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CONTINUITY OF SCALE PARAMETER ESTIMATORS WITH RESPECT TO STOCHASTIC ORDERS

BY

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Abstract. Lehmann and Rojo [8] proposed a concept of invariance of stochastic orders and related probability metrics with respect to increasing transformations of random variables. Bartoszewicz and Benduch [3] and Bartoszewicz and Frąszczak [4] applied a concept of Lehmann and Rojo to new settings. In the paper these results are applied to the problem of robustness in the sense of Zieliński [11], [12]. Metrics related to some stochastic orders are used to study the continuity (robustness) of scale parameter estimators when contaminations of the models are generated by stochastic orders. The exponential model is considered in detail.

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1. PRELIMINARIES

Let X and Y be two random variables, F and G their respective probability distribution functions, and f and g their respective density functions, if they exist. Denote by $\overline{F} = 1 - F$ the tail (or survival function) of F, by $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $u \in (0,1)$, the quantile function, and by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper respective bounds of the support of F being an interval; for G analogously. We identify distribution functions F and G with respective probability distributions. We denote by fF^{-1} the composition of f and F^{-1}, and by $G^{-1}F$ the composition of G^{-1} and F (similarly for other densities, distributions and quantile functions). We use *increasing* in place of *nondecreasing*, and *decreasing* in place of *nonincreasing*.

Let $(X_1, X_2, \ldots, X_n)$ be a sample from the distribution F. Let $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ be order statistics of the sample. We denote by $F_{i:n}$ the distribution function of $X_{i:n}$ and by $f_{i:n}$ its density, if exists, $i = 1, 2, \ldots, n$.

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Let $\mathcal{F}$ denote a class of distributions on $\mathbb{R}$. Define a preorder $\mathcal{S}$ of distributions from $\mathcal{F}$ as follows:

$$F \leqslant_S F \quad \text{for all } F \in \mathcal{F},$$

$$F \leqslant_S G \text{ and } G \leqslant_S H \Rightarrow F \leqslant_S H.$$

In the literature of the subject one defines equivalently preorders between relative random variables and uses the notation $X \leqslant_S Y$ if and only if $F \leqslant_S G$.

Some of the preorders will be considered in this paper. Recall their definitions and some properties for completeness. For more details and properties we refer to Shaked and Shanthikumar [10].

We say that F is *smaller than G in the likelihood ratio order* ($F \leqslant_{\text{lr}} G$) if $g(x)/f(x)$ is increasing. F is said to be *smaller than G in the hazard rate order* ($F \leqslant_{\text{hr}} G$) if $\overline{G}(x)/\overline{F}(x)$ is increasing or $r_F(x) \geqslant r_G(x)$ for every x if F and G are absolutely continuous, where $r_F(x) = f(x)/\overline{F}(x)$ is the *hazard rate function* of F (for r_G analogously). We say that F is *stochastically smaller than G* ($F \leqslant_{\text{st}} G$) if $F(x) \geqslant G(x)$ for every x or, equivalently, $\overline{F}(x) \leqslant \overline{G}(x)$ for every x.

We say that F is *smaller than G in the dispersive order* ($F \leqslant_{\text{disp}} G$) if it follows that $G^{-1}F(x) - x$ is increasing in x.

Lehmann and Rojo [8] proposed a concept of invariant stochastic orders and invariant probabilistic metrics consistent with the stochastic orders.

Let $\mathbf{\Psi}$ be a class of operations $\psi : \mathcal{F} \to \mathcal{F}$. The preorder $\mathcal{S}$ is *invariant* under $\mathbf{\Psi}$ if

$$(F \leqslant_S G) \Rightarrow \big(\psi(F) \leqslant_S \psi(G)\big) \quad \text{for all } \psi \in \mathbf{\Psi}.$$

The metric $d_S(F,G)$, invariant under transforms from $\mathbf{\Psi}$, satisfies the following conditions:

(a) if $F \leqslant_S G$ and $G \leqslant_S H$, then $d_S(F,G) \leqslant d_S(F,H)$;

(b) $d_S(F,G) = d_S\big(\psi(F),\psi(G)\big)$ for all $\psi \in \mathbf{\Psi}$.

