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ON THE INSTANTANEOUS FREQUENCY OF GAUSSIAN STOCHASTIC PROCESSES*

BY

PATRIK WAHLBERG (TURIN) AND PETER J. SCHREIER (PADERBORN)

Abstract. We study the instantaneous frequency (IF) of continuous-time, complex-valued, zero-mean, proper, mean-square differentiable, nonstationary Gaussian stochastic processes. We compute the probability density function for the IF for fixed time, which generalizes a result known for wide-sense stationary processes to nonstationary processes. For a fixed point in time, the IF has either zero or infinite variance. For harmonizable processes, we obtain as a consequence the result that the mean of the IF, for fixed time, is the normalized first-order frequency moment of the Wigner spectrum.

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1. INTRODUCTION

This paper concerns the instantaneous frequency (IF) of mean-square differentiable Gaussian complex-valued zero-mean proper nonstationary stochastic processes defined on $\mathbb{R}$. The IF of a stochastic process is the derivative of the phase function. It is a real-valued stochastic process. Our main result is a formula for the probability density function (pdf) for the IF, for an arbitrary fixed point in time. The pdf is parameterized by the covariance function of the process and its partial derivatives, of order not greater than two, evaluated at the same point in time.

The time axis may be divided into three subsets where the variance of the IF process exhibits completely different behavior: In the first subset, the pdf has heavy tails and behaves like x^{-3} for large x, which implies that the IF variance is infinite. In the second subset, the IF has a degenerate pdf consisting of a Dirac measure at its mean. In the third subset, the IF is $+\infty$ with probability one. (Alternatively,

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one may say that the IF is not defined on the third subset.) Hence, the IF has either infinite or zero variance where it is well defined. For wide-sense stationary (WSS) processes we show that the first subset is either $\mathbb{R}$ or empty. In the latter case, the covariance function has real part $\rho_x(t) = \alpha\cos(\beta t)$, $t \in \mathbb{R}$, for $\alpha > 0$ and $\beta \geqslant 0$. For harmonizable but not WSS processes, we give examples showing that the first set may be $\mathbb{R}$ or may be empty, but not ruling out other possibilities.

Our main result is a generalization of results by Miller [17] (Theorem 9.3) and Broman [3], who derived the formula for the pdf of the IF for fixed time for Gaussian WSS processes. The generalization consists of a relaxation of their assumption of wide-sense stationarity to a class of nonstationary processes, and a proof of the formula under this weaker assumption.

The result is formulated for Gaussian complex-valued zero-mean proper processes that are mean-square differentiable and have mean-square continuous derivative. When we specialize it to certain harmonizable stochastic processes $z(t)$, $t \in \mathbb{R}$, the formula for the pdf of the IF for fixed time implies the identity

$$\mathbb{E}\frac{d}{dt}\arg z(t) = \frac{\int_{\mathbb{R}} \xi\, \widetilde{W}_z(t,\xi)d\xi}{\int_{\mathbb{R}} \widetilde{W}_z(t,\xi)d\xi} \quad \forall t:\ r_z(t,t) > 0. \tag{1.1}$$

Here r_z is the covariance function of z, and $\widetilde{W}_z$ denotes the Wigner spectrum,

$$\widetilde{W}_z(t,\xi) = \int_{\mathbb{R}} \mathbb{E}\big(z(t+\tau/2)\overline{z(t-\tau/2)}\big)e^{-i\tau\xi}d\tau, \quad t,\xi \in \mathbb{R}. \tag{1.2}$$

For WSS processes, the Wigner spectrum does not depend on t and reduces to the (nonnegative) spectral density measure, i.e. the Fourier transform of the covariance function. In this case (1.1) says that the expected value of the IF is the normalized first-order frequency moment of the spectral density, and it does not depend on time.

The formula (1.1) generalizes to stochastic processes the corresponding well-known formula for deterministic functions:

$$\frac{d}{dt}\arg f(t) = \frac{\int_{\mathbb{R}} \xi\, W_f(t,\xi)d\xi}{\int_{\mathbb{R}} W_f(t,\xi)d\xi} \quad \forall t:\ f(t) \neq 0, \tag{1.3}$$

where the Wigner distribution W_f is defined by

$$W_f(t,\xi) = \int_{\mathbb{R}} f(t+\tau/2)\overline{f(t-\tau/2)}e^{-i\tau\xi}d\tau, \quad t,\xi \in \mathbb{R}, f \in L^2(\mathbb{R}). \tag{1.4}$$

Research on the IF has a long history in telecommunications, signal processing and time-frequency analysis (see [4]; [5], Chapter 2; [6], p. 303; and [14]). In analog frequency modulation, the IF (minus a constant carrier frequency) represents the information in a modulated signal ([19], Chapter 8).

