PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 30, Fasc. 2 (2010), pp. 223–245

COMPUTING VaR AND AVaR IN INFINITELY DIVISIBLE DISTRIBUTIONS*

BY

YOUNG SHIN KIM (KARLSRUHE), SVETLOZAR T. RACHEV** (KARLSRUHE),
MICHELE LEONARDO BIANCHI*** (ROME) AND FRANK J. FABOZZI (NEW HAVEN)

Abstract. In this paper we derive closed-form solutions for the cumulative distribution function and the average value-at-risk for five subclasses of the infinitely divisible distributions: classical tempered stable distribution, Kim–Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. We present empirical evidence using the daily performance of the S&P 500 for the period January 2, 1997 through December 29, 2006.

2000 AMS Mathematics Subject Classification: 60E07, 91B28.

Key words and phrases: Tempered stable distribution, infinitely divisible distribution, VaR, CVaR, AVaR.

1. INTRODUCTION

In finance, numerous studies of return and price distributions of different asset classes and national financial markets reject the notion that the distributions are normal. The most popular alternative to the normal distribution is the class α-stable and tempered stable distributions. Although the α-stable distribution does not have finite moments, generally, tempered stable distributions have finite moments for all orders and finite exponential moments. Moreover, tempered stable distributions include non-Gaussian α-stable distributions as the limiting case. For this reason, tempered stable distributions have been preferred to the normal and used as extension of α-stable distributions for modeling the distribution of asset returns.

---

* The authors are grateful to Hyung Keun Koo for his helpful comments and suggestions.

** Rachev gratefully acknowledges research support by grants from the Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst.

*** The views expressed in this paper are those of the author and should not be attributed to the institution to which he belongs.

There is ample empirical evidence that daily asset returns are skewed and leptokurtic. These well-documented findings reported for asset returns are not mere academic conclusions that hold little interest for practitioners. Rather, they have important implications for asset managers and risk managers. Not properly accounting for these stylized facts can result in models that result in inferior investment performance by asset managers and disastrous financial consequences for financial institutions that rely upon them for risk management. More specifically, a thorough understanding of the tail loss distribution for a portfolio or trading position is critical for the design of stress tests. The failure of stress tests in identifying potential losses has been identified by several researchers as the cause of the failure of risk management systems to identify the losses suffered by the major dealers in the subprime mortgage market in 2007−2008. Although the risk management systems of these financial entities were structured such that they were compatible with what was thought to be the historical performance of subprime mortgage returns, they proved to be inadequate because of their failure to focus on the distribution in the tails. Better modeling of asset return distributions is an essential component of stress testing and should be considered in bank stress tests that are currently being formulated by bank regulators.

It is important to mention that random variables with tempered heavy tails, which are still infinitely divisible distributed, retain many of the properties of random variables with usual heavy tails such as α-stable random variable (see Grabchak and Samorodnitsky [10], Klebanov et al. [17], and Rachev and Mittnik [20]).

In particular, risk calculations with tempered heavy tails will have much in common with risk calculations with the usual heavy tails, although they are not identical. In particular, the density functions of a tempered stable random variable and an α-stable random variable are comparable in the center, even if the tail behavior is slightly different. Furthermore, at the level of processes, if the time scale increases, a tempered stable process converges to a Brownian motion, while if the time scale decreases, it converges to an α-stable one. This property seems to be common for financial asset return processes. For this reason, in the class of infinitely divisible distributions we select distributions that belong to the tempered stable family.

The value-at-risk (VaR) measure has been adopted as a standard risk measure in the financial industry. Nevertheless, it has a number of well-known limitations as a risk measure. For example, it does not satisfy the subadditivity property, and hence VaR is not a coherent risk measure.[1]

The average value-at-risk (AVaR) is the average of VaRs larger than the VaR for a given tail probability.[2] AVaR is a superior alternative to VaR because it sat-

[1]The notion of a coherent risk measure was introduced by Artzner et al. [2].

[2]AVaR is also known as conditional value-at-risk (CVaR). See Pflug [19] and Rockafellar and Uryasev [22], [23].

isfies all axioms of coherient risk measures and it is consistent with preference relations of risk-averse investors (see Rachev et al. [21]).[3] Moreover, AVaR is still a coherent measure, while VaR is not. Consequently, in dealing with risk management and portfolio optimization problems, it is important to compute AVaR accurately for non-normal distributions. The closed-form solutions for AVaR for the α-stable distribution and on the skewed-t distribution have been presented by Stoyanov et al. [24] and Dokov et al. [9], respectively. Explicit formulas for VaR and AVaR are of great importance in operational risk assessment because of the need to calculate these risk measures at the extreme tail when the use of Monte Carlo methods is impractical.

In this paper, we develop a closed-form solution for the calculation of VaR and AVaR on some subclass of infinitely divisible distributions. We apply this formula to five classes of tempered stable distributions, which are a parametric subclass of infinitely divisible distributions. The remainder of this paper is organized as follows. The integral representation of the cumulative distribution function and AVaR are presented in Section 2. Section 3 discusses the computational issues. Section 4 reviews the five classes of tempered stable distributions and applies the formula for VaR and AVaR to each class. The empirical results are reported in Section 5. Section 6 summarizes the principal conclusions of the paper. In the Appendix we give the proofs of our propositions and tables containing formulas for the discussed functions for tempered stable and log tempered stable distributions.

2. VaR AND AVaR ON INFINITELY DIVISIBLE DISTRIBUTIONS

In this section, the random variable X represents the loss of a portfolio, and $F_X(x) = P(X \leqslant x)$, $\bar{F}_X(x) = P(X \geqslant x)$, $f_X(x) = \frac{d}{dx}F_X(x)$, $\phi_X(u) = E[e^{iuX}]$ stand for the cumulative distribution function (CDF), the survival function (SF), the probability density function (PDF), and the characteristic function (Ch.F) of X, respectively. For convenience, in this paper we put $(x)^+ = \max(x, 0)$, and $\Re(z)$ and $\Im(z)$ represent the real part and imaginary part of a complex number z, respectively.

We first investigate an integral representation of F_X.

PROPOSITION 2.1. *Suppose a random variable X is infinitely divisible.*

(i) *If there is $\rho > 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$F_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_X(u+i\rho)}{\rho - ui}du\right) \quad \text{for } x \in \mathbb{R}. \tag{2.1}$$

---

[3] AVaR and another popular risk measure, expected tail loss (ETL), coincide if the loss distribution is continuous at the corresponding VaR level. However, if there is discontinuity, then AVaR and ETL differ.

(ii) *If there is $\rho < 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$\bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_X(u+i\rho)}{ui-\rho}du\right) \quad \text{for } x \in \mathbb{R}. \tag{2.2}$$

The VaR of X at tail probability ε is defined as

$$\begin{aligned}\mathrm{VaR}_\varepsilon(X) &= \inf\{y \in \mathbb{R} : P(X \geqslant y) \leqslant (1-\varepsilon)\}\\ &= \inf\{y \in \mathbb{R} : F_X(y) \geqslant \varepsilon\}.\end{aligned}$$

The AVaR at tail probability ε is defined as the average of the VaRs which are larger than $\mathrm{VaR}_\varepsilon(X)$, that is

$$\mathrm{AVaR}_\varepsilon(X) = \frac{1}{1-\varepsilon}\int_\varepsilon^1 \mathrm{VaR}_t(X)dt. \tag{2.3}$$

If $F_X(x)$ is continuous, then we have

$$\mathrm{VaR}_\varepsilon(X) = F_X^{-1}(\varepsilon) = \bar{F}_X^{-1}(1-\varepsilon) \tag{2.4}$$

and

$$\int_\varepsilon^1 \mathrm{VaR}_t(X)dt = \int_\varepsilon^1 F_X^{-1}(t)dt = \int_{F_X^{-1}(\varepsilon)}^\infty s\,dF_X(s) = E[X1_{\{X \geqslant F_X^{-1}(\varepsilon)\}}].$$

By (2.3), we obtain

$$\begin{aligned}\mathrm{AVaR}_\varepsilon(X) &= \frac{1}{1-\varepsilon}E[X1_{\{X\geqslant \mathrm{VaR}_\varepsilon(X)\}}]\\ &= \frac{1}{1-\varepsilon}E\big[\mathrm{VaR}_\varepsilon(X)1_{\{X\geqslant \mathrm{VaR}_\varepsilon(X)\}} + \big(X-\mathrm{VaR}_\varepsilon(X)\big)^+\big]\\ &= \mathrm{VaR}_\varepsilon(X) + \frac{1}{1-\varepsilon}E\big[\big(X-\mathrm{VaR}_\varepsilon(X)\big)^+\big].\end{aligned} \tag{2.5}$$

Therefore, we obtain the closed-form solution of $\mathrm{AVaR}_\varepsilon(X)$ for the infinitely divisible random variable X.