Lehmann and Rojo [8] considered the class $\mathbf{\Phi}$ of all continuous increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$ of random variables X and Y. Then we define:

$$\psi(F) = F\varphi^{-1}, \quad F \in \mathcal{F}.$$

It is well known (see, e.g., Shaked and Shanthikumar [10]) that the likelihood, hazard rate and usual stochastic orders are invariant under these transformations. Rojo and Lehmann [8] defined and studied the following metrics related to these orders:

$$d_{\text{lr}}(F,G) = \sup_{x\in\mathbb{R}} \left|\log \frac{g(x)}{f(x)}\right|,$$

$$d_{\text{hr}}(F,G) = \sup_{x\in\mathbb{R}} \left|\log \frac{\bar{G}(x)}{\bar{F}(x)}\right|,$$

and

$$d_{\text{st}}(F,G) = \sup_{t\in(0,1)} |G^{-1}(t) - F^{-1}(t)|. \tag{1.1}$$

Bartoszewicz and Benduch [3] gave a characterization of some stochastic orders of absolutely continuous distributions on $[0, \infty)$ in terms of *invariance under the GTTT transform*

$$H_F^{-1}(t;\varphi) = \int_0^{F^{-1}(t)} \varphi F(x)dx, \quad F \in \mathcal{F},\ t \in (0,1),$$

with respect to the class $\mathbf{\Phi}_0$ of continuous functions $\varphi : (0,1) \to \mathbb{R}^+ \setminus \{0\}$. Here $\psi(F) = H_F(t;\varphi)$. Particularly, they proved that the dispersive order is invariant under the GTTT transform and the respective metric is of the form

$$d_{\mathrm{disp}}(F,G) = \sup_{t\in(0,1)} \left| \log \frac{fF^{-1}(t)}{gG^{-1}(t)} \right|.$$

Bartoszewicz and Frąszczak [4] considered also the invariance of some stochastic orders under transformations being compositions of distribution functions $\varphi \in \mathbf{\Phi}_1$ with supports on $[0,1]$ with underlying distributions, i.e. $\psi(F) = \varphi F$, $F \in \mathcal{F}$. Particularly, for the usual stochastic order and the dispersive one the respective metric is $d_{\mathrm{st}}(F,G)$ defined by (1.1).

We use the above metrics to define a continuity of estimators with respect to the stochastic orders. The invariance of these orders and metrics does not have any importance in our considerations.

2. RESULTS

2.1. Continuity of estimators with respect to stochastic orders. Zieliński [11] proposed the following concept of robustness. Consider a statistical model $M = (\mathcal{X}, \mathcal{A}, \mathcal{F}_0)$, $\mathcal{F}_0 \subset \mathcal{F}$, where $\mathcal{F}$ is the family of all probability distributions on the sample space $(\mathcal{X}, \mathcal{A})$. Define a function $\pi : \mathcal{F}_0 \to 2^{\mathcal{F}}$, called the *contamination of the model* M, satisfying the following two conditions: $F \in \pi(F)$, $F \in \mathcal{F}_0$, and $\pi(F) \cap \mathcal{F}_0 = \{F\}$. Let $\mathcal{F}_1 = \bigcup_{F\in\mathcal{F}_0} \pi(F)$. The model $M_1 = (\mathcal{X}, \mathcal{A}, \mathcal{F}_1)$ is called the *extension* of M or the *supermodel*.

Bartoszewicz [1], [2], studied robustness of estimators in some statistical models, using Zieliński's concept of robustness, when contaminations of the model are generated by some stochastic order S on $\mathcal{F}_1$. For $F \in \mathcal{F}_0$ we define the contamination of F generated by the order S as the set of distributions from $\mathcal{F}_1$:

$$\pi_{H,K}(F) = \{G : H \leqslant_S G \leqslant_S K\}, \tag{2.1}$$

where H and K are fixed distributions from $\mathcal{F}_1$, $F \in \pi_{H,K}(F)$ and $\pi_{H,K}(F) \cap \mathcal{F}_0 = \{F\}$; see [2] for details.

