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ON THE RATE OF CONVERGENCE IN NON-CENTRAL ASYMPTOTICS OF THE HERMITE VARIATIONS OF FRACTIONAL BROWNIAN SHEET

BY

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Abstract. The Hermite variations of the anisotropic fractional Brownian sheet enjoy similar behaviour to that for the fractional Brownian motion: central (convergence to a normal distribution) or non-central (convergence to a Hermite-type distribution). In this note, we investigate the rate of convergence in the non-central case.

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1. INTRODUCTION

The asymptotic behaviour of Hermite variations of the fractional Brownian motion (fBm) has been completely described in the eighties by [5]–[7], [12]. Recall that the fBm with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = (B^H_t)_{t\in[0,1]}$ with covariance $K_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, $t, s \in [0,1]$. It is the only self-similar Gaussian process with stationary increments. The Hermite variations of the fBm have interesting applications such as, for instance, in parameter estimation; see [1], [4], [8], [14] and references therein. Their asymptotics, roughly speaking, are described as follows: for $q \geqslant 2$ a positive integer, the properly normalised q-Hermite variations of fBm behave in law like $\mathcal{N}(0,1)$ when $H \leqslant 1 - 1/(2q)$ and like a Hermite-type distribution when $H > 1 - 1/(2q)$. For statistical purpose, the rates of these convergences are of great interest since they allow to control the error in the approximation of a statistics by its limit. In the central limit case, the rate of convergence is derived in [9] by using the Gaussian approximation results of Nourdin and Peccati based on a combination of Stein's method and Malliavin calculus. In the non-central limit case, interpreting the Hermite variations of fBm as Wiener–Itô integrals, the rate of convergence is derived in [2], based on an estimation of the distance in variation of the law of Wiener–Itô integrals by Davydov and Martynova; see Theorem 3.1.

Recently, Réveillac et al. have investigated in [11] similar behaviour for (two-parametric) anisotropic fractional Brownian sheet (fBs) B^{H_1,H_2} (a precise definition is given below): Setting

$$(1.1)\quad V_{N_1,N_2} = \sum_{i_1=0}^{N_1-1}\sum_{i_2=0}^{N_2-1} H_q\big(N_1^{H_1}N_2^{H_2}(B^{H_1,H_2}_{(i_1+1)/N_1,(i_2+1)/N_2} - B^{H_1,H_2}_{i_1/N_1,(i_2+1)/N_2} - B^{H_1,H_2}_{(i_1+1)/N_1,i_2/N_2} + B^{H_1,H_2}_{i_1/N_1,i_2/N_2})\big),$$

the authors of [11] show that, as $N_1, N_2 \to +\infty$, with an appropriate normalisation $\varphi(H_1, H_2, N_1, N_2)$ it follows that

$$(1.2)\qquad \varphi(H_1, H_2, N_1, N_2)V_{N_1,N_2} \Rightarrow \mathcal{N}(0, 1)$$

when $H_1 \leqslant 1 - 1/(2q)$ or $H_2 \leqslant 1 - 1/(2q)$, while

$$(1.3)\qquad \varphi(H_1, H_2, N_1, N_2)V_{N_1,N_2} \Rightarrow \mathcal{L}(Z)$$

when both H_1 and H_2 are in $\big(1 - 1/(2q), 1\big)$. Here $Z = Z^{(q)}_{1,1}$ stands for the value at $(1, 1)$ of a Hermite-type sheet Z and H_q is the qth Hermite polynomial given by

$$H_q(x) = \frac{(-1)^q}{q!}\exp\left(\frac{x^2}{2}\right)\frac{d^q}{dx^q}\left(\exp\left(-\frac{x^2}{2}\right)\right).$$

We refer to [11] for precise statements. The central limit (1.2) is derived by using the Gaussian approximation results in [9] and this allows, in particular, the authors of [11] to provide a rate of convergence for (1.2). In this short note, we focus on the non-central behaviour of fBs and propose a rate of convergence for (1.3). Doing so, we complete [11] and we generalise [2] like [11] generalises [9] and [2]. We deal directly with multi-parametric anisotropic Brownian sheet. The rest of the paper is organised as follows: we give the notation and state the main result in Section 2; Section 3 is devoted to the description of the set-up where the proof is done after; technical results are postponed to the Appendix.