PROPOSITION 2.2. *Suppose X is infinitely divisible and $F_X(x)$ is continuous. If there is $\rho < 0$ such that $|\phi_X(z)| < \infty$ for all $z \in \mathbb{C}$ with $\Im(z) = \rho$, then*

$$\begin{aligned}&\mathrm{AVaR}_\varepsilon(X) \\ &= \mathrm{VaR}_\varepsilon(X) - \frac{\exp\big(\mathrm{VaR}_\varepsilon(X)\rho\big)}{\pi(1-\varepsilon)}\Re\left(\int_0^\infty \exp\big(-iu\mathrm{VaR}_\varepsilon(X)\big)\frac{\phi_X(u+i\rho)}{(u+i\rho)^2}du\right).\end{aligned} \tag{2.6}$$

In operational risk management, the loss is always positive, and its distribution is right skewed and has a heavy right tail; see Chernobai et al. [7]. For this reason, the log-normal and log-α-stable random variable have been often used to model operational loss. We will derive the closed-form solution of the AVaR for a more general class of distributions, including the log-normal distribution.

Consider a random variable Y such that $\log Y$ is infinitely divisible. Then the random variable Y is referred to as the *log infinitely divisible random variable*. Since a normal random variable is infinitely divisible, the log-normal random variable is also log infinitely divisible. Using Proposition 2.1, we obtain the following corollary.

COROLLARY 2.1. *Assume a random variable Y is log infinitely divisible and $\phi_{\log Y}$ is the Ch.F of $\log Y$.*

(i) *If there is $\rho > 0$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$F_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\log Y}(u+i\rho)}{\rho - ui}du\right), \quad y > 0. \tag{2.7}$$

(ii) *If there is $\rho < 0$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$\bar{F}_Y(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\log Y}(u+i\rho)}{ui - \rho}du\right), \quad y > 0. \tag{2.8}$$

P r o o f. Since

$$\begin{aligned} F_Y(y) = \mathrm{P}(Y \leqslant y) &= \mathrm{P}(\log Y \leqslant \log y) \\ &= F_{\log Y}(\log y), \quad y > 0, \end{aligned}$$

where $F_{\log Y}$ is the CDF of $\log Y$, we can prove (i) and (ii) by substituting $x = \log y$ into (i) and (ii) of Proposition 2.1, respectively. ■

If the CDF $F_Y(y)$ of a log infinitely divisible random variable Y is continuous, then we have

$$\mathrm{VaR}_\varepsilon(Y) = F_Y^{-1}(\varepsilon) = \bar{F}_Y^{-1}(1-\varepsilon). \tag{2.9}$$

A closed-form solution of $\mathrm{AVaR}_\varepsilon(Y)$ for the log infinitely divisible random variable Y is obtained as follows.

PROPOSITION 2.3. *Let Y be a log infinitely divisible random variable, and F_Y and $\phi_{\log Y}$ be the CDF of Y and the Ch.F of $\log Y$, respectively. If $F_Y(x)$ is continuous for $x > 0$ and there is $\rho < -1$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$\begin{aligned}(2.10)\quad &\mathrm{AVaR}_\varepsilon(Y)\\ &= \mathrm{VaR}_\varepsilon(Y) - \frac{\big(\mathrm{VaR}_\varepsilon(Y)\big)^{1+\rho}}{\pi(1-\varepsilon)}\Re\Bigg(\int_0^\infty \frac{\big(\mathrm{VaR}_\varepsilon(Y)\big)^{-iu}\phi_{\log Y}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\Bigg).\end{aligned}$$

3. COMPUTATIONAL ISSUES

According to Proposition 2.1, the CDF and the SF of an infinitely divisible random variable X are equal to

$$F_X(x) = \frac{e^{x\rho}}{\pi}\Re\Bigg(\int_0^\infty e^{-ixu}g_1(u)du\Bigg),$$
$$\bar{F}_X(x) = \frac{e^{x\rho}}{\pi}\Re\Bigg(\int_0^\infty e^{-ixu}g_2(u)du\Bigg),$$

where

$$g_1(u) = \frac{\phi_X(u+i\rho)}{\rho - ui} \quad \text{and} \quad g_2(u) = \frac{\phi_X(u+i\rho)}{ui-\rho}.$$

By Proposition 2.2, AVaR of X is also obtained by

$$\mathrm{AVaR}_\varepsilon(X) = x - \frac{e^{x\rho}}{\pi(1-\varepsilon)}\Re\Bigg(\int_0^\infty e^{-iux}g_3(u)du\Bigg),$$

where $x = \mathrm{VaR}_\varepsilon(X)$ and

$$g_3(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)^2}.$$

By Proposition 2.3, AVaR of a log infinitely divisible random variable Y is also obtained by

$$\mathrm{AVaR}_\varepsilon(Y) = e^x - \frac{e^{x(1-\rho)}}{\pi(1-\varepsilon)}\Re\Bigg(\int_0^\infty e^{-iux}g_4(u)du\Bigg),$$

where $x = \log \mathrm{VaR}_\varepsilon(Y)$ and

$$g_4(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}.$$

Therefore, we can obtain the CDF, the SF, and AVaR, if we can compute the integral

$$\int_0^{\infty} e^{-ixu} g(u)du = 2\pi \int_0^{\infty} e^{-2\pi ixu} g(2\pi u)du.$$

Let

$$f(x) = \int_0^{\infty} e^{-2\pi ixu} g(2\pi u)du.$$

Then we can approximate the value of $f(x)$ using the discrete numerical integration:

$$f(x) \approx \hat{f}(x) = \sum_{n=0}^{N-1} \exp\left(-2\pi ix\left(\frac{nK}{N}\right)\right) g\left(\frac{2\pi nK}{N}\right)\frac{K}{N},$$

where K and N are large positive integers with $N > K$. If $x_k = (k - N)/K$, $k = 0, 1, 2, \dots, N-1$, then we have

$$\exp\left(-2\pi ix_k\left(\frac{nK}{N}\right)\right) = (-1)^n \exp\left(-2\pi i\left(\frac{nk}{N}\right)\right),$$

and hence

$$\hat{f}(x_k) = \frac{K}{N} \sum_{n=0}^{N-1} w^{nk} g_n,$$

where $w = e^{-2\pi i/N}$ and $g_n = (-1)^n g\big((2\pi nK)/N\big)$. To compute $\sum_{n=0}^{N-1} w^{nk} g_n$ we can use the fast Fourier transform, which is implemented by many numerical software packages. If $x_k < x < x_{k+1}$, then $\hat{f}(x)$ can be obtained by the interpolation of $\hat{f}(x_k)$ and $\hat{f}(x_{k+1})$.