In the statistical theory of robustness in the sense of Huber [7] and Hampel et al. [6] there is a *continuity of estimators*, which plays an important role and means often *robustness*. It is necessary that distributions of an estimator in the model and in the extended (contaminated) model should not essentially differ when sample distributions do not essentially differ in these models. Such comparisons are usually due using neighborhoods (contaminations) generated by week topologies and probability metrics. To make the robustness considered by Zieliński [11] and Bartoszewicz [2], [3] more close to the robustness in the sense of Huber and Hampel we use probability metrics given in [8] as well as in [3] and [4] and propose the following definition of continuity of estimators with respect to a given stochastic order.

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a sample from the distribution $F \in \mathcal{F}_0$. The statistical model of the sample is of the product form $(\mathbb{R}, \mathcal{B}, \mathcal{F}_0)^n$, and therefore we take into account in our considerations simply the model $(\mathbb{R}, \mathcal{B}, \mathcal{F}_0)$ of one observation. Let $T = T(\boldsymbol{X})$ be an estimator (a statistical functional). Denote by $\mathcal{L}(T, F)$ the distribution of T and by $\mathcal{F}_1^T$ the set of these distributions when $F \in \mathcal{F}_1$. Let the stochastic order S and the relative metric d_S be defined on $\mathcal{F}_1$ as well as on $\mathcal{F}_1^T$. Denote by $\pi_{H,K}(F)$ a contamination of the model generated by the order S. Similarly as Hampel et al. [6], Chapter 2, Definition 3, we propose the following definition.

DEFINITION 2.1. The estimator T is *continuous* in the metric d_S with respect to the order $\mathcal{S}$ in the contamination $\pi_{H,K}(F)$ if for every distribution $F \in \mathcal{F}_0$ and $G \in \pi_{H,K}(F)$:

(i) $F \leqslant_S G \Rightarrow \mathcal{L}(T, F) \leqslant_S \mathcal{L}(T, G)$,

(ii) $\forall_{\varepsilon>0} \exists_{\delta>0}\, d_S(F, G) < \delta \Rightarrow d_S\big(\mathcal{L}(T, F), \mathcal{L}(T, G)\big) < \varepsilon$.

Let $(X_1, X_2, \ldots, X_n)$ be a sample from the distribution $F \in \mathcal{F}_0 = \{F(\cdot; \theta) : \theta > 0\}$, $F(x; \theta) = F(x/\theta)$, where F is a fixed distribution on the positive half-line, and the estimators $T_i = X_{i:n}/E(X_{i:n})$, $i = 1, 2, \ldots, n$, of the parameter θ. Let H and K be also fixed distributions on the positive half-line, and define the contamination $\pi_{H,K}(F) = \{G : H \leqslant_{\mathrm{st}} G \leqslant_{\mathrm{st}} K\}$ generated by the usual stochastic order and the contamination $\tilde{\pi}_{H,K}(F) = \{G : H \leqslant_{\mathrm{disp}} G \leqslant_{\mathrm{disp}} K\}$ generated by the dispersive order. We can formulate the following result.

THEOREM 2.1. *The estimators* $T_i = X_{i:n}/E_1(X_{i:n})$, $i = 1, 2, \ldots, n$, *are continuous in the metric* d_{st}:

(a) *with respect to the usual stochastic order in the contamination* $\pi_{H,K}(F)$;

(b) *with respect to the dispersive order in the contamination* $\tilde{\pi}_{H,K}(F)$.

Proof. The theorem is implied by preservation of these two orders by order statistics. Let G be any distribution from $\pi_{H,K}$ ($\tilde{\pi}_{H,K}$, respectively). Since $\mathcal{L}(X_{i:n}, F) = F_{i:n} = B_{i:n}F$ and $\mathcal{L}(X_{i:n}, G) = G_{i:n} = B_{i:n}G$, where $B_{i:n}$ is the beta distribution function $B(i, n-i+1)$, $i = 1, 2, \ldots, n$, we have

$$d_{\mathrm{st}}\big(\mathcal{L}(T_i, F), \mathcal{L}(T_i, G)\big) = d_{\mathrm{st}}(F, G), \quad i = 1, 2, \ldots, n.$$