The Wigner distribution (1.4) gives a time-frequency description of a function f. It was introduced in Quantum Mechanics as a candidate for a pdf of a particle in phase space. It is well known that W_f is only rarely a nonnegative function (Hudson's theorem; cf. [11], Theorem 4.4.1). Therefore, W_f may neither be interpreted as an energy distribution nor as a pdf, in general. This is consistent with the Uncertainty Principle (cf. [11], Chapters 2–4), which gives upper bounds on the resolution of phase space localization of particles, or, in the mathematical interpretation, of the time-frequency resolution of functions. Nevertheless, if the Wigner distribution W_f is convolved with a sufficiently wide Gaussian function, it becomes nonnegative, so domains of sufficiently large area in the time-frequency (phase) plane admit localization. Formula (1.3) supports the interpretation of W_f as a time-frequency distribution, since the right-hand side is a normalized first-order frequency moment of the Wigner distribution for a fixed time instant, which delivers the center frequency of a narrowband function.

A consequence of our result is a partition of the time axis into two parts where the IF variance is zero and infinite, respectively. The question of the dichotomy zero/infinite variance of the IF process is left somewhat open in this paper. We do not know if one process may exhibit nonempty time sets of both kinds. We show by examples that there exist processes whose IF process has zero variance constantly, and there exist processes whose IF process has infinite variance constantly. Each of these cases can occur both inside and outside the class of WSS processes.

The paper is organized as follows. After fixing some definitions and notation in Section 2, we introduce the framework of mean-square differentiable Gaussian proper stochastic processes in Section 3. Here we also give some background on harmonizable processes. In Section 4, we define the IF and discuss formula (1.3), expressing the IF of a deterministic function as a normalized first-order frequency moment of the Wigner distribution (for fixed time). Then, in Section 5, we prove our main result, which is a formula for the pdf of the IF for fixed time and its relation to the Wigner spectrum. Finally, we show by examples in Section 6 that a process may have infinite-variance IF for all time instants, or it may have zero-variance IF for all time instants.

2. PRELIMINARIES

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and let $X : \Omega \mapsto \mathbb{R}^d$ be a random variable. The probability measure on $\mathcal{B}(\mathbb{R}^d)$ (the Borel σ-algebra) induced by X is defined by $P_X(A) = \mathbb{P}\big(X^{-1}(A)\big)$, $A \in \mathcal{B}(\mathbb{R}^d)$. If the probability measure P_X is absolutely continuous with respect to the Lebesgue measure, we have $P_X(A) = \int_A p_X(x)dx$, where p_X is the probability density function (pdf) of X. Sometimes, by abuse of notation, we will say that a probability measure which is a Dirac measure at a point $a \in \mathbb{R}^d$, denoted by δ_a, has pdf δ_a. We denote the expectation of a random variable X by $\mathbb{E}X$, and by $L_0^2(\Omega) = L_0^2(\Omega, \mathcal{B}, \mathbb{P})$ we understand the Hilbert space of complex-valued zero-mean finite-variance random variables. The space of second-order random variables with nonzero mean is denoted by $L^2(\Omega)$.

Given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, a continuous-time stochastic process on $\mathbb{R}$ is defined as a family of complex-valued $\mathcal{B}$-measurable functions $z_t(\omega)$ indexed by $t \in \mathbb{R}$. We often suppress the variable $\omega \in \Omega$ and write $z(t) = z_t(\omega)$, $t \in \mathbb{R}$, and occasionally we also suppress the t variable. Sometimes we write $z_t(\omega) = z(t, \omega)$ to emphasize the fact that $z : \mathbb{R} \times \Omega \mapsto \mathbb{C}$ is a function of two variables.

For a space U, we denote by χ_A the indicator function of the subset $A \subseteq U$, that is, $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \in U \setminus A$. The space of continuous functions on $\mathbb{R}$ is denoted by $C(\mathbb{R})$, and the space of continuously differentiable functions on $\mathbb{R}$ is denoted by $C^1(\mathbb{R})$. The derivative with respect to time t is denoted by $\dot{f} = df/dt$. This notation is also used for stochastic processes, where we define the derivative in the mean-square sense. This means that a process z is mean-square differentiable at $t = t_0$ if there exists $\dot{z}(t_0) \in L^2(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \mathbb{E} \left| \frac{z(t_0 + \varepsilon) - z(t_0)}{\varepsilon} - \dot{z}(t_0) \right|^2 = 0. \tag{2.1}$$

For a function $f(x, y)$ of two variables, we write the partial derivative with respect to the first variable as $\partial_1 f(x, y) = \partial f(x, y)/\partial x$, and with respect to the second variable as $\partial_2 f(x, y) = \partial f(x, y)/\partial y$. The normalization of the Fourier transform for functions $f \in L^1(\mathbb{R})$ is

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-it\xi} dt, \quad \xi \in \mathbb{R}.$$

This gives the inverse Fourier transform

$$f(t) = \mathscr{F}^{-1} \widehat{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}.$$

We denote by $\mathscr{F} L^1(\mathbb{R})$ the space of functions f with Fourier transform $\widehat{f} \in L^1(\mathbb{R})$. For functions of several variables, the partial Fourier transform with respect to variable j is denoted by $\mathscr{F}_j$. The Wigner(–Ville) distribution (see, e.g., [8], Section 2.1.3) for $f \in L^2(\mathbb{R})$ is defined as in (1.4), where $\overline{f}$ denotes complex conjugation of f. Finally, we put $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, the determinant of a square matrix M is $\det(M) = |M|$, and the transpose of a vector x is x^T.