The value $\mathrm{VaR}_\varepsilon(X) = \bar{F}_X^{-1}(1-\varepsilon)$ is a solution to the following equation:

$$\bar{F}_X(x) + \varepsilon - 1 = 0.$$

We can find the solution by various numerical methods such as the Newton–Raphson method. Using the Newton–Raphson method, we iterate the following

$$x_{i+1} = x_i + \frac{\bar{F}_X(x_i) + \varepsilon - 1}{\bar{F}'_X(x_i)},$$

until the relative error between x_j and x_{j+1} becomes sufficiently small. In this case, $\bar{F}'_X(x_i) = -f_X(x_i)$ is the PDF of X, and it can be obtained numerically. The Newton–Raphson method is also implemented by many numerical software packages.

4. TEMPERED STABLE DISTRIBUTIONS

In this section, we present five subclasses of infinitely divisible distributions for modeling a portfolio loss distribution: classical tempered stable distribution, Kim–Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. In the literature, these distributions have been referred to as tempered stable distributions. In general, these distributions do not have closed-form solution for the probability density function. Instead, they are defined by their Ch.Fs.

Below we will let a random variable X denote a tempered stable distributed random variable. Consider a random variable Y such that $\log Y$ is a tempered stable distribution. Then the random variable Y is referred to as the *log tempered stable random variable*.

4.1. Classical tempered stable distribution. For the definition see Koponen [18], Boyarchenko and Levendorskiĭ [5], and Carr et al. [6]. Let $\alpha \in (0,2)$, $C > 0$, $\lambda_+, \lambda_- > 0$, and $m \in \mathbb{R}$. X is said to follow the *classical tempered stable* (*CTS*) *distribution* if the Ch.F of X is given by

$$\begin{aligned}\phi_{\text{CTS}}(u) &:= \phi_X(u)\\ &= \exp\big[ium - iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})\\ &\quad + C\Gamma(-\alpha)\big((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha\big)\big],\end{aligned}$$

and we put $X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$. The mean of X is m, and cumulants $c_n(X) = (d^n/du^n)\log\phi_X(u)|_{u=0}$ of X are

$$c_n(X) = C\Gamma(n-\alpha)\big(\lambda_+^{\alpha-n} + (-1)^n\lambda_-^{\alpha-n}\big)$$

for $n = 2, 3, \ldots$

By analytic continuation in complex analysis, the function $\phi_{\text{CTS}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : \Im(z) \in (-\lambda_+, \lambda_-)\}$, i.e. $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) \in (-\lambda_+, \lambda_-)$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 2.1, equation (2.4), and Proposition 2.2 for $-\lambda_+ < \rho < 0$.

If a random variable Y is a log infinitely divisible distribution such that

$$\log Y \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-CTS random variable*. If $\lambda_+ > 1$, then there exists $\rho < -1$ such that $|\phi_{\text{CTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 2.1, equation (2.9) for $-\lambda_+ < \rho < 0$, and Proposition 2.3 for $-\lambda_+ < \rho < -1$.

4.2. Kim–Rachev distribution. For the definition see Kim et al. [11], [12]. Let $\alpha \in (0,2) \setminus \{1\}$, $k_+, k_-, r_+, r_- > 0$, $p_+, p_- \in \{p > -\alpha \mid p \neq -1, p \neq 0\}$, and $m \in \mathbb{R}$. X is said to follow the *Kim–Rachev* (*KR*) *distribution* if the Ch.F of X is given by

$$\begin{aligned}\phi_{\text{KR}}(u) &:= \phi_X(u)\\ &= \exp\biggl[ium - iu\Gamma(1-\alpha)\biggl(\frac{k_+r_+}{p_+ + 1} - \frac{k_-r_-}{p_- + 1}\biggr)\\ &\quad + k_+H(iu;\alpha,r_+,p_+) + k_-H(-iu;\alpha,r_-,p_-)\biggr]\end{aligned}$$

with

$$H(x;\alpha,r,p) = \frac{\Gamma(-\alpha)}{p}\bigl({}_2F_1(p,-\alpha;1+p;rx) - 1\bigr),$$

where ${}_2F_1$ is the hypergeometric function (see Andrews [1]) and we write

$$X \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m).$$

The mean of X is m, and cumulants of X are

$$c_n(X) = \Gamma(n-\alpha)\biggl(\frac{k_+r_+^n}{p_+ + n} + (-1)^n\frac{k_-r_-^n}{p_- + n}\biggr)$$

for $n = 2, 3, \ldots$ If p_+ and p_- approach infinity, then KR distribution converges to the CTS distribution.

The function $\phi_{\text{KR}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : \Im(z) \in (-r_+^{-1}, r_-^{-1})\}$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{KR}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 2.1, equation (2.4), and Proposition 2.2 for $-r_+^{-1} < \rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m),$$

then Y is referred to as a *log-KR random variable*. If $1/r_+ > 1$, then there exists $\rho < -1$ such that $|\phi_{\text{KR}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 2.1, equation (2.9) for $-r_+^{-1} < \rho < 0$, and Proposition 2.3 for $-r_+^{-1} < \rho < -1$.

4.3. Modified tempered stable distribution. For the definition see Kim et al. [16]. Let $\alpha \in (0,2) \setminus \{1\}$, $C, \lambda_+, \lambda_- > 0$, and $m \in \mathbb{R}$. X is said to follow the *modified tempered stable* (*MTS*) *distribution* if the Ch.F of X is given by

$$\begin{aligned}\phi_{\text{MTS}}(u) &:= \phi_X(u)\\ &= \exp\bigl[ium + C\bigl(G_R(u;\alpha,C,\lambda_+) + G_R(u;\alpha,C,\lambda_-)\bigr)\\ &\quad + iuC\bigl(G_I(u;\alpha,\lambda_+) - G_I(u;\alpha,\lambda_-)\bigr)\bigr],\end{aligned}$$

where for $u \in \mathbb{R}$

$$G_R(x;\alpha,\lambda) = 2^{-(\alpha+3)/2}\sqrt{\pi}\Gamma\left(-\frac{\alpha}{2}\right)\left((\lambda^2+x^2)^{\alpha/2}-\lambda^\alpha\right)$$

and

$$G_I(x;\alpha,\lambda) = 2^{-(\alpha+1)/2}\Gamma\left(\frac{1-\alpha}{2}\right)\lambda^{\alpha-1}\left[{}_2F_1\left(1,\frac{1-\alpha}{2};\frac{3}{2};-\frac{x^2}{\lambda^2}\right)-1\right],$$

and we write $X \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$. The mean of X is m, and cumulants of X are equal to

$$c_n(X) = 2^{n-(\alpha+3)/2}C\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\left(\lambda_+^{\alpha-n}+(-1)^n\lambda_-^{\alpha-n}\right)$$

for $n = 2, 3, \ldots$

The function $\phi_{\text{MTS}}(u)$ can be extended analytically to the region $\{z \in \mathbb{C} : |\Im(z)| < \min\{\lambda_+, \lambda_-\}\}$, that is $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $|\Im(z)| < \min\{\lambda_+, \lambda_-\}$. Therefore, there exists $\rho < 0$ such that $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\text{VaR}_\varepsilon(X)$, and $\text{AVaR}_\varepsilon(X)$ are obtained by Proposition 2.1, equation (2.4), and Proposition 2.2 for $-\min\{\lambda_+, \lambda_-\} < \rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-MTS random variable*. If $\lambda_+ > 1$ and $\lambda_- > 1$, then there exists $\rho < -1$ such that $|\phi_{\text{MTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_Y(x)$, $\text{VaR}_\varepsilon(Y)$, and $\text{AVaR}_\varepsilon(Y)$ are obtained by Corollary 2.1 and equation (2.9) for $-\min\{\lambda_+, \lambda_-\} < \rho < 0$, and Proposition 2.3 for $-\min\{\lambda_+, \lambda_-\} < \rho < -1$.