Thus for every $\varepsilon > 0$ there exists $\delta = \varepsilon$ and for every $G \in \pi_{H,K}$ ($G \in \tilde{\pi}_{H,K}$, respectively) we have

$$d_{\mathrm{st}}(F,G) < \delta \Rightarrow d_{\mathrm{st}}\big(\mathcal{L}(T_i,F),\mathcal{L}(T_i,G)\big) < \varepsilon, \quad i = 1,2,\ldots,n,$$

and the proof is complete. ■

Let now the family $\mathcal{F}_0 = \{F(\cdot;\theta) : \theta > 0\}$, where $F(x;\theta) = F(x/\theta)$ for a fixed F on the positive half-line, be dominated by the Lebesgue measure. Let H and K be also fixed absolutely continuous distributions on the positive half-line, and define the contamination $\tilde{\pi}_{H,K}(F) = \{G : H \leqslant_{\mathrm{disp}} G \leqslant_{\mathrm{disp}} K\}$ generated by the dispersive order. The following theorem also holds.

THEOREM 2.2. *The estimators $T_i = X_{i:n}/E_1(X_{i:n})$, $i = 1,2,\ldots,n$, are continuous in the metric d_{disp} with respect to the dispersive orders in the contamination $\tilde{\pi}_{H,K}(F)$.*

Proof. Since $F_{i:n}(x) = B_{i:n}F(x)$, $x > 0$, $F_{i:n}^{-1}(t) = F^{-1}B_{i:n}^{-1}(t)$, $t \in [0,1]$, and the density of $F_{i:n}$ is equal to $f_{i:n}(x) = b_{i:n}F(x)f(x)$, we have

$$f_{i:n}F_{i:n}^{-1}(t) = b_{i:n}B_{i:n}^{-1}(t)fF^{-1}\big(B_{i:n}^{-1}(t)\big), \quad t \in [0,1],$$

where $b_{i:n}$ is the density of the distribution $B_{i:n}$. Similarly,

$$g_{i:n}G_{i:n}^{-1}(t) = b_{i:n}B_{i:n}^{-1}(t)gG^{-1}\big(B_{i:n}^{-1}(t)\big), \quad t \in [0,1].$$

Thus we have

$$\begin{aligned} d_{\mathrm{disp}}(F_{i:n},G_{i:n}) &= \sup_{t\in(0,1)}\left|\log\frac{f_{i:n}F_{i:n}^{-1}(t)}{g_{i:n}G_{i:n}^{-1}(t)}\right| = \sup_{t\in(0,1)}\left|\log\frac{fF^{-1}\big(B_{i:n}(t)\big)}{gG^{-1}\big(B_{i:n}(t)\big)}\right| \\ &= d_{\mathrm{disp}}(F,G). \end{aligned}$$

The metric d_{disp} is invariant with respect to multiplying by the same scale parameter, and then $d_{\mathrm{disp}}(F_{i:n},G_{i:n}) = d_{\mathrm{disp}}\big(\mathcal{L}(T_i,F),\mathcal{L}(T_i,G)\big)$. Therefore, for every $\varepsilon > 0$ there exists $\delta = \varepsilon$ and for every $G \in \tilde{\pi}_{F,K}$:

$$d_{\mathrm{disp}}(F,G) < \delta \Rightarrow d_{\mathrm{disp}}\big(\mathcal{L}(T_i,F),\mathcal{L}(T_i,G)\big) < \varepsilon, \quad i = 1,2,\ldots,n,$$

which completes the proof. ■

REMARK 2.1. The estimators given in Theorems 2.1 and 2.2 were constructed by Bartoszewicz [1] in the context of bias-robustness of estimators when contaminations of the model are generated by stochastic orders. He proved [1] that the statistic $T_n = X_{n:n}/E_1(X_{n:n})$ is the uniformly most robust estimate of the scale parameter θ in the class of nonnegative L-statistics, unbiased estimators in the model, under contaminations $\pi_{F,K}(F)$ and $\tilde{\pi}_{F,K}(F)$. Now we see that T_n is also continuous in the respective metrics with respect to considered orders.