3. MEAN-SQUARE DIFFERENTIABLE AND HARMONIZABLE GAUSSIAN STOCHASTIC PROCESSES

Let $z(t) = x(t) + iy(t)$, $t \in \mathbb{R}$, be a continuous-time, complex-valued, zero-mean, Gaussian stochastic process $z : \mathbb{R} \mapsto L_0^2(\Omega)$, not necessarily WSS. The assumption that z is Gaussian means the following (cf., e.g., [12], Chapter 1). For any finite vector of time instants $t = (t_j)_{j=1}^n$, the real-valued $2n$-vector of samples $Z_t := \big(x(t_1), \dots, x(t_n), y(t_1), \dots, y(t_n)\big)^T$ has the pdf

$$p_{Z_t}(u) = (2\pi)^{-n}|M|^{-1/2} \exp\bigg(-\frac{1}{2}u^T M^{-1} u\bigg), \quad u \in \mathbb{R}^{2n},$$

provided the covariance matrix $M = \mathbb{E}(Z_t Z_t^T)$ is invertible. A more general definition, which works also when M is singular, is the requirement that the sampled process Z_t is a random variable with characteristic function

$$\phi_{Z_t}(u) = \mathbb{E}\big(\exp(iu^T Z_t)\big) = \exp\bigg(-\frac{1}{2}u^T M u\bigg), \quad u \in \mathbb{R}^{2n}.$$

The covariance function of the process z is denoted by

$$r_z(t, s) = \mathbb{E}\big(z(t)\overline{z(s)}\big), \quad t, s \in \mathbb{R}.$$

We assume that z is *proper* [18] (cf. [15], Section 33.3, and [7], p. 74). This means that $\mathbb{E}\big(z(t)z(s)\big) = 0$ for all $t, s \in \mathbb{R}$. It follows that we have

$$\begin{aligned} r_x(t, s) &= r_y(t, s), \quad r_{yx}(t, s) = -r_{yx}(s, t), \\ r_z(t, s) &= 2r_x(t, s) + 2ir_{yx}(t, s), \quad t, s \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where $r_x(t, s) = \mathbb{E}\big(x(t)x(s)\big)$, $r_y(t, s) = \mathbb{E}\big(y(t)y(s)\big)$, $r_{yx}(t, s) = \mathbb{E}\big(y(t)x(s)\big)$. A complex-valued zero-mean Gaussian random variable Z such that $\mathbb{E}Z^2 = 0$ is sometimes called *(circularly) symmetric* ([12], Chapter 1). The condition $\mathbb{E}Z^2 = 0$ is equivalent to the property that Z and $e^{i\theta}Z$ have identical probability distributions for all $\theta \in \mathbb{R}$.

We require that the process z be mean-square continuous, and have a continuous mean-square derivative according to the following definition.

DEFINITION 3.1. A mean-square continuous process z is *mean-square differentiable* with continuous derivative $\dot{z}$ provided the following holds: For all $t_0 \in \mathbb{R}$, there exists $\dot{z}(t_0) \in L^2(\Omega)$ such that (2.1) is satisfied, and

$$\lim_{\varepsilon \to 0} \mathbb{E}|\dot{z}(t_0 + \varepsilon) - \dot{z}(t_0)|^2 = 0 \quad \forall t_0 \in \mathbb{R}.$$

This definition guarantees that $\partial_1 r_z$, $\partial_1\partial_2 r_z$ are continuous functions (see, e.g., [15], Section 34.2), and we have

$$\mathbb{E}\big(\dot{z}(t)\overline{z(s)}\big) = \partial_1 r_z(t, s), \quad \mathbb{E}\big(\dot{z}(t)\overline{\dot{z}(s)}\big) = \partial_1\partial_2 r_z(t, s), \quad t, s \in \mathbb{R}. \tag{3.2}$$

REMARK 3.1. Note that in Definition 3.1, as well as in the rest of the paper, we discuss the derivative process in the mean-square sense only. The definition does not imply that each realization is continuously differentiable with probability one. Conditions that are sufficient for continuously differentiable realizations with probability one are considerably more subtle and difficult. See, e.g., [6], Chapter 9, and [17], Chapter 2.

For some results we will assume that z is strongly harmonizable (abbreviated in this paper as *harmonizable*); see [13], Section 1.2; [15], Section 34.4; and [20]. This means that r_z has a Fourier–Stieltjes representation

$$r_z(t,s) = \iint_{\mathbb{R}^2} e^{i(t\xi - s\eta)} m_z(d\xi, d\eta), \quad t,s \in \mathbb{R}. \tag{3.3}$$

Here m_z is a measure of bounded variation on $\mathbb{R}^2$, called the *spectral measure*. This assumption implies that the process z has a Fourier transform representation

$$z(t) = \int_{\mathbb{R}} e^{it\xi} Z(d\xi), \quad t \in \mathbb{R},$$

where the so-called spectral process $Z : \mathcal{B}(\mathbb{R}) \mapsto L_0^2(\Omega)$ is a vector-valued measure of bounded semivariation ([13], Chapter 1).

