}$, and $x_k = 2^{-k/2} r_k$, $k \in N$. Assume ξ is a symmetric real-valued random variable with

$$P[|\xi| \geqslant x] = ((k+1) r_k)^{-s_k}, \quad x_k < x \leqslant x_{k+1},$$

and, for each fixed $k \in N$, choose $y_k > 0$ so that

$$P[2^{k/2} r_k \leqslant |\xi| \leqslant y_k] \geqslant \tfrac{1}{2} P[2^{k/2} r_k \leqslant |\xi|].$$

Moreover, let ξ_i, $i \in N$, be stochastically independent observations on ξ and introduce $i_k = r_0^{s_0} + \ldots + r_k^{s_k}$ and

$$\lambda_i = r_k^{-1}, \quad \eta_i = \xi_i 1_{[|\xi_i| \leqslant y_k]}, \quad i_{k-1} < i \leqslant i_k.$$

In the following $(e_i)_{i \in N}$ denotes the standard basis for c_0. We claim that the series

$$\sum_2^\infty (\lambda_i \eta_i) e_i \tag{4.3}$$

converges a.s. in c_0, i.e.,

$$\sum_2^\infty P[\lambda_i |\eta_i| \geqslant \varepsilon] < \infty \quad \text{for all } \varepsilon\ (0 < \varepsilon < 1).$$

However, for each fixed ε $(0 < \varepsilon < 1)$ we have $x_k < \varepsilon r_k \leqslant x_{k+1}$, where k is large, so the above claim is a direct consequence of the estimate

$$\sum_{i_{k-1}+1}^{i_k} P[\lambda_i |\eta_i| \geqslant \varepsilon] \leqslant r_k^{s_k} P[|\xi| \geqslant \varepsilon r_k].$$

Let X denote the sum of the series in (4.3). We shall prove that $X \notin L_q(c_0)$ for all $q > 0$, so, in view of Lemma 4.1, it is enough to show

$$\sum_2^\infty \int_z^\infty y^{q-1} P[\lambda_i |\eta_i| \geqslant y]\, dy = \infty \quad \text{for all } z, q\ (0 < z, q < \infty). \tag{4.4}$$

To this end, first note that for fixed z, q $(0 < z, q < \infty)$, the sum does not fall below

$$r_k^{s_k} \int_z^{2^{k/2}} y^{q-1} P[yr_k \leqslant |\xi| \leqslant y_k]\, dy$$

$$\geqslant r_k^{s_k} P[2^{k/2} r_k \leqslant |\xi|] (2^{(kq)/2} - z^q)/2q, \quad k \geqslant 2(\ln z)/(\ln 2).$$

Since $x_k < 2^{k/2} r_k \leqslant x_{k+1}$, $k \in N_+$, the sum of the series in (4.4) dominates

$$\lim_{k\to\infty} (k+1)^{-s_k} (2^{(kq)/2} - z^q)/2q = \infty,$$

and hence $X \notin L_q(c_0)$ for all $q > 0$.

Finally, setting $V_i = \operatorname{span}\{\eta_{i+2}\}$, $i \in N$, and observing that

$$\xi \in \bigcap_{q<\infty} L_q(R),$$

we obtain at once the statements in (i)-(v).

5. Series expansions and polynomial chaos. Throughout this section T stands for a compact metric space and $C(T)$ denotes the Banach space of all real-valued continuous functions on T.

Suppose $a_\nu \in C(T)$, $\nu \in N_{dn}$, and set

$$X_m = \sum_{N_{dn}(m)} a_\nu \xi^\nu, \quad m \in N. \tag{5.1}$$

Clearly, $(X_m)_{m\in N}$ constitutes a martingale. Since $L_p(C(T))$-convergence of a martingale implies a.s. convergence, we concentrate below on the first type of convergence.

THEOREM 5.1. *Suppose $X \in L_p(C(T))$ and $X(t) \in \mathscr{P}_n(\mathscr{C}; R)$, $t \in T$. Then*

$$X = \lim_{m\to\infty} \sum_{N_{dn}(m)} \mathrm{E}[X\xi^\nu]\xi^\nu \text{ in } L_p(C(T)).$$

In particular, $X \in \mathscr{P}_n(C(T))$.

Proof. Obviously,

$$X(t) = \sum_{N_{dn}} \mathrm{E}[X(t)\xi^\nu]\xi^\nu \text{ in } L_2(R) \tag{5.2}$$

for each fixed $t \in T$ by Theorem 3.1 and Lemma 2.1. Now we set

$$\mathscr{B}_m = \sigma\big(\xi;\, \xi \in \bigcup_0^m V_i\big), \quad m \in N,$$

and conclude that X is measurable if Ω is equipped with the P-completion of the σ-algebra $\sigma\big(\bigcup_0^\infty \mathscr{B}_m\big)$. Using the martingale theory it therefore suffices to show that

$$\mathrm{E}[X \mid \mathscr{B}_m] = \sum_{N_{dn}(m)} \mathrm{E}[X\xi^\nu]\xi^\nu, \quad m \in N,$$

which, however, is a consequence of (5.2). This completes the proof of Theorem 5.1.

Now we turn to the more subtle problem of giving sufficient conditions to ensure $L_p(C(T))$-convergence of the martingale in (5.1). Our conditions are given in terms of so-called majorizing measures, originally introduced by Fernique [4] for Gaussian stochastic processes.