2.2. Continuity of estimators in the exponential model. Let $(X_1, X_2, \ldots, X_n)$ be a sample from the distribution $F \in \mathcal{F}$, where $\mathcal{F}$ is the family of exponential distributions $F_1(x;\theta) = 1 - e^{-x/\theta}$, $x > 0$, with mean $\theta > 0$ and its contamination of the form

$$\pi_{\gamma_1,k}(\theta) = \{F_\gamma(x;\theta) = \gamma F_1(x;\theta) + (1-\gamma)F_1(x;\theta/k)\colon x > 0, 0 < \gamma_1 \leqslant \gamma \leqslant 1\},$$

where γ_1 is a fixed number from $(0, 1]$ and $k \geqslant 2$ is a fixed integer. The contamination $\pi_{\gamma_1,k}(\theta)$ is a set of mixtures of two exponential distributions with densities:

$$f_\gamma(x;\theta) = \frac{\gamma}{\theta}e^{-x/\theta} + \frac{k(1-\gamma)}{\theta}e^{-kx/\theta}, \quad x > 0, 0 < \gamma_1 \leqslant \gamma \leqslant 1. \tag{2.2}$$

It is obvious that any density of the form (2.2) is not exponential except for the case $\gamma = 1$. It is easy to verify that $\pi_{\gamma_1,k}(\theta)$ is ordered with respect to likelihood ratio order for every θ, i.e. $F_\gamma(\cdot;\theta) \leqslant_{\rm lr} F_{\gamma'}(\cdot;\theta)$ whenever $\gamma \leqslant \gamma'$, and therefore also $F_\gamma(\cdot;\theta) \leqslant_{\rm hr} F_{\gamma'}(\cdot;\theta)$ and $F_\gamma(\cdot;\theta) \leqslant_{\rm st} F_{\gamma'}(\cdot;\theta)$ for every $\theta > 0$. Thus $\pi_{\gamma_1,k}(\theta)$ is a contamination of the form (2.1) where $H = F_{\gamma_1}$ and $K = F_1$ for S being the likelihood ratio order, the hazard order and the usual stochastic one.

We shall prove the following result.

THEOREM 2.3. *The estimator $T_n = X_{n:n}/E_1(X_{n:n})$ is continuous in metrics $d_{\rm lr}$ and $d_{\rm hr}$ with respect to likelihood ratio and hazard orders in the contamination $\pi_{\gamma_1,k}(\theta)$.*

P r o o f. It is well known that order statistics preserve the likelihood ratio and hazard orders (see Shaked and Shanthikumar [10], Theorems 1.C.33 and 1.B.35). Then it suffices to verify the condition (ii) in Definition 2.1.

Let us calculate $d_{\rm lr}(F_1, F_\gamma)$, $\gamma \in [\gamma_1, 1)$. We have

$$\begin{aligned} \log\frac{f_\gamma(x;\theta)}{f_1(x;\theta)} &= \log\frac{\gamma\theta^{-1}e^{-x/\theta} + k(1-\gamma)\theta^{-1}e^{-kx/\theta}}{\theta^{-1}e^{-x/\theta}} \\ &= \log\,[\gamma + k(1-\gamma)e^{-(k-1)x/\theta}]. \end{aligned} \tag{2.3}$$

The function (2.3) is decreasing with maximum at zero and minimum at infinity, which are equal to

$$\log[\gamma + (1-\gamma)k] > 0 \quad \text{and} \quad \log\gamma < 0,$$

respectively. It suffices to verify when

$$|\log[\gamma + (1-\gamma)k]| \leqslant (>)\, |\log\gamma| = \left|-\log\frac{1}{\gamma}\right|,$$

which is equivalent to

$$k \leqslant (>) \frac{1+\gamma}{\gamma}.$$

Therefore

$$d_{\rm lr}(F_1,F_\gamma)=\sup_{x>0}\left|\log\frac{f_\gamma(x;\theta)}{f_1(x;\theta)}\right|=\begin{cases}\log[\gamma+(1-\gamma)k] & \text{for } k>(1+\gamma)/\gamma,\\ |\log\gamma| & \text{for } k\leqslant(1+\gamma)/\gamma.\end{cases}$$

Let $c_n=E_1(X_{n:n})=1+1/2+1/3+\ldots+1/n$ and F_{γ,T_n} be the distribution function of T_n when F_γ (particularly, $\gamma=1$) is the distribution of the observed random variable. Let f_{γ,T_n} be a density of F_{γ,T_n}. It is obvious that $F_{\gamma;T_n}(x;\theta)=F_{\gamma,n:n}(c_nx;\theta)=[F_\gamma(c_nx;\theta)]^n$ and

$$f_{\gamma;T_n}(x;\theta)=c_nf_{\gamma,n:n}(c_nx;\theta)=nc_nf_\gamma(c_nx;\theta)[F_\gamma(c_nx;\theta)]^{n-1}.$$