An important special case of harmonizable processes are mean-square continuous WSS processes for which there exists, by definition, a continuous positive definite function ρ_z such that $r_z(t,s) = \rho_z(t-s)$ ([15], Section 34.5). In the spectral domain the mean-square continuous WSS processes are characterized by $m_z(A,B) = \mu_z(A \cap B)$, $A, B \in \mathcal{B}(\mathbb{R})$, for a nonnegative bounded measure of one variable μ_z.

For harmonizable processes, the following requirement in the spectral domain is sufficient to guarantee that the process z is differentiable in the sense of Definition 3.1.

DEFINITION 3.2. A harmonizable process z with spectral measure m_z has *spectral moments of order one* if

$$\iint_{\mathbb{R}^2} (1+|\xi|^2)^{1/2} (1+|\eta|^2)^{1/2} |m_z|(d\xi, d\eta) < \infty. \tag{3.4}$$

Here $|m_z|$ denotes the total variation measure of the complex measure m_z (see [21], Chapter 6). Note that this definition implies that the following four integrals

$$\iint_{\mathbb{R}^2} |m_z|(d\xi, d\eta), \quad \iint_{\mathbb{R}^2} |\xi||\eta||m_z|(d\xi, d\eta),$$
$$\iint_{\mathbb{R}^2} |\xi||m_z|(d\xi, d\eta), \quad \iint_{\mathbb{R}^2} |\eta||m_z|(d\xi, d\eta)$$

are all finite. Definition 3.2 guarantees that we may take partial derivatives under the integral in (3.3) as

$$\begin{aligned}
\partial_1 r_z(t,s) &= \iint_{\mathbb{R}^2} i\xi e^{i(t\xi - s\eta)} m_z(d\xi, d\eta), \\
\partial_2 r_z(t,s) &= \iint_{\mathbb{R}^2} (-i\eta) e^{i(t\xi - s\eta)} m_z(d\xi, d\eta), \\
\partial_1 \partial_2 r_z(t,s) &= \iint_{\mathbb{R}^2} \xi\eta e^{i(t\xi - s\eta)} m_z(d\xi, d\eta), \quad t,s \in \mathbb{R},
\end{aligned} \tag{3.5}$$

due to Lebesgue's dominated convergence theorem ([21], Chapter 1). Moreover, the process satisfies Definition 3.1 and (3.2) (cf. [15], Section 34.2).

The Wigner distribution (1.4) may be written as $W_f = \mathscr{F}_2(f \otimes \overline{f} \circ \kappa)$, where $\mathscr{F}_2$ denotes the partial Fourier transform in the second variable, and κ denotes the coordinate transformation,

$$\kappa(x,y) = (x + y/2, x - y/2) \Leftrightarrow \kappa^{-1}(x,y) = \big((x+y)/2, x-y\big). \tag{3.6}$$

The Wigner (–Ville) spectrum ([8], Section 2.4.3; [16]) of a harmonizable process z is defined by

$$\widetilde{W}_z = \mathscr{F}_2(r_z \circ \kappa). \tag{3.7}$$

Since r_z may not be an integrable function, the partial Fourier transform in (3.7) is in general defined with r_z understood as a tempered distribution. However, in the case when $r_z \in S_0(\mathbb{R}^2)$, which means Feichtinger's algebra (see [11], Chapters 11 and 12), we can write (3.7) as the partial Fourier integral

$$\widetilde{W}_z(t,\xi) = \int_{\mathbb{R}} r_z(t+\tau/2, t-\tau/2) e^{-i\tau\xi} d\tau, \quad t \in \mathbb{R}. \tag{3.8}$$

Under the same assumption (plus Gaussianity) we may interchange the order of integration and expectation. Thus

$$\widetilde{W}_z(t,\xi) = \mathbb{E}\big(\int_{\mathbb{R}} z(t+\tau/2)\overline{z(t-\tau/2)} e^{-i\tau\xi} d\tau\big) = \mathbb{E}\big(W_z(t,\xi)\big), \quad t,\xi \in \mathbb{R},$$

that is, the Wigner spectrum is the expected value of the Wigner distribution of the stochastic process z defined by (1.4) (see [23]).

Using (3.6) we may write the representation (3.3) as

$$r_z \circ \kappa(t,s) = \iint_{\mathbb{R}^2} e^{i(s\xi + t\eta)} m_z \circ \kappa(d\xi, d\eta). \tag{3.9}$$

Thus, by identification with (3.7), it follows that

$$\widetilde{W}_z(t, d\xi) = 2\pi \int_{\eta \in \mathbb{R}} e^{it\eta} m_z \circ \kappa(d\xi, d\eta). \tag{3.10}$$

4. THE INSTANTANEOUS FREQUENCY, THE WIGNER DISTRIBUTION AND THE WIGNER SPECTRUM

The argument (or phase) of a complex number $z = x + iy \in \mathbb{C} \setminus \{0\}$ is defined by $z = |z| e^{i \arg z}$, where we impose the restriction $-\pi < \arg z \leqslant \pi$. We have

$$\arg z = \begin{cases} \arctan(y/x), & x > 0, \\ \pi \operatorname{sgn}(y) + \arctan(y/x), & x < 0, y \neq 0, \\ (\pi/2)\operatorname{sgn}(y), & x = 0, y \neq 0, \\ \pi, & x < 0, y = 0, \\ \text{undefined}, & x = y = 0. \end{cases} \tag{4.1}$$