2. NOTATION AND MAIN RESULT

In the whole note, we fix $d \in \mathbb{N} \setminus \{0\}$ and consider multi-parametric processes indexed by $[0, 1]^d$. We shall use bold notation for multi-indexed quantities, e.g., $\mathbf{a} = (a_1, \dots, a_d)$, $\mathbf{ab} = (a_1b_1, \dots, a_db_d)$, $\mathbf{a}/\mathbf{b} = (a_1/b_1, \dots, a_d/b_d)$, $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d (a_i, b_i)$, $\sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}]} a_{\mathbf{i}} = \sum_{i_1=0}^{N_1} \cdots \sum_{i_d=0}^{N_d} a_{i_1,\dots,i_d}$, $[\mathbf{a}]^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$. The (*d-parametric*) *anisotropic fractional Brownian sheet* is the centered Gaussian process $\{B^{\mathbf{H}}_{\mathbf{t}} : \mathbf{t} = (t_1, \dots, t_d) \in [0, 1]^d\}$ with Hurst multi-index $\mathbf{H} = (H_1, \dots, H_d) \in (0, 1)^d$. It is equal to zero on the hyperplanes

$\{\mathbf{t} : t_i = 0\}$, $1 \leqslant i \leqslant d$, and its covariance function is given by
(2.1)

$$K_{\mathbf{H}}(\mathbf{t},\mathbf{s}) = \mathbb{E}[B^{\mathbf{H}}_{\mathbf{t}} B^{\mathbf{H}}_{\mathbf{s}}] = \prod_{k=1}^{d} K_{H_k}(t_k, s_k) = \prod_{k=1}^{d} \frac{t_k^{2H_k} + s_k^{2H_k} - |t_k - s_k|^{2H_k}}{2}.$$

We are interested in the Hermite variation of $B^{\mathbf{H}}$ defined by

$$V_{\mathbf{N}} := \sum_{\mathbf{i} \in [\mathbf{0}, \mathbf{N}-\mathbf{1}]} H_q\big([\mathbf{N}]^{\mathbf{H}} \Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})\big), \tag{2.2}$$

where $\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})$ stands for the generalised increments of $B^{\mathbf{H}}$ on the block

$$\Delta_{\mathbf{i},\mathbf{N}} := \left[\frac{\mathbf{i}}{\mathbf{N}}, \frac{\mathbf{i}+\mathbf{1}}{\mathbf{N}}\right] = \prod_{k=1}^{d} \left[\frac{i_k}{N_k}, \frac{i_k+1}{N_k}\right]$$

given by

$$\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}) := \sum_{\epsilon \in \{0,1\}^d} (-1)^{d - \sum_{k=1}^{d} \epsilon_k} B^{\mathbf{H}}_{(i_1+\epsilon_1)/N_1, \ldots, (i_d+\epsilon_d)/N_d}.$$

Note that $\mathbb{E}[\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})^2] = [\mathbf{N}]^{-2\mathbf{H}}$ (see Lemma 4.1) so that $[\mathbf{N}]^{\mathbf{H}}\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})$ has L^2-norm equal to one. For instance, when $d = 2$, we recover (with obvious changes in the notation) the Hermite variations of fBs investigated in [11], see (1.1). Our main result concerns the behaviour of the fBs in the high frequencies regime. In this case, the limit expresses in terms of a Hermite-type random variable $Z = Z^{(q)}(1, \ldots, 1)$ which is the value at $(1, \ldots, 1)$ of a d-parametric Hermite-type sheet defined by $Z^{(q)}(\mathbf{t}) = I_q\big(h(\mathbf{t})\big)$, $\mathbf{t} \in [0,1]^d$, where $h(\mathbf{t})$ is the limit in $(\mathfrak{H}^{\mathbf{H}})^{\odot q}$ of the expression

$$h_{\mathbf{N}}(\mathbf{t}) = \frac{[\mathbf{N}]^{q-1}}{q!} \sum_{\mathbf{i} \in [\mathbf{0}, (\mathbf{N}-\mathbf{1})\mathbf{t}]} \mathbf{1}^{\otimes q}_{\Delta_{\mathbf{i},\mathbf{N}}}.$$

In particular, $Z = I_q\big(h(1, \ldots, 1)\big)$. Note that, here and after, the limit $\mathbf{N} \to +\infty$ means $N_k \to +\infty$ for each $k = 1, \ldots, d$. We refer to Proposition 3.1 below for a proof of the convergence of $h_{\mathbf{N}}(\mathbf{t})$ when $\mathbf{t} = (1, \ldots, 1)$; see also [11], Remark 2, when $d = 2$.