Hence after easy calculations we obtain

$$\begin{aligned}\frac{f_{\gamma,T_n}(x;\theta)}{f_{1,T_n}(x;\theta)}&=\left[\gamma+(1-\gamma)\frac{1-e^{-c_nkx/\theta}}{1-e^{-c_nx/\theta}}\right]^{n-1}[\gamma+k(1-\gamma)e^{-c_n(k-1)x/\theta}]\\&=\big[\gamma+(1-\gamma)\sum_{j=0}^{k-1}e^{-jc_nx/\theta}\big]^{n-1}[\gamma+k(1-\gamma)e^{-c_n(k-1)x/\theta}].\end{aligned}$$

The function $\log[f_{\gamma,T_n}(x;\theta)/f_{1,T_n}(x;\theta)]$ is decreasing with maximum at zero and minimum at infinity, which are equal to

$$\log\{[\gamma+(1-\gamma)k]^n\}>0\quad\text{and}\quad\log\gamma^n<0,$$

respectively. Similarly as previously we obtain

$$\begin{aligned}d_{\rm lr}(F_{1,T_n},F_{\gamma,T_n})&=\sup_{x>0}\left|\log\frac{f_{\gamma,T_n}(x;\theta)}{f_{1,T_n}(x;\theta)}\right|\\&=\begin{cases}n\log[\gamma+(1-\gamma)k] & \text{for } k>(1+\gamma)/\gamma,\\ n|\log\gamma| & \text{for } k\leqslant(1+\gamma)/\gamma.\end{cases}\end{aligned}$$

Thus $d_{\rm lr}(F_{1,T_n},F_{\gamma,T_n})=nd_{\rm lr}(F_1,F_\gamma)$ and for every $\varepsilon>0$ there exists $\delta=\varepsilon/n$ such that for every distribution $G\in\pi_{\gamma_1,k}(\theta)$ and every $\theta>0$,

$$d_{\rm lr}(F_1,G)<\delta\Rightarrow d_{\rm lr}\big(\mathcal{L}(T_n,F_1),\mathcal{L}(T_n,G)\big)<\varepsilon.$$

Now we verify the condition (ii) in Definition 2.1 for the metric $d_{\rm hr}$. We have

$$\begin{aligned}d_{\rm hr}(F_\gamma,F_1)&=\sup_{x>0}\left|\log\frac{\bar F_\gamma(x;\theta)}{\bar F_1(x;\theta)}\right|=\sup_{x>0}\left|\log\frac{\gamma e^{-x/\theta}+(1-\gamma)e^{-xk/\theta}}{e^{-x/\theta}}\right|\\&=\sup_{x>0}|\log[\gamma+(1-\gamma)e^{-x(k-1)/\theta}]|=\log\gamma^{-1}.\end{aligned}$$

It is easy to notice that $\bar{F}_{\gamma,T_n}(x;\theta) = 1 - [F_\gamma(c_n x;\theta)]^n$ with $c_n = E_1(X_{n:n})$ is the survival function of T_n when F_γ is the distribution of the observed random variable. The ratio

$$\frac{\bar{F}_{\gamma,T_n}(x;\theta)}{\bar{F}_{1,T_n}(x;\theta)} = \frac{1 - [\gamma(1 - e^{-xc_n/\theta}) + (1-\gamma)(1 - e^{-xc_n k/\theta})]^n}{1 - [1 - e^{-xc_n/\theta}]^n}$$

is decreasing with maximum equal to one at $x = 0$ and minimum equal to γ at infinity. Hence

$$d_{\mathrm{hr}}(F_{1,T_n}, F_{\gamma,T_n}) = \sup_{x>0} \left| \log \frac{\bar{F}_{\gamma,T_n}(x;\theta)}{\bar{F}_{1,T_n}(x;\theta)} \right| = \log \frac{1}{\gamma} = d_{\mathrm{hr}}(F_1, F_\gamma).$$