For $z \in U := \mathbb{C} \setminus \{(-\infty, 0] + i0\}$ we have $\arg z = \text{Im}\,(\log z) = -i \log(z/|z|)$, where the principal branch of the logarithm function is understood, i.e. $-\pi < \text{Im}\,(\log z) < \pi$. The function $\log z$ is holomorphic from the domain U onto $\mathbb{R} + i(-\pi, \pi) \subseteq \mathbb{C}$, and therefore $z \mapsto \arg z$ is a smooth function $U \mapsto (-\pi, \pi)$.

Let $f(t) = x(t) + iy(t)$, $t \in \mathbb{R}$, be a function $f : \mathbb{R} \mapsto \mathbb{C}$. If $f(t) \neq 0$, we reserve the notation

$$\varphi(t) = \arg f(t), \quad t \in \mathbb{R},$$

for the phase function. For $f \in C(\mathbb{R})$, the set $U_f := \{t \in \mathbb{R} : f(t) \in U\} \subseteq \mathbb{R}$ is open and $\varphi = \arg \circ f : U_f \mapsto (-\pi, \pi)$ is continuous. If, moreover, $f \in C^1(\mathbb{R})$, it follows from the above that φ is differentiable on U_f. Since $(d/dt) \arctan(t) = 1/(1 + t^2)$, the derivative is

$$\dot{\varphi}(t) = \frac{d}{dt} \arg f(t) = \frac{1}{1 + y^2(t)/x^2(t)} \cdot \frac{x(t)\dot{y}(t) - \dot{x}(t)y(t)}{x^2(t)} = \frac{x(t)\dot{y}(t) - \dot{x}(t)y(t)}{x^2(t) + y^2(t)} \tag{4.2}$$

for $t \in U_f$. In fact, for $\{t \in \mathbb{R} : x(t) > 0\}$ and $\{t : x(t) < 0,\ y(t) \neq 0\}$, this follows from (4.1). In the remaining case, i.e., $\{t \in \mathbb{R} : x(t) = 0,\ y(t) \neq 0\}$, we may use the following modified definition, equivalent to (4.1) in $\{t \in \mathbb{R} : y(t) \neq 0\}$,

$$\arg z = \frac{\pi}{2} \operatorname{sgn}(y) - \arctan\left(\frac{x}{y}\right),$$

which gives $\dot{\varphi}(t) = -\dot{x}(t)/y(t)$ for $\{t \in \mathbb{R} : x(t) = 0,\ y(t) \neq 0\}$. This also leads to the expression on the right-hand side of (4.2).

Clearly, (4.2) can be extended from the domain U_f to $\{t \in \mathbb{R} : f(t) \neq 0\}$ and $\dot{\varphi}$ is still a continuous function $\{t \in \mathbb{R} : f(t) \neq 0\} \mapsto \mathbb{R}$. The *instantaneous frequency* (IF) of f (see [4]; [5], Chapter 2; [6], p. 303; [19], p. 267; [8], Section 2.3.2) is defined by (4.2) as the derivative $\dot{\varphi}$ with domain $\{t \in \mathbb{R} : f(t) \neq 0\}$. For an exponential function $t \mapsto e^{i\xi t}$ with frequency $\xi \in \mathbb{R}$, the IF is thus ξ constantly, which means that the term *instantaneous frequency* is an extension of the concept of a constant (global) frequency. For $\{t \in \mathbb{R} : f(t) = 0\}$ it will turn out to be convenient to *define* (by abuse of notation) $\dot{\varphi}(t) = +\infty$. In summary, we have for $f \in C^1(\mathbb{R})$

$$\dot{\varphi}(t) = \begin{cases} \big(x(t)\dot{y}(t) - \dot{x}(t)y(t)\big)\big(x^2(t) + y^2(t)\big)^{-1} & \text{if } x^2(t) + y^2(t) > 0, \\ +\infty & \text{if } x^2(t) + y^2(t) = 0. \end{cases} \tag{4.3}$$

If f is real-valued and continuous, then $f(t) \neq 0$ implies that f has constant sign, that is, $\varphi(t) = 0$ or $\varphi(t) = \pi$, in a neighborhood of t. Hence $\dot{\varphi}(t)$ is well-defined and equals zero in this neighborhood of t. The derivative of the phase function of real-valued functions is thus not interesting. However, the Hilbert transform (see, e.g., [6], Section 7.8; [19], Section 4.2) of a real-valued function gives rise to a complex-valued function with a nonzero IF. Since this transformation from a real-valued signal to a so-called *analytic* signal transforms $\cos(\xi t)$ into $e^{i\xi t}$, it gives a natural definition of the IF of a real-valued function, commonly used in the signal analysis literature.