THEOREM 2.1. *Let $q \geqslant 2$ and $B^{\mathbf{H}}$ be an fBs with d parameters and multi-Hurst index $\mathbf{H} \in \big(\mathbf{1} - \mathbf{1}/(2q), \mathbf{1}\big)$ (i.e. $H_k \in \big(1 - 1/(2q), 1\big)$, $k = 1, \ldots, d$). Then, as $\mathbf{N} \to +\infty$,*

$$d_{TV}\big(\mathfrak{L}([\mathbf{N}]^{q-1-q\mathbf{H}} V_{\mathbf{N}}), \mathfrak{L}(Z)\big) \leqslant O\Big(\sum_{k=1}^{d} N_k^{(2q-1-2qH_k)/(2q)}\Big). \tag{2.3}$$

REMARK 2.1.

• When $d = 1$, (2.3) recovers the non-central case of [2]. When $d = 2$, the inequality (2.3) completes the rate of convergence of [11].

• The d-parametric Hermite-type sheet $(Z_{\mathbf{t}}^{(q)})_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]}$ enjoys analogous properties as in [11] where $d = 2$. In particular, its covariance structure is given by

$$\mathbb{E}[Z_{\mathbf{t}}^{(q)} Z_{\mathbf{s}}^{(q)}] = K_{q(\mathbf{H}-\mathbf{1})+\mathbf{1}}(\mathbf{t}, \mathbf{s})$$

and is similar to that of the fBs. In particular, it is self-similar with (multi) exponent $q(\mathbf{H} - \mathbf{1}) + \mathbf{1}$, it has stationary increments and has Hölder continuous path of (multi) order $\mathbf{H}' < \mathbf{H}$. See [11], Proposition 1. The proof is a straightforward adaptation of that of Proposition 1 in [11] and is left to the reader. Moreover, the Hermite-type random variable $Z = Z_{1,\dots,1}^{(q)}$ appearing at the limit in (2.3) is a q-multiple Wiener–Itô integral $I_q(h)$ whose $L^2(\Omega)$-norm is given in (3.6) below.

• The anisotropic fBs is one of several possible extensions of fBm with multi-parameters considered in the literature; see references in [11]. The product structure of the covariance in (2.1) is the crucial characteristic of the fBs considered, that allows, roughly speaking, to describe the behaviour of the (Hermite variation of the) fBs as the "tensorisation" of the behaviour of the fBm resulting in each axis direction.

3. SET-UP AND PROOF

The proof follows similar lines to those in [2] with a special attention paid to the order of complicated multiple sums of multiple products: roughly speaking, it consists in rewriting $V_{\mathbf{N}}$ in (2.2) as a Wiener–Itô integral with respect to fBs and in using the control of the distance in total variation of the law of Wiener–Itô integrals (see Theorem 3.1).

In the sequel, we freely use some properties of Malliavin calculus for fBs. We refer to [10] for any details regarding Malliavin calculus or to [11], [13] in the special context of fBs. We denote by $\mathfrak{H}^{\mathbf{H}}$ the canonical Hilbert space generated by the Gaussian process $B^{\mathbf{H}}$. It is defined as the closure of the linear space spanned by indicator functions on $[0, 1]^d$ with respect to the scalar product $\langle \mathbf{1}_{[\mathbf{0},\mathbf{t}]}, \mathbf{1}_{[\mathbf{0},\mathbf{s}]} \rangle_{\mathfrak{H}^{\mathbf{H}}} = K_{\mathbf{H}}(\mathbf{t}, \mathbf{s})$. Note that the mapping $\mathbf{1}_{[\mathbf{0},\mathbf{t}]} \mapsto B_{\mathbf{t}}^{\mathbf{H}}$ extends to an isometry between $\mathfrak{H}^{\mathbf{H}}$ and the space generated by $B^{\mathbf{H}}$ (the so-called first chaos of $B^{\mathbf{H}}$) so that most of the computations for $B^{\mathbf{H}}$ are transferred in the sequel to the Hilbert space $\mathfrak{H}^{\mathbf{H}}$. We denote by $(\mathfrak{H}^{\mathbf{H}})^{\otimes q}$ (resp., $(\mathfrak{H}^{\mathbf{H}})^{\odot q}$) the q-tensor product (resp., symmetric tensor product) of $\mathfrak{H}^{\mathbf{H}}$.