4.4. Normal tempered stable distribution. For the definition see Barndorff-Nielsen and Levendorskii [3] and Kim et al. [15]. Let $\alpha \in (0, 2)$, $C, \lambda > 0$, $|\beta| < \lambda$, and $m \in \mathbb{R}$. X is said to follow the *normal tempered stable* (*NTS*) *distribution* if the Ch.F of X is given by

$$\begin{aligned}\phi_{\text{NTS}}(u) &:= \phi_X(u)\\ &= \exp\left[ium + \frac{C\sqrt{\pi}\,\Gamma(-\alpha/2)}{2^{(\alpha+1)/2}}iu\alpha\beta(\lambda^2-\beta^2)^{\alpha/2-1}\right.\\ &\quad\left.+\frac{C\sqrt{\pi}\,\Gamma(-\alpha/2)}{2^{(\alpha+1)/2}}\left(\left(\lambda^2-(\beta+iu)^2\right)^{\alpha/2}-(\lambda^2-\beta^2)^{\alpha/2}\right)\right],\end{aligned}$$

and we write $X \sim \text{NTS}(\alpha, C, \lambda, \beta, m)$. The mean of X is m. The general expressions for cumulants of X are omitted since they are rather complicated. Instead of the general form, we present three cumulants:

$$\begin{aligned}
c_2(X) &= \kappa\alpha(\lambda^2-\beta^2)^{\alpha/2-2}(\alpha\beta^2-\lambda^2-\beta^2),\\
c_3(X) &= -\kappa\alpha\beta(\lambda^2-\beta^2)^{\alpha/2-3}(\alpha^2\beta^2-3\alpha\lambda^2-3\alpha\beta^2+6\lambda^2+2\beta^2),\\
c_4(X) &= \kappa\alpha(\alpha-2)(\lambda^2-\beta^2)^{\alpha/2-4}\\
&\qquad\times(\alpha^2\beta^4-6\alpha\lambda^2\beta^2-4\alpha\beta^4+3\beta^4+18\lambda^2\beta^2+3\lambda^4),
\end{aligned}$$

where $\kappa = 2^{-(\alpha+1)/2}C\sqrt{\pi}\,\Gamma(-\alpha/2)$.

The function $\phi_{\mathrm{NTS}}(u)$ can be extended analytically to the region $\{z\in\mathbb{C} : \Im(z)\in(-\lambda+\beta,\lambda+\beta)\}$, that is $|\phi_{\mathrm{NTS}}(z)|<\infty$ for all complex z with $\Im(z)\in(-\lambda+\beta,\lambda+\beta)$. Therefore, there exists $\rho<0$ such that $|\phi_{\mathrm{NTS}}(z)|<\infty$ for all complex z with $\Im(z)=\rho$. Hence, $\bar{F}_X(x)$, $\mathrm{VaR}_\varepsilon(X)$, and $\mathrm{AVaR}_\varepsilon(X)$ are obtained by Proposition 2.1, equation (2.4), and Proposition 2.2 for $-\lambda+\beta<\rho<0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \mathrm{NTS}(\alpha, C, \lambda, \beta, m),$$

then Y is referred to as a *log-NTS random variable*. If $\lambda-\beta>1$, then there exists $\rho<-1$ such that $|\phi_{\mathrm{NTS}}(z)|<\infty$ for all complex z with $\Im(z)=\rho$. Hence, $\bar{F}_Y(x)$, $\mathrm{VaR}_\varepsilon(Y)$, and $\mathrm{AVaR}_\varepsilon(Y)$ are obtained by Corollary 2.1 and equation (2.9) for $-\lambda+\beta<\rho<0$, and Proposition 2.3 for $-\lambda+\beta<\rho<-1$.

4.5. Rapidly decreasing tempered stable distribution. For the definition see Bianchi et al. [4] and Kim et al. [14]. Let $\alpha\in(0,2)\setminus\{1\}$, $C,\lambda_+,\lambda_->0$, and $m\in\mathbb{R}$. X is said to follow the *rapidly decreasing tempered stable* (*RDTS*) *distribution* if the Ch.F of X is given by

$$\phi_{\mathrm{RDTS}}(u)=\phi_X(u)=\exp\big[ium + C\big(G(iu;\alpha,\lambda_+)+G(-iu;\alpha,\lambda_-)\big)\big]$$

with

$$\begin{aligned}
G(x;\alpha,\lambda) &= 2^{-\alpha/2-1}\lambda^\alpha\,\Gamma\Big(-\frac{\alpha}{2}\Big)\Big(M\Big(-\frac{\alpha}{2},\frac{1}{2};\frac{x^2}{2\lambda^2}\Big)-1\Big)\\
&\qquad + 2^{-\alpha/2-1/2}\lambda^{\alpha-1}x\,\Gamma\Big(\frac{1-\alpha}{2}\Big)\Big(M\left(\frac{1-\alpha}{2},\frac{3}{2};\frac{x^2}{2\lambda^2}\right)-1\Big),
\end{aligned}$$

where M is the confluent hypergeometric function (see Andrews [1]), and we write $X\sim\mathrm{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$. The mean of X is m, and cumulants of X are

$$c_n(X) = 2^{(n-\alpha-2)/2}C\,\Gamma\Big(\frac{n-\alpha}{2}\Big)\left(\lambda_+^{\alpha-n}+(-1)^n\lambda_-^{\alpha-n}\right)$$

for $n=2,3,\ldots$

The function $\phi_{\mathrm{RDTS}}(u)$ is expandable to an entire function on $\mathbb{C}$. Hence, $\mathrm{AVaR}_{\varepsilon}(X)$ is obtained by equation (2.6) if $\rho < 0$, that is $|\phi_{\mathrm{RDTS}}(z)| < \infty$ for all complex z. Therefore, there exists $\rho < 0$ such that $|\phi_{\mathrm{RDTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho$. Hence, $\bar{F}_X(x)$, $\mathrm{VaR}_{\varepsilon}(X)$, and $\mathrm{AVaR}_{\varepsilon}(X)$ are obtained by Proposition 2.1, equation (2.4), and Proposition 2.2 for $\rho < 0$.

If Y is a log infinitely divisible random variable such that

$$\log Y \sim \mathrm{RDTS}(\alpha, C, \lambda_+, \lambda_-, m),$$

then Y is referred to as a *log-RDTS random variable*. Since $|\phi_{\mathrm{RDTS}}(z)| < \infty$ for all complex z, we have $|\phi_{\mathrm{RDTS}}(z)| < \infty$ for all complex z with $\Im(z) = \rho < -1$. Hence, $\bar{F}_Y(x)$, $\mathrm{VaR}_{\varepsilon}(Y)$, and $\mathrm{AVaR}_{\varepsilon}(Y)$ are obtained by Corollary 2.1, equation (2.9) for $\rho < 0$, and Proposition 2.3 for $\rho < -1$.

Ch.Fs, SFs, VaRs, and AVaRs of tempered stable and log tempered stable random variables are presented in Tables 1 and 2 with ranges of ρ.

5. EMPIRICAL EXAMPLE

In this section, we present a simple empirical example for calculating the VaR and the AVaR values using equations (2.4) and (2.6). Our application in this paper is to the U.S. equity market. Because in this empirical study we do not focus on operational risk, the closed-form solution for AVaR for log infinitely divisible random variables as given by equation (2.10) will not be estimated here. We use daily closing prices for the S&P 500 index (a proxy for the U.S. equity market) from January 2, 1997 through December 29, 2006 obtained from Yahoo! Finance. We use maximum likelihood estimation to estimate the parameters.

The VaR and the AVaR values for confidence levels $\{90\%, 91\%, \ldots, 99\%, 99.1\%, 99.2\%, \ldots, 99.9\%\}$ are provided in the upper and the lower panels of Figure 1, respectively. We plot the values of the normal, the KR, and the RDTS distributions[4] in the two figures and compare the values with empirical VaR and empirical AVaR.[5] From the results shown in the upper panel of Figure 1, the normal VaR is larger than the empirical VaR, and the two tempered stable VaRs are smaller if the confidence level is less than or equal to 95%. If the confidence level is larger than 96%, the two tempered stable VaRs are larger than the empirical VaR, and the normal VaR is smaller. Moreover, by the results shown in the lower panel of Figure 1, the AVaRs of the two tempered stable distributions are relatively similar to the empirical AVaR compared to the normal distribution. More details for this empirical investigation including parameter estimation results and calculated values of the VaR and the AVaR for the five tempered stable distributions can be found in Kim et al. [13].