The IF can be written as a normalized first-order frequency moment of the Wigner distribution ([4]; [5], Section 8.5; [8], Section 2.1.3). The heuristic version of this result is well known ([5], Section 8.5). The following more precise version can be shown (for a proof, see [2], Proposition 4.1). To state the result, we need the definition of the Sobolev space of order $s \in \mathbb{R}$, denoted by $f \in H^s(\mathbb{R})$ (cf. [10], p. 115). A tempered distribution f belongs to $H^s(\mathbb{R})$ provided its Fourier transform $\widehat{f}$ is locally square-integrable and satisfies

$$\int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty.$$

The Sobolev scale is a smoothness scale, since for $f \in H^s(\mathbb{R})$ and s large, a certain amount of asymptotic decay at infinity of the Fourier transform is required. This implies that f will be differentiable to a degree that increases with s.

PROPOSITION 4.1. *Suppose* $\varepsilon > 0$, $f \in H^{3/2+\varepsilon}(\mathbb{R})$, $f = x + iy$ *and* $\dot{\varphi}$ *is defined by* (4.3). *Then for any* $t \in \mathbb{R}$ *such that* $f(t) \neq 0$ *we have*

$$\dot{\varphi}(t) = \frac{\int_{\mathbb{R}} \xi \, W_f(t, \xi) d\xi}{\int_{\mathbb{R}} W_f(t, \xi) d\xi}. \tag{4.4}$$

One of the goals of this paper is to generalize Proposition 4.1 from deterministic functions to certain stochastic processes defined on $\mathbb{R}$, denoted by $z(t) = x(t) + iy(t) = |z(t)|e^{i\varphi(t)}$, $t \in \mathbb{R}$. More precisely, we would like to prove the formula

$$\mathbb{E}\big(\dot{\varphi}(t)\big) = \frac{\int_{\xi \in \mathbb{R}} \xi \, \widetilde{W}_z(t, d\xi)}{\int_{\xi \in \mathbb{R}} \widetilde{W}_z(t, d\xi)} \quad \forall t: \ r_z(t, t) > 0. \tag{4.5}$$

In fact, we will compute the pdf of the random variable $\dot{\varphi}(t)$ for fixed $t \in \mathbb{R}$, and then, as a consequence, derive formula (4.5). This problem has been studied by Miller [17] (Theorem 9.3) and Broman [3] for WSS Gaussian proper stochastic processes. Miller and Broman independently derived the probability density function for $\dot{\varphi}(t)$ for fixed t, using either (in [17]) the assumption that the process is proper or (in [3]) the more restrictive assumption that the signal is analytic ([19], Section 4.2). Our aim here is to generalize their results from WSS to Gaussian, proper, *nonstationary* processes that have mean-square continuous derivative. As a special case, we will study certain harmonizable processes and prove that the identity (4.5) holds for them.

Assume that $z(t) = x(t) + iy(t) = |z(t)|e^{i\varphi(t)}$, $t \in \mathbb{R}$, is a zero-mean, complex-valued, proper, Gaussian stochastic process that is differentiable in the sense of Definition 3.1. We define the stochastic IF process by (4.3), which implies that $\dot{\varphi}$ is an $\mathbb{R} \cup \{+\infty\}$-valued stochastic process defined on $\mathbb{R}$.

REMARK 4.1. A trivial case of a second-order zero-mean stochastic process consists of a random variable $X \in L_0^2(\Omega)$ times a function $f \in C^1(\mathbb{R})$, that is, $z(t) = Xf(t)$, $t \in \mathbb{R}$. For such a process we have

$$z(t, \omega) = |X(\omega)||f(t)|e^{i(\arg X(\omega) + \varphi(t))},$$

where $f(t) = |f(t)|e^{i\varphi(t)}$, provided $f(t) \neq 0$ and $X(\omega) \neq 0$. It follows that the IF of $z(t, \omega)$ is $\dot{\varphi}(t)$ provided that $f(t) \neq 0$ and $X(\omega) \neq 0$. That is, the IF of $z(t)$ is $\dot{\varphi}(t)\chi_{\{\omega: X(\omega) \neq 0\}}(\omega) + \infty\chi_{\{\omega: X(\omega) = 0\}}(\omega)$, $t \in \mathbb{R}$. This means that the IF is essentially the deterministic IF of f. It is only stochastic in the sense that for certain $\omega \in \Omega$ it is $+\infty$ for all $t \in \mathbb{R}$, and for the remaining ω it does not depend on ω. This is true also when X is non-Gaussian and improper, that is, $\mathbb{E}X^2 \neq 0$.

5. THE PROBABILITY DENSITY FUNCTION FOR THE STOCHASTIC IF FOR FIXED TIME

In this section we will derive the pdf of the IF stochastic process. More precisely, for a fixed arbitrary $t \in \mathbb{R}$, we will compute the pdf of the random variable $\dot{\varphi}(t)$, defined by (4.3) for a mean-square differentiable stochastic process $z(t) = x(t) + iy(t) = |z(t)|e^{i\varphi(t)}$, $t \in \mathbb{R}$. As a consequence, we will obtain formula (4.5) for $\mathbb{E}\big(\dot{\varphi}(t)\big)$ provided the process is harmonizable and satisfies (3.4).