In the sequel, $I_q(f)$ denotes the q-multiple Wiener–Itô integral of $f \in (\mathfrak{H}^{\mathbf{H}})^{\odot q}$ with respect to $B^{\mathbf{H}}$. In this framework, we rewrite $\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}) = I_1(\mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}})$ and, since $H_q\big(I_1(h)\big) = (1/q!) I_q(h^{\otimes q})$ when $\|h\|_{\mathfrak{H}^{\mathbf{H}}} = 1$, we have

$$[\mathbf{N}]^{q-1-q\mathbf{H}} V_{\mathbf{N}} = I_q(h_{\mathbf{N}})$$

with

$$h_{\mathbf{N}} = \frac{[\mathbf{N}]^{q-1}}{q!} \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]} \mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}}^{\otimes q}. \tag{3.1}$$

PROPOSITION 3.1. *The sequence* $(h_{\mathbf{N}})_{\mathbf{N}}$ *(resp.,* $I_q(h_{\mathbf{N}})$*) is a Cauchy sequence in* $(\mathfrak{H}^{\mathbf{H}})^{\odot q}$ *(resp., in* $L^2(\Omega)$*).*

Proof. Set $a(\mathbf{H}) = \mathbf{H}(2\mathbf{H}-1) := \prod_{k=1}^d H_k(2H_k-1)$. We have

$$\begin{aligned}
\langle h_{\mathbf{N}}, h_{\mathbf{M}}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} &= \frac{[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2}\Big\langle \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]} \mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}}^{\otimes q}, \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{M}-\mathbf{1}]} \mathbf{1}_{\Delta_{\mathbf{j},\mathbf{M}}}^{\otimes q}\Big\rangle \\
&= \frac{[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2} \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{M}-\mathbf{1}]} \langle \mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}}^{\otimes q}, \mathbf{1}_{\Delta_{\mathbf{j},\mathbf{M}}}^{\otimes q}\rangle \\
&= \frac{[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2} \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{M}-\mathbf{1}]} \langle \mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}}, \mathbf{1}_{\Delta_{\mathbf{j},\mathbf{M}}}\rangle_{\mathfrak{H}^{\mathbf{H}}}^q \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2} \\
&\quad\times \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{M}-\mathbf{1}]} \Big(\int_{[0,1]^{2d}} \mathbf{1}_{\Delta_{\mathbf{i},\mathbf{N}}}(\mathbf{u})\mathbf{1}_{\Delta_{\mathbf{j},\mathbf{M}}}(\mathbf{v})|\mathbf{u}-\mathbf{v}|^{2\mathbf{H}-2} d\mathbf{u}d\mathbf{v}\Big)^q \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2} \\
&\quad\times \sum_{\mathbf{i}\in[\mathbf{0},\mathbf{N}-\mathbf{1}]}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{M}-\mathbf{1}]} \prod_{k=1}^d \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/M_k}^{(j_k+1)/M_k} |u_k-v_k|^{2H_k-2} du_k dv_k\Big)^q \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{q-1}[\mathbf{M}]^{q-1}}{(q!)^2} \\
&\quad\times \prod_{k=1}^d \sum_{i_k=0}^{N_k-1}\sum_{j_k=0}^{M_k-1} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/M_k}^{(j_k+1)/M_k} |u_k-v_k|^{2H_k-2} du_k dv_k\Big)^q,
\end{aligned} \tag{3.2}$$

which converges to

$$\frac{a(\mathbf{H})^q}{(q!)^2} \prod_{k=1}^d \int_{[0,1]^2} |u_k-v_k|^{2qH_k-2q} du_k dv_k \tag{3.3}$$

as $\mathbf{N},\mathbf{M}\to+\infty$. Finally, $\big(I_q(h_{\mathbf{N}})\big)_{\mathbf{N}}$ is a Cauchy sequence since, by isometry,

$$\begin{aligned}
\|I_q(h_{\mathbf{N}}) - I_q(h_{\mathbf{M}})\|_{L^2(\Omega)}^2 &= \|h_{\mathbf{N}} - h_{\mathbf{M}}\|_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}^2 \\
&= \|h_{\mathbf{N}}\|_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}^2 + \|h_{\mathbf{M}}\|_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}^2 - 2\langle h_{\mathbf{N}}, h_{\mathbf{M}}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}
\end{aligned}$$

goes to zero as $\mathbf{N},\mathbf{M}\to+\infty$. ■

We derive the rate of convergence (2.3) by applying the following result of Davydov and Martynova [5]:

THEOREM 3.1 (cf. [5]). *Let $q \geqslant 2$ be fixed and $f \in \mathfrak{H}^{\odot q} \setminus \{0\}$. Then, for any sequence $(f_n)_{n\geqslant 1} \subset \mathfrak{H}^{\odot q}$ converging to f, there exists a constant $c_{q,f}$, depending only on q and f, such that*:

$$d_{TV}\Big(\mathfrak{L}\big(I_q(f_n)\big), \mathfrak{L}\big(I_q(f)\big)\Big) \leqslant c_{q,f} \|f_n - f\|_{\mathfrak{H}^{\odot q}}^{1/q}.$$

Proof of Theorem 2.1. In the sequel, it remains to assess $\|h_{\mathbf{N}} - h\|$. To that purpose, we have

$$\|h_{\mathbf{N}} - h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} = \|h_{\mathbf{N}}\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} - 2\langle h_{\mathbf{N}}, h\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} + \|h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}. \tag{3.4}$$

Then (3.2) and (3.3) give successively:

$$\|h_{\mathbf{N}}\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} \tag{3.5}$$
$$= \frac{a(\mathbf{H})^q [\mathbf{N}]^{2q-2}}{(q!)^2} \prod_{k=1}^d \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/N_k}^{(j_k+1)/N_k} |u_k - v_k|^{2H_k-2} du_k dv_k\Big)^q,$$

$$\|h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} = \frac{a(\mathbf{H})^q}{(q!)^2} \prod_{k=1}^d \int_{[0,1]^2} |u_k - v_k|^{2qH_k-2q} du_k dv_k, \tag{3.6}$$

(3.7)
$$\|h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} = \frac{a(\mathbf{H})^q}{(q!)^2} \prod_{k=1}^d \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/N_k}^{(j_k+1)/N_k} |u_k - v_k|^{2qH_k-2q} du_k dv_k.$$

Similar computations yield, for $\phi = \phi_1 \otimes \ldots \otimes \phi_d$, $\phi_k \in \mathfrak{H}^{H_k}$,

$$\langle h_{\mathbf{N}}, \phi^{\otimes q}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}}$$
$$= \frac{a(\mathbf{H})^q [\mathbf{N}]^{q-1}}{q!} \prod_{k=1}^d \sum_{i_k=0}^{N_k-1} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} \int_{[0,1]} |u_k - v_k|^{2H_k-2} \phi_k(v_k) du_k dv_k\Big)^q$$

and, taking the limit as $\mathbf{N} \to +\infty$, we get

$$\langle h, \phi^{\otimes q}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} = \frac{a(\mathbf{H})^q}{q!} \prod_{k=1}^d \int_{[0,1]} \Big(\int_{[0,1]} |u_k - v_k|^{2H_k-2} \phi_k(v_k) dv_k\Big)^q du_k,$$

from which we derive

$$
\begin{aligned}
&\langle h, h_{\mathbf{N}}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} \\
&= \frac{a(\mathbf{H})^q}{(q!)^2}[\mathbf{N}]^{q-1} \sum_{\mathbf{i}\in[0,\mathbf{N}-1]} \prod_{k=1}^{d} \int_{[0,1]} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} |u_k - v_k|^{2H_k-2} dv_k\Big)^q du_k \\
&= \frac{a(\mathbf{H})^q}{(q!)^2}[\mathbf{N}]^{q-1} \sum_{\mathbf{i}\in[0,\mathbf{N}-1]} \prod_{k=1}^{d} \sum_{j=0}^{N_k-1} \int_{j_k/N_k}^{(j_k+1)/N_k} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} |u_k - v_k|^{2H_k-2} dv_k\Big)^q du_k,
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
\text{(3.8)}\quad \langle h, h_{\mathbf{N}}\rangle_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} &= \frac{a(\mathbf{H})^q}{(q!)^2}[\mathbf{N}]^{q-1} \\
&\times \sum_{\mathbf{i}\in[0,\mathbf{N}-1]} \sum_{\mathbf{j}\in[0,\mathbf{N}-1]} \prod_{k=1}^{d} \int_{j_k/N_k}^{(j_k+1)/N_k} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} |u_k - v_k|^{2H_k-2} dv_k\Big)^q du_k.
\end{aligned}
$$