[4] Since the VaR and the AVaR values of the CTS, the MTS, the NTS, and the KR distributions are very similar, and the values of the RDTS distribution are more or less different from the KR case, we plot only the values of the KR and the RDTS distributions in the two figures.

[5] We use *empirical AVaR* provided in Rachev et al. [21].

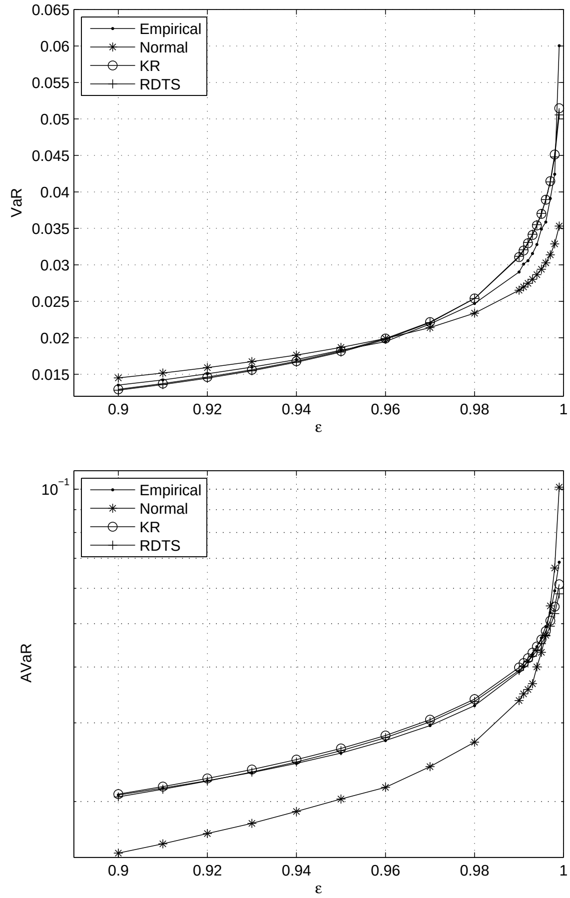

FIGURE 1. One-day VaR (upper) and AVaR (lower)

6. CONCLUSION

In this paper, we derive closed-form solutions for the AVaR for five subclasses of the infinitely divisible distribution. If a loss distribution is infinitely divisible and the Ch.F of the loss distribution is defined on the complex subset $\{z \in \mathbb{C} \colon \Im(z) = \rho\}$ for some $\rho < 0$, then we can obtain the closed-form solution of AVaR. If a loss distribution is log infinitely divisible and its Ch.F is defined on the complex subset $\{z \in \mathbb{C} : \Im(z) = \rho\}$ for some $\rho < -1$, then we can also obtain closed-form solutions of the CDF and the AVaR. In order to apply the closed-form solution we derived, we considered five tempered stable distributions: classical tempered stable distribution, Kim–Rachev distribution, modified tempered stable distribution, normal tempered stable distribution, and rapidly decreasing tempered stable distribution. We calculated VaR and AVaR values using closed-form solutions for the S&P 500 index. In our investigation, the estimated values for the tempered stable VaR and AVaR are more realistic than the normal VaR and the normal AVaR.

7. APPENDIX

7.1. Proof of Proposition 2.1. (i) By the definition of the CDF, we have

$$F_X(x) = P(X \leqslant x) = \int_{-\infty}^{x} f_X(t)dt.$$

The probability density function $f_X(t)$ can be obtained from the Ch.F ϕ_X by the complex inverse formula (see [8]). Thus, we have

$$f_X(t) = \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-itz}\phi_X(z)dz,$$

and consequently

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-itz}\phi_X(z)dzdt = \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \int_{-\infty}^{x} e^{-itz}dt\,\phi_X(z)dz.$$

Note that if $\rho > 0$, then

$$\lim_{t\to-\infty} |e^{-it(a+i\rho)}| = \lim_{t\to\infty} |e^{it(a+i\rho)}| = \lim_{t\to\infty} e^{-\rho t} = 0, \quad a \in \mathbb{R},$$

and hence

$$\int_{-\infty}^{x} e^{-itz}dt = -\frac{1}{iz}[e^{-itz}]_{-\infty}^{x} = -\frac{1}{iz}e^{-ixz},$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Thus, we have

$$\begin{aligned} F_X(x) &= -\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{1}{iz}e^{-ixz}\phi_X(z)dz \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i(u+i\rho)}e^{-ix(u+i\rho)}\phi_X(u+i\rho)du \\ &= \frac{e^{x\rho}}{2\pi} \int_{-\infty}^{\infty} e^{-ixu}\frac{\phi_X(u+i\rho)}{\rho - iu}du. \end{aligned}$$

Let

$$g_\rho(u) = \frac{\phi_X(u+i\rho)}{\rho - iu};$$

then we can show that $g_\rho(-u) = \overline{g_\rho(u)}$ with $u \in \mathbb{R}$, and hence we have

$$\int_{-\infty}^{\infty} e^{-ixu}g_\rho(u)du = 2\Re\Big(\int_{0}^{\infty} e^{-ixu}g_\rho(u)du\Big).$$

Therefore we obtain (2.1).

(ii) By the definition of the SF and the complex inverse formula, we have

$$\begin{aligned}\bar{F}_X(x) &= \int_x^\infty f_X(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho}\int_x^\infty e^{-itz}dt\,\phi_X(z)dz.\end{aligned}$$

Note that if $\rho < 0$, then

$$\lim_{t\to\infty}|e^{-it(a+i\rho)}| = \lim_{t\to\infty} e^{\rho t} = 0, \quad a \in \mathbb{R},$$

and hence

$$\int_x^\infty e^{-itz}dt = -\frac{1}{iz}[e^{-itz}]_x^\infty = \frac{1}{iz}e^{-ixz},$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Using similar arguments to those in the proof of (i), we can prove (ii).

7.2. Proof of Proposition 2.2. In order to prove Proposition 2.2, we need the following lemma.

LEMMA 7.1. *Let $K \in \mathbb{R}$. If X is infinitely divisible and there is $\rho < 0$ such that $|\phi_X(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$E[(X-K)^+] = -\frac{e^{K\rho}}{\pi}\Re\left(\int_0^\infty e^{-iuK}\frac{\phi_X(u+i\rho)}{(u+i\rho)^2}du\right). \tag{7.1}$$

By Lemma 7.1, we obtain the closed-form solution of AVaR for continuous and infinitely divisible random variable as follows:

Proof of Lemma 7.1. Since the Ch.F $\phi_X(z)$ of X is defined for all complex z with $\Im(z) = \rho$, the probability density function $f_X(x)$ of X equals

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz}\phi_X(z)dz$$

by the complex inversion formula. Thus, we have

$$\begin{aligned}E[(X-K)^+] &= \int_K^\infty (x-K)f_X(x)dx \\ &= \int_K^\infty (x-K)\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz}\phi_X(z)dzdx \\ &= \frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho}\Big(\int_K^\infty (x-K)e^{-ixz}dx\Big)\phi_X(z)dz.\end{aligned}$$

Note that if $\rho < 0$, then

$$\lim_{x\to\infty}\left|\frac{1+ix(a+i\rho)}{(a+i\rho)^2}e^{-ix(a+i\rho)} - \frac{iK}{(a+i\rho)}e^{-ix(a+i\rho)}\right| = 0$$

for $a \in \mathbb{R}$. We have

$$\int_K^\infty (x-K)e^{-ixz}dx = \left[\frac{1+ixz}{z^2}e^{-ixz} - \frac{iK}{z}e^{-ixz}\right]_K^\infty = -\frac{e^{-iKz}}{z^2},$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Therefore,

$$E[(X-K)^+] = -\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{e^{-iKz}}{z^2}\phi_X(z)dz.$$