THEOREM 5.2. *Let* $a_\nu \in C(T)$, $\nu \in N_{dn}$, *and suppose the series* $\sum_{N_{dn}} a_\nu^2$ *converges in* $C(T)$. *Set*

$$\varrho(s, t) = \Big(\sum_{N_{dn}} (a_\nu(s) - a_\nu(t))^2\Big)^{1/2}, \quad s, t \in T,$$

and

$$B(t; r) = \{s \in T;\ \varrho(s, t) < r\}, \quad t \in T,\ r \geq 0.$$

(a) *Assume there exists a Radon probability measure* μ *on* (T, ϱ) *such that*

$$\lim_{\varepsilon \downarrow 0} \sup_{u \in T} \int_0^\varepsilon [\mu(B(u; r))]^{-2/p}\, dr = 0.$$

Then

$$\text{(5.3)} \quad \sup_{m \in N} |X_m(s) - X_m(t)| \leq \gamma \sup_{u \in T} \int_0^{\varrho(s,t)/2} [\mu(B(u; r))]^{-2/p}\, dr, \quad s, t \in T,$$

for an appropriate $\gamma \in L_p(\mathbf{R})$. *In particular,* $(X_m)_{m \in N}$ *converges in* $L_p(C(T))$.

(b) *Suppose that* $M_\infty(\mathscr{C}) < \infty$ *and that there exists a Radon probability measure* μ *on* (T, ϱ) *satisfying*

$$\text{(5.4)} \quad \lim_{\varepsilon \downarrow 0} \sup_{u \in T} \int_0^\varepsilon \big[\ln\big(1/\mu(B(u; r))\big)\big]^{n/2}\, dr = 0.$$

Set $y_1 = 0$, $y_n = [(n-2)/\alpha]^{n/2}$, $n \geq 2$, *where* $\alpha > 0$ *is as in* (4.2). *Then*

$$\sup_{m \in N} |X_m(s) - X_m(t)| \leq 20 \sup_{u \in T} \int_0^{\varrho(s,t)/2} \left(\left[\frac{2}{\alpha} \ln \frac{\delta}{\mu^2(B(u; r))}\right]^{n/2} - y_n\right) dr, \quad s, t \in T,$$

for a suitable $\delta \in L_2(\mathbf{R})$. *In particular,* $(X_m)_{m \in N}$ *converges in each* $L_q(C(T))$, $0 < q < \infty$.

A minor variation of Theorem 5.2 (b) is proved by Fernique ([4], Théorème 6.2.1 and Corollaire 6.2.3) for Gaussian stochastic processes and $n = 1$ in (5.4). Further, Heinkel ([6], Théorème 4) shows that condition (5.4) with $n = 1$ is still sufficient for sample continuity of subgaussian stochastic processes.

Proof. The first part of the proof depends on a very ingenious construction of Heinkel ([6], Proposition 1, and [7], p. 205) based on the Jensen inequality.

Suppose φ: $[0, \infty[\to [0, \infty[$ is a strictly increasing convex function and let μ be a Radon probability measure on (T, ϱ) with topological support T. Setting

$$\theta_m = \int \varphi\left(\frac{|X_m(s)-X_m(t)|}{\varrho(s, t)} 1_{\{\varrho \neq 0\}}(s, t)\right) d(\mu \otimes \mu)$$

and using the ϱ-continuity of $X_m(\cdot, \omega)$ for each fixed $(m, \omega) \in N \times \Omega$, we infer from the work of Heinkel that

$$|X_m(s)-X_m(t)| \leqslant 20 \sup_{u \in T} \int_0^{\varrho(s,t)/2} \varphi^{-1}\left(\frac{\theta_m}{\mu^2(B(u;r))}\right) dr, \qquad s, t \in T.$$

(a) We apply the above inequality to the function $\varphi(x) = x^p$, $x \geqslant 0$. To show (5.3) it remains only to prove that $\sup_{m \in N} \theta_m$ is integrable. However, by the standard martingale estimates and Theorem 3.1, the expectation of the random variable

$$\sup_{m \in N} \left[\frac{|X_m(s)-X_m(t)|}{\varrho(s, t)} 1_{\{\varrho \neq 0\}}(s, t)\right]^p$$

does not exceed

$$\left[\frac{p}{p-1}(A p^{n/2} M_p^n(\mathscr{C}))\right]^p,$$

where A is as in (3.1). This proves (5.3).

The assumptions in Theorem 5.1 ensure that ϱ: $T \times T \to \boldsymbol{R}$ is continuous and, by Theorem 3.1, the martingale $(X_m(t))_{m \in N}$ converges in $L_p(\boldsymbol{R})$ for each fixed $t \in T$. These properties together with (5.3) establish immediately the last statement in part (a).

(b) To begin with, we introduce the convex function

$$\varphi(x) = \exp[\alpha(x+y_n)^{2/n}/2], \qquad x \geqslant 0.$$

The submartingale estimates and Theorem 4.1 now promise that the expectation of the random variable

$$\sup_{m \in N} \varphi^2\left(\frac{|X_m(s)-X_m(t)|}{\varrho(s, t)} 1_{\{\varrho \neq 0\}}(s, t)\right)$$

does not exceed $8\exp(\alpha y_n^{2/n})$ and part (b) follows in the same way as part (a).

This completes the proof of Theorem 5.1.

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