Let $z = x + iy$ be a proper Gaussian stochastic process which is differentiable according to Definition 3.1. Fix $t \in \mathbb{R}$ and define the random $\mathbb{R}^4$-valued random variable $X = \big(x(t), \dot{y}(t), y(t), \dot{x}(t)\big)^T$. Then X is zero-mean and Gaussian, since Gaussianity is preserved under mean-square limits ([12], Theorem 1.3). Because z is proper, (3.1) holds, and thus $\mathbb{E}x(t)y(t) = \mathbb{E}\dot{y}(t)\dot{x}(t) = \partial_1\partial_2 r_{yx}(t,t) = 0$ and $\mathbb{E}\dot{y}(t)y(t) = \partial_1 r_y(t,t) = \partial_1 r_x(t,t) = \mathbb{E}x(t)\dot{x}(t)$. It follows that the vector X has covariance matrix $M = \mathbb{E}XX^T \in \mathbb{R}^{4\times 4}$ with the structure

$$M = \begin{pmatrix} a & b & 0 & c \\ b & d & c & 0 \\ 0 & c & a & -b \\ c & 0 & -b & d \end{pmatrix}, \tag{5.1}$$

and parameter values

$$\begin{aligned} a &= \mathbb{E}x(t)^2 = r_x(t,t), & b &= \mathbb{E}x(t)\dot{y}(t) = \partial_1 r_{yx}(t,t), \\ c &= \mathbb{E}\dot{y}(t)y(t) = \partial_1 r_x(t,t), & d &= \mathbb{E}\dot{y}(t)\dot{y}(t) = \partial_1\partial_2 r_x(t,t). \end{aligned} \tag{5.2}$$

We have $|M| = (ad - c^2 - b^2)^2$ and

$$ad - c^2 - b^2 \geqslant 0, \tag{5.3}$$

which follows from taking the determinant of the upper left 3×3 submatrix of the nonnegative definite matrix M.

Our main technical result concerns real Gaussian zero-mean four-vectors with covariance matrix (5.1). In the proof we will need the following small lemma.

LEMMA 5.1. *Let $X : \Omega \mapsto \mathbb{R}^d$ be a random variable. Suppose that $N \in \mathcal{B}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ are two disjoint null sets for P_X, i.e. $P_X(N) = P_X(\{x_0\}) = 0$, and $x_0 \notin N$. If we define $\widetilde{X} : \Omega \mapsto \mathbb{R}^d \setminus N$ by*

$$\widetilde{X}(\omega) = \begin{cases} X(\omega), & \omega \in \Omega \setminus X^{-1}(N), \\ x_0, & \omega \in X^{-1}(N), \end{cases} \tag{5.4}$$

then $P_{\widetilde{X}}(A) = P_X(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$.

Proof. Let $A \in \mathcal{B}(\mathbb{R}^d)$. We write $A = A_1 \cup A_2 \cup A_3$ as a pairwise disjoint union with $A_1 = A \cap N$, $A_2 = (A \cap \{x_0\}) \setminus N$ and $A_3 = A \setminus (N \cup \{x_0\})$. Since $\widetilde{X}$ takes values in $\mathbb{R}^d \setminus N$, we have $P_{\widetilde{X}}(A_1) \leqslant P_{\widetilde{X}}(N) = 0$, and moreover

$$P_{\widetilde{X}}(A_2) \leqslant P_{\widetilde{X}}(\{x_0\}) = \mathbb{P}\big(X^{-1}(N) \cup X^{-1}(x_0)\big) \leqslant P_X(N) + P_X(\{x_0\}) = 0.$$

Thus

$$\begin{aligned} P_{\widetilde{X}}(A) &= P_{\widetilde{X}}(A_1) + P_{\widetilde{X}}(A_2) + P_{\widetilde{X}}(A_3) \\ &= P_{\widetilde{X}}\big(A \setminus (N \cup \{x_0\})\big) = P_X\big(A \setminus (N \cup \{x_0\})\big), \end{aligned}$$

where the final equality is obtained as follows:

$$\begin{aligned} \widetilde{X}(\omega) \in A \setminus (N \cup \{x_0\}) &\Leftrightarrow \omega \in X^{-1}(A) \setminus \big(X^{-1}(N) \cup X^{-1}(x_0)\big) \\ &\Leftrightarrow X(\omega) \in A \setminus (N \cup \{x_0\}). \end{aligned}$$

This finally gives

$$\begin{aligned} P_X(A) &= P_X\big(A \cap (N \cup \{x_0\})\big) + P_X\big(A \setminus (N \cup \{x_0\})\big) \\ &= P_X\big(A \setminus (N \cup \{x_0\})\big) = P_{\widetilde{X}}(A) \end{aligned}$$

because $P_X\big(A \cap (N \cup \{x_0\})\big) \leqslant P_X(N) + P_X(\{x_0\}) = 0$. ∎

Next we give the crucial result about Gaussian four-vectors with covariance matrix M according to (5.1).