By using $u + i\rho$ instead of z, we obtain

$$E[(X-K)^+] = -\frac{e^{K\rho}}{2\pi}\int_{-\infty}^{\infty} e^{-iKu}\frac{\phi_X(u+i\rho)}{(u+i\rho)^2}du.$$

Let

$$h_\rho(u) = \frac{\phi_X(u+i\rho)}{(u+i\rho)^2};$$

then we can show that $h_\rho(-u) = \overline{h_\rho(u)}$ with $u \in \mathbb{R}$, and hence

$$\int_{-\infty}^{\infty} e^{-iuK}h_\rho(u)du = 2\Re\Big(\int_0^\infty e^{-iuK}h_\rho(u)du\Big),$$

which completes the proof. ■

Proof of Proposition 2.2. Equation (2.5) leads to (2.6) by substituting $K = \mathrm{VaR}_\varepsilon(X)$ into (7.1). ■

7.3. Proof of Proposition 2.3. In order to obtain Proposition 2.3, we need the following lemma.

LEMMA 7.2. *Assume a random variable Y is the log infinitely divisible and $\phi_{\log Y}$ is the Ch.F of $\log Y$. If there is $\rho < -1$ such that $|\phi_{\log Y}(z)| < \infty$ for all complex z with $\Im(z) = \rho$, then*

$$E[(Y-K)^+] = -\frac{K^{1+\rho}}{\pi}\Re\left(\int_0^\infty \frac{K^{-iu}\phi_{\log Y}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right) \tag{7.2}$$

for $K > 0$.

Proof of Lemma 7.2. Let $X = \log Y$. Since the Ch.F $\phi_X(z)$ of X is defined for all complex z with $\Im(z) = \rho$, the probability density function $f_X(x)$

of X is equal to

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz} \phi_X(z) dz$$

by the complex inversion formula. Thus, we have

$$\begin{aligned} E[(Y-K)^+] = E[(e^X - K)^+] &= \int_{\log K}^{\infty} (e^x - K) f_X(x) dx \\ &= \int_{\log K}^{\infty} (e^x - K) \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} e^{-ixz} \phi_X(z) dz dx \\ &= \frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \Big(\int_{\log K}^{\infty} (e^x - K) e^{-ixz} dx \Big) \phi_X(z) dz. \end{aligned}$$

Note that if $\rho < -1$, then

$$\lim_{x\to\infty} \left| \frac{e^{(1-i(a+i\rho))x}}{1 - i(a+i\rho)} + \frac{K e^{-i(a+i\rho)x}}{i(a+i\rho)} \right| = 0$$

for $a \in \mathbb{R}$. We have

$$\int_{\log K}^{\infty} (e^x - K) e^{-ixz} dx = \left[\frac{e^{(1-iz)x}}{1 - iz} + \frac{K e^{-izx}}{iz} \right]_{\log K}^{\infty} = -\frac{K^{1-iz}}{z(i+z)},$$

where $z \in \mathbb{C}$ with $\Im(z) = \rho$. Therefore,

$$E[(Y-K)^+] = -\frac{1}{2\pi} \int_{-\infty+i\rho}^{\infty+i\rho} \frac{K^{1-iz}}{z(i+z)} \phi_X(z) dz.$$

By using $u + i\rho$ instead of z, we obtain

$$\begin{aligned} E[(Y-K)^+] &= -\frac{K}{2\pi} \int_{-\infty}^{\infty} \frac{K^{-i(u+i\rho)} \phi_X(u+i\rho)}{(u+i\rho)\big(u + i(1+\rho)\big)} du \\ &= -\frac{K^{1+\rho}}{2\pi} \int_{-\infty}^{\infty} \frac{K^{-iu} \phi_X(u+i\rho)}{(u+i\rho)\big(u + i(1+\rho)\big)} du. \end{aligned}$$

Let

$$h_\rho(u) = \frac{K^{-iu} \phi_X(u+i\rho)}{(u+i\rho)\big(u + i(1+\rho)\big)};$$

then we can show that $h_\rho(-u) = \overline{h_\rho(u)}$ with $u \in \mathbb{R}$, and hence

$$\int_{-\infty}^{\infty} h_\rho(u) du = 2\Re\Big(\int_0^{\infty} h_\rho(u) du \Big),$$

which completes the proof. ∎

P r o o f o f P r o p o s i t i o n 2.3. Equation (2.5) leads to (2.10) by substituting $K = \text{VaR}_\varepsilon(Y)$ into (7.2). ∎

TABLE 1. The Ch.F, SF, VaR, and AVaR for the tempered stable distribution

CTS	$X \sim \text{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$
Ch.F	$\phi_{\text{CTS}}(u) = \exp\big[ium - iuC\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) + C\Gamma(-\alpha)\big((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha\big)\big]$
SF	$\bar{F}_{\text{CTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\bigg(\int\limits_0^\infty e^{-ixu}\frac{\phi_{\text{CTS}}(u+i\rho)}{ui-\rho}du\bigg), \quad -\lambda_+ < \rho < 0$
VaR	$\text{VaR}_{\text{CTS}}(\varepsilon) = \bar{F}_{\text{CTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{CTS}}(\varepsilon) = \text{VaR}_{\text{CTS}}(\varepsilon) - \frac{\exp\big(\rho \text{VaR}_{\text{CTS}}(\varepsilon)\big)}{\pi(1-\varepsilon)} \times \Re\bigg(\int\limits_0^\infty e^{-iu\text{VaR}_{\text{CTS}}(\varepsilon)}\frac{\phi_{\text{CTS}}(u+i\rho)}{(u+i\rho)^2}du\bigg), \quad -\lambda_+ < \rho < 0$
MTS	$X \sim \text{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$
Ch.F	$\phi_{\text{MTS}}(u) = \exp\big[ium + C\big(G_R(u;\alpha,C,\lambda_+) + G_R(u;\alpha,C,\lambda_-)\big) + iuC\big(G_I(u;\alpha,\lambda_+) - G_I(u;\alpha,\lambda_-)\big)\big]$, where $G_R(x;\alpha,\lambda) = 2^{-(\alpha+3)/2}\sqrt{\pi}\Gamma\Big(-\frac{\alpha}{2}\Big)\big((\lambda^2+x^2)^{\alpha/2} - \lambda^\alpha\big)$ and $G_I(x;\alpha,\lambda) = 2^{-(\alpha+1)/2}\Gamma\Big(\frac{1-\alpha}{2}\Big)\lambda^{\alpha-1}\Big[{}_2F_1\Big(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{x^2}{\lambda^2}\Big) - 1\Big]$
SF	$\bar{F}_{\text{MTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\bigg(\int\limits_0^\infty e^{-ixu}\frac{\phi_{\text{MTS}}(u+i\rho)}{ui-\rho}du\bigg), \quad -\min\{\lambda_+, \lambda_-\} < \rho < 0$
VaR	$\text{VaR}_{\text{MTS}}(\varepsilon) = \bar{F}_{\text{MTS}}^{-1}(1-\varepsilon)$
AVaR	$\text{AVaR}_{\text{MTS}}(\varepsilon) = \text{VaR}_{\text{MTS}}(\varepsilon) - \frac{\exp\big(\rho \text{VaR}_{\text{MTS}}(\varepsilon)\big)}{\pi(1-\varepsilon)} \times \Re\left(\int\limits_0^\infty e^{-iu\text{VaR}_{\text{MTS}}(\varepsilon)}\frac{\phi_{\text{MTS}}(u+i\rho)}{(u+i\rho)^2}du\right), \quad -\min\{\lambda_+, \lambda_-\} < \rho < 0$
NTS	$X \sim \text{NTS}(\alpha, C, \lambda, \beta, m)$
Ch.F	$\phi_{\text{NTS}}(u) = \exp\big\{ium + iu\kappa\beta(\lambda^2 - \beta^2)^{\alpha/2-1} + \kappa\big[\big(\lambda^2 - (\beta+iu)^2\big)^{\alpha/2} - (\lambda^2-\beta^2)^{\alpha/2}\big]\big\}$, where $\kappa = 2^{-(\alpha+1)/2}C\sqrt{\pi}\Gamma\Big(-\frac{\alpha}{2}\Big)$
SF	$\bar{F}_{\text{NTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\bigg(\int\limits_0^\infty e^{-ixu}\frac{\phi_{\text{NTS}}(u+i\rho)}{ui-\rho}du\bigg), \quad -\lambda + \beta < \rho < 0$