Plugging (3.5), (3.7), (3.8) in (3.4), we have

$$
\begin{aligned}
&\|h_N - h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{2q-2}}{(q!)^2} \prod_{k=1}^{d} \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/N_k}^{(j_k+1)/N_k} |u_k - v_k|^{2H_k-2} du_k dv_k\Big)^q \\
&\quad - 2\frac{a(\mathbf{H})^q}{(q!)^2}[\mathbf{N}]^{q-1} \\
&\quad \times \sum_{\mathbf{i}\in[0,\mathbf{N}-1]} \sum_{\mathbf{j}\in[0,\mathbf{N}-1]} \prod_{k=1}^{d} \int_{j_k/N_k}^{(j_k+1)/N_k} \Big(\int_{i_k/N_k}^{(i_k+1)/N_k} |u_k - v_k|^{2H_k-2} dv_k\Big)^q du_k \\
&\quad + \frac{a(\mathbf{H})^q}{(q!)^2} \prod_{k=1}^{d} \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \int_{i_k/N_k}^{(i_k+1)/N_k} \int_{j_k/N_k}^{(j_k+1)/N_k} |u_k - v_k|^{2qH_k-2q} du_k dv_k \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{2q-2-2q\mathbf{H}}}{(q!)^2} \Big\{ \prod_{k=1}^{d} \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \Big(\int_0^1\int_0^1 |i_k - j_k + x_k - y_k|^{2H_k-2} dx_k dy_k\Big)^q \\
&\quad - 2\prod_{k=1}^{d} \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \int_0^1 \Big(\int_0^1 |i_k - j_k + x_k - y_k|^{2H_k-2} dy_k\Big)^q dx_k \\
&\quad + \prod_{k=1}^{d} \sum_{i_k=0}^{N_k-1} \sum_{j_k=0}^{N_k-1} \int_0^1\int_0^1 |i_k - j_k + x_k - y_k|^{2qH_k-2q} dx_k dy_k \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{2q-2-2q\mathbf{H}}}{(q!)^2} \\
&\quad \times \Big\{ \prod_{k=1}^{d} \sum_{|r_k|\leqslant N_k-1} (N_k - |r_k|)\big(\int_0^1\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dx_k dy_k\big)^q \\
&\quad - 2\prod_{k=1}^{d} \sum_{|r_k|\leqslant N_k-1} (N_k - |r_k|)\int_0^1 \big(\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dy_k\big)^q dx_k \\
&\quad + \prod_{k=1}^{d} \sum_{|r_k|\leqslant N_k-1} (N_k - |r_k|)\int_0^1\int_0^1 |r_k + x_k - y_k|^{2qH_k-2q}dx_k dy_k \Big\} \\
&= \frac{a(\mathbf{H})^q[\mathbf{N}]^{2q-2-2q\mathbf{H}}}{(q!)^2} \\
&\quad \times \Big\{ \sum_{|\mathbf{r}|\leqslant \mathbf{N}-1} \prod_{k=1}^{d} (N_k - |r_k|)\big(\int_0^1\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dx_k dy_k\big)^q \\
&\quad - 2\sum_{|\mathbf{r}|\leqslant \mathbf{N}-1} \prod_{k=1}^{d} (N_k - |r_k|)\int_0^1 \big(\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dy_k\big)^q dx_k \\
&\quad + \sum_{|\mathbf{r}|\leqslant \mathbf{N}-1} \prod_{k=1}^{d} (N_k - |r_k|)\int_0^1\int_0^1 |r_k + x_k - y_k|^{2qH_k-2q}dx_k dy_k \Big\}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
(3.9)\quad \|h_N - h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} &= \frac{a(\mathbf{H})^q[\mathbf{N}]^{2q-1-2q\mathbf{H}}}{(q!)^2} \\
&\quad \times \sum_{|\mathbf{r}|\leqslant \mathbf{N}-1} \left[\mathbf{1} - \frac{|\mathbf{r}|}{\mathbf{N}}\right] \Big\{ \prod_{k=1}^{d} \big(\int_0^1\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dx_k dy_k\big)^q \\
&\quad - 2\prod_{k=1}^{d} \int_0^1 \big(\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dy_k\big)^q dx_k \\
&\quad + \prod_{k=1}^{d} \int_0^1\int_0^1 |r_k + x_k - y_k|^{2qH_k-2q}dx_k dy_k \Big\}.
\end{aligned}
$$