Thus for every $\varepsilon > 0$ there exists $\delta = \varepsilon$ and for every distribution $G \in \pi_{\gamma_1,k}(\theta)$ and every $\theta > 0$ we have

$$d_{\mathrm{hr}}(F_1, G) < \delta \Rightarrow d_{\mathrm{hr}}\big(\mathcal{L}(T_n, F_1), \mathcal{L}(T_n, G)\big) < \varepsilon,$$

which completes the proof. ■

REMARK 2.2. The estimator $X_{n:n}/E_1(X_{n:n})$ is the uniformly most bias-robust estimator of θ in the class of nonnegative linear combinations of order statistics, unbiased estimators in the exponential model, under the contamination $\pi_{\gamma_1,k}(\theta)$; see [2].

Consider a sample $(X_1, X_2, \ldots, X_n)$ from the distribution $F \in \mathcal{F}$, where $\mathcal{F}$ is the Marshall–Olkin family of distributions with survival function (see [9]),

$$\bar{F}_\alpha(t) = \frac{\alpha \bar{F}(t)}{1 - (1-\alpha)\bar{F}(t)}, \quad \alpha > 0,$$

where $F(t) = F_1(t;\theta) = 1 - e^{-t/\theta}$. In the sequel, the distribution F_α from this family will be denoted by

$$F_\alpha(t;\theta) = \frac{1 - e^{-t/\theta}}{1 - (1-\alpha)e^{-t/\theta}},$$

and its density by

$$f_\alpha(t;\theta) = \frac{\alpha\theta^{-1}e^{-t/\theta}}{[1 - (1-\alpha)e^{-t/\theta}]^2}.$$

Consider a contamination of the exponential model of the form:

$$\pi_{\alpha_0}(\theta) = \{F_\alpha(\cdot;\theta)\colon 0 < \alpha_0 \leqslant \alpha \leqslant 1\},$$

where α_0 is fixed. It is easy to verify that this family of distributions is ordered with respect to the likelihood ratio order for every θ, i.e. $F_\alpha(\cdot;\theta) \leqslant_{\mathrm{lr}} F_{\alpha'}(\cdot;\theta)$ if $\alpha \leqslant \alpha'$, and hence they are also ordered with respect to the hazard and usual stochastic order. From Theorem 1 in Benduch-Frąszczak [5] we immediately infer that $\pi_{\alpha_0}(\theta)$ is also ordered with respect to the dispersive order. Therefore, the contamination $\pi_{\alpha_0}(\theta)$ is of the form (2.1), where $H = F_{\alpha_0}$, $K = F_1$ and S being the likelihood ratio order or the dispersive one.

We shall prove the following result.

THEOREM 2.4. *The estimator* $T_n = X_{n:n}/E_1(X_{n:n})$ *is continuous*:

(i) *in the metric* d_{lr} *with respect to likelihood ratio order in the contamination* $\pi_{\alpha_0}(\theta)$;

(ii) *in the metric* d_{disp} *with respect to dispersive order in the contamination* $\pi_{\alpha_0}(\theta)$.

Proof. It is well known that order statistics preserve the likelihood ratio order as well as the dispersive one; see, e.g., Shaked and Shanthikumar [10]. Then it suffices to verify only the condition (ii) in Definition 2.1. First calculate $d_{\mathrm{lr}}(F_1, F_\alpha)$, $\alpha_0 \leqslant \alpha < 1$:

$$d_{\mathrm{lr}}(F_1, F_\alpha) = \sup_{x>0} \left| \log \frac{f_\alpha(x;\theta)}{f_1(x;\theta)} \right| = \sup_{x>0} \left| \log \frac{\alpha}{[1-(1-\alpha)e^{-x/\theta}]^2} \right| = \log \frac{1}{\alpha}.$$