NTS $X \sim \mathrm{NTS}(\alpha, C, \lambda, \beta, m)$

VaR $\mathrm{VaR}_{\mathrm{NTS}}(\varepsilon) = \bar{F}_{\mathrm{NTS}}^{-1}(1-\varepsilon)$

AVaR
$$\mathrm{AVaR}_{\mathrm{NTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{NTS}}(\varepsilon) - \frac{\exp\big(\rho \mathrm{VaR}_{\mathrm{NTS}}(\varepsilon)\big)}{\pi(1-\varepsilon)} \times \Re\left(\int_0^\infty e^{-iu\mathrm{VaR}_{\mathrm{NTS}}(\varepsilon)} \frac{\phi_{\mathrm{NTS}}(u+i\rho)}{(u+i\rho)^2} du\right), \quad -\lambda + \beta < \rho < 0$$

KR $X \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$

Ch.F
$$\phi_{\mathrm{KR}}(u) = \exp\left[ium - iu\Gamma(1-\alpha)\left(\frac{k_+ r_+}{p_+ + 1} - \frac{k_- r_-}{p_- + 1}\right) + k_+ H(iu; \alpha, r_+, p_+) + k_- H(-iu; \alpha, r_-, p_-)\right],$$

where $H(x; \alpha, r, p) = \dfrac{\Gamma(-\alpha)}{p}\big({}_2F_1(p, -\alpha; 1+p; rx) - 1\big)$

SF
$$\bar{F}_{\mathrm{KR}}(x) = \frac{e^{x\rho}}{\pi}\mathbb{R}\left(\int_0^\infty e^{-ixu}\frac{\phi_{\mathrm{KR}}(u+i\rho)}{ui-\rho} du\right), \qquad -1/r_+ < \rho < 0$$

VaR $\mathrm{VaR}_{\mathrm{KR}}(\varepsilon) = \bar{F}_{\mathrm{KR}}^{-1}(1-\varepsilon)$

AVaR
$$\mathrm{AVaR}_{\mathrm{KR}}(\varepsilon) = \mathrm{VaR}_{\mathrm{KR}}(\varepsilon) - \frac{\exp\big(\rho \mathrm{VaR}_{\mathrm{KR}}(\varepsilon)\big)}{\pi(1-\varepsilon)} \times \Re\left(\int_0^\infty e^{-iu\mathrm{VaR}_{\mathrm{KR}}(\varepsilon)} \frac{\phi_{\mathrm{KR}}(u+i\rho)}{(u+i\rho)^2} du\right), \quad -1/r_+ < \rho < 0$$

RDTS $X \sim \mathrm{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$

Ch.F $\phi_{\mathrm{RDTS}}(u) = \exp\big[ium + C\big(G(iu; \alpha, \lambda_+) + G(-iu; \alpha, \lambda_-)\big)\big]$

where
$$G(x; \alpha, \lambda) = 2^{-\alpha/2-1}\lambda^\alpha \Gamma\left(-\frac{\alpha}{2}\right)\left(M\left(-\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right) + 2^{-\alpha/2-1/2}\lambda^{\alpha-1} x \Gamma\left(\frac{1-\alpha}{2}\right)\left(M\left(\frac{1-\alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2}\right) - 1\right)$$

SF
$$\bar{F}_{\mathrm{RDTS}}(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-ixu}\frac{\phi_{\mathrm{RDTS}}(u+i\rho)}{ui-\rho} du\right), \qquad \rho < 0$$

VaR $\mathrm{VaR}_{\mathrm{RDTS}}(\varepsilon) = \bar{F}_{\mathrm{RDTS}}^{-1}(1-\varepsilon)$

AVaR
$$\mathrm{AVaR}_{\mathrm{RDTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{RDTS}}(\varepsilon) - \frac{\exp\big(\rho \mathrm{VaR}_{\mathrm{RDTS}}(\varepsilon)\big)}{\pi(1-\varepsilon)} \times \Re\left(\int_0^\infty e^{-iu\mathrm{VaR}_{\mathrm{RDTS}}(\varepsilon)} \frac{\phi_{\mathrm{RDTS}}(u+i\rho)}{(u+i\rho)^2} du\right), \quad \rho < 0$$

TABLE 2. The SF, VaR, and AVaR for the log tempered stable distribution

log-CTS $\log Y \sim \mathrm{CTS}(\alpha, C, \lambda_+, \lambda_-, m)$

SF
$$\bar{F}_{\mathrm{logCTS}}(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\mathrm{CTS}}(u+i\rho)}{ui-\rho}du\right), \qquad -\lambda_+ < \rho < 0$$

VaR
$$\mathrm{VaR}_{\mathrm{logCTS}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logCTS}}(1-\varepsilon)$$

AVaR
$$\mathrm{AVaR}_{\mathrm{logCTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logCTS}}(\varepsilon) - \frac{\big(\mathrm{VaR}_{\mathrm{logCTS}}(\varepsilon)\big)^{1-\rho}}{\pi(1-\varepsilon)} \times\Re\left(\int_0^\infty \frac{\big(\mathrm{VaR}_{\mathrm{logCTS}}(\varepsilon)\big)^{iu}\phi_{\mathrm{CTS}}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right), \quad -\lambda_+ < \rho < -1$$

---

log-MTS $\log Y \sim \mathrm{MTS}(\alpha, C, \lambda_+, \lambda_-, m)$

SF
$$\bar{F}_{\mathrm{logMTS}}(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\mathrm{MTS}}(u+i\rho)}{ui-\rho}du\right), \qquad -\min\{\lambda_+, \lambda_-\} < \rho < 0$$

VaR
$$\mathrm{VaR}_{\mathrm{logMTS}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logMTS}}(1-\varepsilon)$$

AVaR
$$\mathrm{AVaR}_{\mathrm{logMTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logMTS}}(\varepsilon) - \frac{\big(\mathrm{VaR}_{\mathrm{logMTS}}(\varepsilon)\big)^{1-\rho}}{\pi(1-\varepsilon)} \times\Re\left(\int_0^\infty \frac{\big(\mathrm{VaR}_{\mathrm{logMTS}}(\varepsilon)\big)^{iu}\phi_{\mathrm{MTS}}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right), \quad -\min\{\lambda_+, \lambda_-\} < \rho < -1$$

---

log-NTS $\log Y \sim \mathrm{NTS}(\alpha, C, \lambda, \beta, m)$

SF
$$\bar{F}_{\mathrm{logNTS}}(y) = \frac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\frac{\phi_{\mathrm{NTS}}(u+i\rho)}{ui-\rho}du\right), \qquad -\lambda+\beta < \rho < 0$$

VaR
$$\mathrm{VaR}_{\mathrm{logNTS}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logNTS}}(1-\varepsilon)$$

AVaR
$$\mathrm{AVaR}_{\mathrm{logNTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logNTS}}(\varepsilon) - \frac{\big(\mathrm{VaR}_{\mathrm{logNTS}}(\varepsilon)\big)^{1-\rho}}{\pi(1-\varepsilon)} \times\Re\left(\int_0^\infty \frac{\big(\mathrm{VaR}_{\mathrm{logNTS}}(\varepsilon)\big)^{iu}\phi_{\mathrm{NTS}}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right), \quad -\lambda+\beta < \rho < -1$$

log-KR $\log Y \sim \mathrm{KR}(\alpha, k_+, k_-, r_+, r_-, p_+, p_-, m)$

SF $\bar{F}_{\mathrm{logKR}}(y) = \dfrac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\dfrac{\phi_{\mathrm{KR}}(u+i\rho)}{ui-\rho}du\right), \qquad -1/r_+ < \rho < 0$