Explicit computations (see [2] for details) ensure

$$
\begin{aligned}
I_1(r_k) &:= \big(\int_0^1\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dx_k dy_k\big)^q = r_k^{2qH_k-2q}\big(1 + u_k(r_k)\big), \\
I_2(r_k) &:= \int_0^1 \big(\int_0^1 |r_k + x_k - y_k|^{2H_k-2}dy_k\big)^q dx_k = r_k^{2qH_k-2q}\big(1 + w_k(r_k)\big), \\
I_3(r_k) &:= \int_0^1\int_0^1 |r_k + x_k - y_k|^{2qH_k-2q}dx_k dy_k = r_k^{2qH_k-2q}\big(1 + v_k(r_k)\big),
\end{aligned}
$$

where

$$u_k(r_k) = O(1/r_k^2), \quad v_k(r_k) = O(1/r_k^2), \quad w_k(r_k) = O(1/r_k).$$

As a consequence, we get

$$\begin{aligned}
&\prod_{k=1}^d I_1(r_k) - 2\prod_{k=1}^d I_2(r_k) + \prod_{k=1}^d I_3(r_k)\\
&= \prod_{k=1}^d r_k^{2qH_k-2q}\big(1+u_k(r_k)\big) - 2\prod_{k=1}^d r_k^{2qH_k-2q}\big(1+w_k(r_k)\big)\\
&\quad + \prod_{k=1}^d r_k^{2qH_k-2q}\big(1+v_k(r_k)\big)\\
&= \prod_{k=1}^d r_k^{2qH_k-2q}\Big[\prod_{k=1}^d \big(1+u_k(r_k)\big) - 2\prod_{k=1}^d \big(1+w_k(r_k)\big) + \prod_{k=1}^d \big(1+v_k(r_k)\big)\Big]\\
&= \sum_{\epsilon\in\{0,1\}^d\setminus\{0\}} O([\mathbf{r}]^{2q\mathbf{H}-2q-\epsilon}).
\end{aligned}$$

Next, note that

$$\begin{aligned}
\sum_{|\mathbf{r}|\leqslant \mathbf{N}-1}\left[\mathbf{1}-\frac{|\mathbf{r}|}{\mathbf{N}}\right]\times[\mathbf{r}]^{2q\mathbf{H}-2q-\epsilon} &= \prod_{k=1}^d\left(\sum_{|r_k|\leqslant N_k-1}\left(1-\frac{|r_k|}{N_k}\right) r_k^{2qH_k-2q-\epsilon_k}\right)\\
&= \prod_{k=1}^d O\big(1\vee (N_k^{2qH_k-2q-\epsilon_k+1})\big)\\
&= \prod_{k=1}^d O(N_k^{(2qH_k-2q-\epsilon_k+1)\vee 0}).
\end{aligned}$$

Plugging this latter bound in (3.9), we have

$$\|h_{\mathbf{N}}-h\|^2_{(\mathfrak{H}^{\mathbf{H}})^{\odot q}} = \sum_{\epsilon\in\{0,1\}^d\setminus\{0\}} O\Big(\prod_{k=1}^d N_k^{-\varepsilon_k\vee(2q-1-2qH_k)}\Big) \tag{3.10}$$

and combining (3.10) with Theorem 3.1, we obtain

$$d_{TV}\big(\mathfrak{L}([\mathbf{N}]^{q-1-q\mathbf{H}}V_{\mathbf{N}}),\mathfrak{L}(Z)\big) \leqslant \sum_{\epsilon\in\{0,1\}^d\setminus\{0\}} O\Big(\prod_{k=1}^d N_k^{[-\varepsilon_k\vee(2q-1-2qH_k)]/2q}\Big).$$

Note that $2q-1-2qH_k \in (-1,0)$. Consequently, when $\epsilon_k = 1$, the exponent of N_k is, in fact, $(2q-1-2qH_k)/(2q)$. Since $\epsilon \neq 0$, this is the case at least for one exponent and the bound goes indeed to 0 as $\mathbf{N}\to+\infty$. This yields (2.3) and concludes the proof of Theorem 2.1. ∎

4. APPENDIX

LEMMA 4.1. *The generalised increment of order* d *of the fBs has the following second moment on the block* $[\mathbf{i}/\mathbf{N}, (\mathbf{i}+\mathbf{1})/\mathbf{N}]$: $\mathbb{E}[\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})^2] = [\mathbf{N}]^{-2\mathbf{H}}$.