Let now f_{α,T_n} denote a density of the distribution of T_n when the observed random variable has the distribution F_α. We have

$$\begin{aligned} f_{\alpha,T_n}(x;\theta) &= nc_n f_\alpha(c_n x;\theta)[F_\alpha(c_n x;\theta)]^{n-1} \\ &= \frac{n\alpha c_n e^{-xc_n/\theta}}{\theta[1-(1-\alpha)e^{-xc_n/\theta}]^2} \left[\frac{1-e^{-xc_n/\theta}}{1-(1-\alpha)e^{-xc_n/\theta}} \right]^{n-1}, \end{aligned}$$

where $c_n = E_1(X_{n:n})$. Thus

$$\frac{f_{\alpha,T_n}(x;\theta)}{f_{1,T_n}(x;\theta)} = \frac{\alpha}{[1-(1-\alpha)e^{-xc_n/\theta}]^{n+1}},$$

and

$$d_{\mathrm{lr}}(F_{\alpha,T_n}, F_{1,T_n}) = \sup_{x>0} \left| \log \frac{\alpha}{[1-(1-\alpha)e^{-xc_n/\theta}]^{n+1}} \right| = \log \frac{1}{\alpha^n} = n d_{\mathrm{lr}}(F_1, F_\alpha).$$

Hence for every $\varepsilon > 0$ there exists $\delta = \varepsilon/n$ such that for every distribution $G \in \pi_{\alpha_0}(\theta)$ and each $\theta > 0$ the following implication holds:

$$d_{\mathrm{lr}}(F_1, G) < \delta \Rightarrow d_{\mathrm{lr}}\big(\mathcal{L}(T_n, F_1), \mathcal{L}(T_n, G)\big) < \varepsilon.$$

The inverse of the density quantile function of F_1 is of the form

$$f_1F_1^{-1}(t;\theta)=\frac{1}{\theta}(1-t),\qquad t\in(0,1).$$

It is easy to calculate

$$f_\alpha F_\alpha^{-1}(t;\theta)=\frac{1}{\alpha(1-t+t\alpha)^2}f_1F_1^{-1}\bigg(\frac{\alpha t}{1-t+\alpha t};\theta\bigg)=\frac{1}{\alpha\theta}(1-t)(1-t+\alpha t).$$

Then we obtain

$$d_{\text{disp}}(F_1,F_\alpha)=\sup_{t\in(0,1)}\left|\log\frac{f_1F_1^{-1}(t;\theta)}{f_\alpha F_\alpha^{-1}(t;\theta)}\right|=\sup_{t\in(0,1)}\left|\log\frac{\alpha}{1-t+\alpha t}\right|=\log\frac{1}{\alpha}.$$

For the distributions of T_n, $F_{\alpha,T_n}(x;\theta)=[F_\alpha(c_nx;\theta)]^n$, we calculate the quantile function of the form

$$F_{\alpha,T_n}^{-1}(t;\theta)=\frac{1}{c_n}F_\alpha^{-1}(t^{1/n},\theta).$$

Therefore, we may calculate a quantile density of F_{α,T_n}:

$$f_{\alpha,T_n}F_{\alpha,T_n}^{-1}(t;\theta)=\frac{nc_n}{\alpha\theta}t^{(n-1)/n}(1-t^{1/n})(1-t^{1/n}+\alpha t^{1/n}),$$

and next obtain

$$\begin{aligned}d_{\text{disp}}(F_{1,T_n},F_{\alpha,T_n})&=\sup_{t\in(0,1)}\left|\log\frac{f_{1,T_n}F_{1,T_n}^{-1}(t;\theta)}{f_{\alpha,T_n}F_{\alpha,T_n}^{-1}(t;\theta)}\right|\\&=\sup_{t\in(0,1)}\left|\log\frac{\alpha}{1-t^{1/n}+\alpha t^{1/n}}\right|=\log\frac{1}{\alpha}=d_{\text{disp}}(F_1,F_\alpha).\end{aligned}$$

Hence for every $\varepsilon>0$ there exists $\delta=\varepsilon$ and for every distribution $G\in\pi_{\alpha_0}(\theta)$ and each $\theta>0$ the implication

$$d_{\text{disp}}(F_1,G)<\delta\Rightarrow d_{\text{disp}}\big(\mathcal{L}(T_n,F_1),\mathcal{L}(T_n,G)\big)<\varepsilon$$

holds. ∎

REMARK 2.3. Using methods presented in [2] one may also prove that the statistic $X_{n:n}/E_1(X_{n:n})$ is the uniformly most bias-robust estimator of θ in the class of nonnegative linear combinations of order statistics, unbiased in the exponential model, under the contamination $\pi_{\alpha_0}(\theta)$.

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