VaR $\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logKR}}(1-\varepsilon)$

AVaR
$$\mathrm{AVaR}_{\mathrm{logKR}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logKR}}(\varepsilon) - \frac{\big(\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon)\big)^{1-\rho}}{\pi(1-\varepsilon)} \times\Re\left(\int_0^\infty \frac{\big(\mathrm{VaR}_{\mathrm{logKR}}(\varepsilon)\big)^{iu}\phi_{\mathrm{KR}}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right), \quad -1/r_+ < \rho < -1$$

log-RDTS $\log Y \sim \mathrm{RDTS}(\alpha, C, \lambda_+, \lambda_-, m)$

SF $\bar{F}_{\mathrm{logRDTS}}(y) = \dfrac{y^\rho}{\pi}\Re\left(\int_0^\infty y^{-iu}\dfrac{\phi_{\mathrm{RDTS}}(u+i\rho)}{ui-\rho}du\right), \qquad \rho < 0$

VaR $\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon) = \bar{F}^{-1}_{\mathrm{logRDTS}}(1-\varepsilon)$

AVaR
$$\mathrm{AVaR}_{\mathrm{logRDTS}}(\varepsilon) = \mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon) - \frac{\big(\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon)\big)^{1-\rho}}{\pi(1-\varepsilon)} \times\Re\left(\int_0^\infty \frac{\big(\mathrm{VaR}_{\mathrm{logRDTS}}(\varepsilon)\big)^{iu}\phi_{\mathrm{RDTS}}(u+i\rho)}{(u+i\rho)\big(u+i(1+\rho)\big)}du\right), \quad \rho < -1$$

REFERENCES

[1] L.D. Andrews, *Special Functions of Mathematics for Engineers*, 2nd edition, Oxford University Press, 1998.

[2] P. Artzner, F. Delbaen, J.-M. Eber and D. Heath, *Coherent measures of risk*, Math. Finance 9 (3) (1999), pp. 203–228.

[3] O. E. Barndorff-Nielsen and S. Levendorskii, *Feller processes of normal inverse Gaussian type*, Quant. Finance 1 (2001), pp. 318–331.

[4] M. L. Bianchi, S. T. Rachev, Y. S. Kim and F. J. Fabozzi, *Tempered infinitely divisible distributions and processes* (in Russian), Teor. Veroyatnost. i Primenen. 55 (1) (2010), pp. 59–86. In English: Theory Probab. Appl. 55 (1) (2011), to appear.

[5] S. I. Boyarchenko and S. Z. Levendorskiǐ, *Option pricing for truncated Lévy processes*, Int. J. Theor. Appl. Finance 3 (2000), pp. 549–552.

[6] P. Carr, H. Geman, D. Madan and M. Yor, *The fine structure of asset returns: An empirical investigation*, J. Business 75 (2) (2002), pp. 305–332.

[7] A. S. Chernobai, S. T. Rachev and F. J. Fabozzi, *Operational Risk: A Guide to Basel II Capital Requirements, Models, and Analysis*, Wiley, 2007.

[8] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer, 1970.

[9] S. Dokov, S. V. Stoyanov and S. T. Rachev, *Computing VaR and AVaR of skewed-t distribution*, J. Appl. Funct. Anal. 3 (1) (2008), pp. 189–208.

[10] M. Grabchak and G. Samorodnitsky, *Do financial returns have finite or infinite variance? A paradox and an explanation*, Quant. Finance 10 (2010), pp. 883–893.

[11] Y. S. Kim, S. T. Rachev, M. L. Bianchi and F. J. Fabozzi, *Financial market models with Lévy processes and time-varying volatility*, J. Banking and Finance 32 (2008), pp. 1363–1378.

[12] Y. S. Kim, S. T. Rachev, M. L. Bianchi and F. J. Fabozzi, *A new tempered stable distribution and its application to finance*, in: *Risk Assessment: Decisions in Banking and Finance*, G. Bol, S. T. Rachev and R. Wuerth (Eds.), Physika Verlag, Springer, 2008, pp. 77–110.

[13] Y. S. Kim, S. T. Rachev, M. L. Bianchi and F. J. Fabozzi, *Computing VaR and AVaR in infinitely divisible distributions*, Technical Report, Chair of Econometrics, Statistics and Mathematical Finance, School of Economics and Business Engineering, University of Karlsruhe, 2009 (http://www.statistik.uni-karlsruhe.de/download/AVaR_ID_KRBF_03-25.pdf).

[14] Y. S. Kim, S. T. Rachev, M. L. Bianchi and F. J. Fabozzi, *Tempered stable and tempered infinitely divisible GARCH models*, J. Banking and Finance 34 (2010), pp. 2096–2109.

[15] Y. S. Kim, S. T. Rachev, D. M. Chung and M. L. Bianchi, *A modified tempered stable distribution with volatility clustering*, in: *New Developments in Financial Modelling*, J. O. Soares, J. P. Pina and M. Catalaõ-Lopes (Eds.), Cambridge Scholars Publishing, 2008, pp. 344–365.

[16] Y. S. Kim, S. T. Rachev, D. M. Chung and M. L. Bianchi, *The modified tempered stable distribution, GARCH models and option pricing*, Probab. Math. Statist. 29 (1) (2009), pp. 91–117.

[17] L. Klebanov, T. Kozubowski and S. T. Rachev, *Ill-Posed Problems in Probability and Stability of Random Sums*, Nova Science Publishers, New York 2006.

[18] I. Koponen, *Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process*, Phys. Rev. E 52 (1995), pp. 1197–1199.

[19] G. Pflug, *Some remarks on the value-at-risk and the conditional value-at-risk*, in: *Probabilistic Constrained Optimization: Methodology and Applications*, S. Uryasev (Ed.), Kluwer Academic Publishers, 2000, pp. 272–281.

[20] S. T. Rachev and S. Mittnik, *Stable Paretian Models in Finance*, Wiley, 2000.

[21] S. T. Rachev, S. Stoyanov and F. J. Fabozzi, *Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures*, Wiley, New Jersey, 2007.

[22] R. T. Rockafellar and S. Uryasev, *Optimization of conditional value-at-risk*, Journal of Risk 2 (3) (2000), pp. 21–41.

[23] R. T. Rockafellar and S. Uryasev, *Conditional value-at-risk for general loss distributions*, J. Banking and Finance 26 (2002), pp. 1443–1471.

[24] S. Stoyanov, G. Samorodnitsky, S. T. Rachev and S. Ortobelli, *Computing the portfolio conditional Value-at-Risk in the α-stable case*, Probab. Math. Statist. 26 (1) (2006), pp. 1–22.

Young Shin Kim
Department of Econometrics, Statistics
and Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe and KIT
Kollegium am Schloss, Bau II, 20.12, R210
D-76128, Karlsruhe, Germany
E-mail: aaron.kim@kit.edu

Svetlozar (Zari) T. Rachev
Chair-Professor of Econometrics, Statistics and
Mathematical Finance
School of Economics and Business Engineering
University of Karlsruhe and KIT
Kollegium am Schloss, Bau II, 20.12, R210
Postfach 6980, D-76128, Karlsruhe, Germany
and
Department of Statistics and Applied Probability
University of California, Santa Barbara, USA
and Chief-Scientist, FinAnalytica Inc.
E-mail: rachev@kit.edu

Michele Leonardo Bianchi
Specialized Intermediaries Supervision Department
Bank of Italy
Via Nazionale, 91, 00184, Rome, Italy

Frank J. Fabozzi
Professor in the Practice of Finance
Yale School of Management
New Haven, CT USA
E-mail: frank.fabozzi@yale.edu

Received on 24.8.2009;
revised version on 11.12.2009