P r o o f. We proceed by induction on the order d. When $d = 1$, the fBs $B^{\mathbf{H}}$ reduces to the fBm B^H, and the generalised increments $\Delta_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})$ to the usual one-dimensional increment $B^H_{(i+1)/N} - B^H_{i/N}$ of B^H for which it follows that $\mathbb{E}[(B^H_{(i+1)/N} - B^H_{i/N})^2] = 1/N^{2H}$ is straightforward.

Next, in order to reason by induction on d, we add in this proof a superscript d in $\Delta^d_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})$ to indicate the order of the increments. By splitting the sum $\sum_{\epsilon\in\{0,1\}^{d-1}}$ according as $\epsilon_d = 1$ or $\epsilon_d = 0$, the increments of order d express in terms of a difference of increments of order $d-1$ with a frozen dth parameter, i.e. with obvious notation:

$$\Delta^d_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}) = \Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,(i_d+1)/N_d}) - \Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,i_d/N_d}).$$

We have

$$\begin{aligned}
(4.1)\quad \mathbb{E}\big[\big(\Delta^d_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})\big)^2\big] &= \mathbb{E}\big[\big(\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,(i_d+1)/N_d})\big)^2\big] + \mathbb{E}\big[\big(\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,i_d/N_d})\big)^2\big] \\
&\quad - 2\mathbb{E}[\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,(i_d+1)/N_d})\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,i_d/N_d})] \\
&= \prod_{k=1}^{d-1}\left(\frac{1}{N_k}\right)^{2H_k}\left(\frac{i_d+1}{N_d}\right)^{2H_d} + \prod_{k=1}^{d-1}\left(\frac{1}{N_k}\right)^{2H_k}\left(\frac{i_d}{N_d}\right)^{2H_d} \\
&\quad - 2\mathbb{E}[\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,(i_d+1)/N_d})\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,i_d/N_d})].
\end{aligned}$$

Considering $B^{H_1,\dots,H_{d-1}}$, a $(d-1)$-parametric fBs and using again the induction hypothesis, we have

$$\begin{aligned}
(4.2)\quad \mathbb{E}[\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,(i_d+1)/N_d})\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}}_{\cdot,i_d/N_d})] &= \sum_{\epsilon,\epsilon'\in\{0,1\}^{d-1}}(-1)^{\sum_{k=1}^{d-1}\epsilon_k+\epsilon'_k} \\
\times\, \mathbb{E}[B^{\mathbf{H}}_{(i_1+\epsilon_1)/N_1,\dots,(i_{d-1}+\epsilon_{d-1})/N_{d-1},(i_d+1)/N_d} & B^{\mathbf{H}}_{(i_1+\epsilon'_1)/N_1,\dots,(i_{d-1}+\epsilon'_{d-1})/N_{d-1},i_d/N_d}] \\
= \sum_{\epsilon,\epsilon'\in\{0,1\}^{d-1}}(-1)^{\sum_{k=1}^{d-1}\epsilon_k+\epsilon'_k} & \\
\times\, \mathbb{E}[B^{H_1,\dots,H_{d-1}}_{(i_1+\epsilon_1)/N_1,\dots,(i_{d-1}+\epsilon_{d-1})/N_{d-1}} & B^{H_1,\dots,H_{d-1}}_{(i_1+\epsilon'_1)/N_1,\dots,(i_{d-1}+\epsilon'_{d-1})/N_{d-1}}] \\
\times\, K_{H_d}\left(\frac{i_d+1}{N_d},\frac{i_d}{N_d}\right) &
\end{aligned}$$

$$= \mathbb{E}[\Delta^{d-1}_{\mathbf{i},\mathbf{N}}(B^{H_1,\ldots,H_{d-1}})^2]K_{H_d}\left(\frac{i_d+1}{N_d}, \frac{i_d}{N_d}\right)$$
$$= \prod_{k=1}^{d-1}\left(\frac{1}{N_k}\right)^{2H_k} \frac{(i_k+1)^{2H_d}+i_k^{2H_d}-1}{2N_d^{H_d}}.$$

Combining (4.1) and (4.2) yields $\mathbb{E}\big[\big(\Delta^d_{\mathbf{i},\mathbf{N}}(B^{\mathbf{H}})\big)^2\big] = \prod_{k=1}^{d}(1/N_k)^{2H_k}$. ■

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