PROPOSITION 5.1. *Let $X = (X_1, X_2, X_3, X_4)$ be a zero-mean Gaussian real-valued vector with covariance matrix M defined in* (5.1). *Define the $\mathbb{R} \cup \{+\infty\}$-valued random variable*

$$Y = \begin{cases} (X_1X_2 - X_3X_4)(X_1^2 + X_3^2)^{-1} & \text{if } X_1^2 + X_3^2 > 0, \\ +\infty & \text{if } X_1^2 + X_3^2 = 0. \end{cases} \tag{5.5}$$

If $ad - c^2 - b^2 > 0$, *then* Y *has probability density function*

$$p_Y(y) = \frac{a}{2}|M|^{1/2}\big((ay-b)^2 + |M|^{1/2}\big)^{-3/2}. \tag{5.6}$$

Consequently, Y *has infinite variance and mean*

$$\mathbb{E}Y = b/a. \tag{5.7}$$

If $ad - c^2 - b^2 = 0$, *we have the two subcases*:

(i) *If* $a > 0$, *then* $Y = b/a$ *a.s.*

(ii) *If* $a = 0$, *then* $Y = +\infty$ *a.s.*

Proof. Suppose first that $ad - c^2 - b^2 > 0$, which means that M is invertible with the inverse

$$M^{-1} = |M|^{-1/2}\begin{pmatrix} d & -b & 0 & -c \\ -b & a & -c & 0 \\ 0 & -c & d & b \\ -c & 0 & b & a \end{pmatrix}. \tag{5.8}$$

Furthermore, $ad - c^2 - b^2 > 0$ implies $a > 0$. Let us define the smooth function $f : \mathbb{R}_+ \times (-\pi,\pi] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^4$, $f(y) = \big(f_1(y), f_2(y), f_3(y), f_4(y)\big)$ by

$$\begin{aligned} f_1(y) &= y_1 \cos y_2, \\ f_2(y) &= y_3 \sin y_2 + y_1 y_4 \cos y_2, \\ f_3(y) &= y_1 \sin y_2, \\ f_4(y) &= y_3 \cos y_2 - y_1 y_4 \sin y_2. \end{aligned} \tag{5.9}$$

The Jacobian of f is

$$Df(y) = \begin{pmatrix} \cos y_2 & -y_1 \sin y_2 & 0 & 0 \\ y_4 \cos y_2 & y_3 \cos y_2 - y_1 y_4 \sin y_2 & \sin y_2 & y_1 \cos y_2 \\ \sin y_2 & y_1 \cos y_2 & 0 & 0 \\ -y_4 \sin y_2 & -y_3 \sin y_2 - y_1 y_4 \cos y_2 & \cos y_2 & -y_1 \sin y_2 \end{pmatrix}$$

whose determinant is $\det Df(y) = y_1^2$. Note that $(y_1, y_2) \mapsto \big(f_1(y), f_3(y)\big)$ is the polar-to-rectangular coordinate transformation on $\mathbb{R}^2$.

We will use f as a coordinate transformation, and then we will need f to be a bijection with a differentiable inverse. Since $f(0, y_2, 0, y_4) = 0$ for any $y_2 \in (-\pi,\pi]$ and any $y_4 \in \mathbb{R}$, it follows that the function f is not injective on the domain $\mathbb{R}_+ \times (-\pi,\pi] \times \mathbb{R} \times \mathbb{R}$. Therefore, we need to restrict the domain of f. Define

$$\begin{aligned} N' &= \big(\{0\} \times (-\pi,\pi] \times \mathbb{R} \times \mathbb{R}\big) \cup (\mathbb{R}_+ \times \{\pi\} \times \mathbb{R} \times \mathbb{R}) \\ &\subseteq \mathbb{R}_+ \times (-\pi,\pi] \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{5.10}$$

and

$$N = \{x \in \mathbb{R}^4 : x_1 \leqslant 0,\ x_3 = 0\} = \mathbb{R}_- \times \mathbb{R} \times \{0\} \times \mathbb{R} \ \subseteq\ \mathbb{R}^4. \tag{5.11}$$

Then it can be verified that $f(N') = N$ and the restriction

$$f : \mathbb{R}_+ \times (-\pi, \pi] \times \mathbb{R} \times \mathbb{R} \setminus N' \mapsto \mathbb{R}^4 \setminus N \tag{5.12}$$

of f to the open set $\mathbb{R}_+ \times (-\pi, \pi] \times \mathbb{R} \times \mathbb{R} \setminus N'$ is surjective and injective. Its inverse is $f^{-1} = g = (g_1, g_2, g_3, g_4)$, where

$$\begin{aligned} g_1(x) &= (x_1^2 + x_3^2)^{1/2}, \\ g_2(x) &= \arg(x_1 + ix_3), \\ g_3(x) &= \frac{x_2x_3 + x_1x_4}{(x_1^2 + x_3^2)^{1/2}}, \\ g_4(x) &= \frac{x_1x_2 - x_3x_4}{x_1^2 + x_3^2}. \end{aligned} \tag{5.13}$$

Since $x \notin N$ implies $x_1^2 + x_3^2 > 0$, it is clear that g_1, g_3, and g_4 are differentiable on $\mathbb{R}^4 \setminus N$. Since the nonpositive x_1-axis in the (x_1, x_3)-plane, in conjunction with any $(x_2, x_4) \in \mathbb{R}^2$, does not belong to $\mathbb{R}^4 \setminus N$, g_2 is differentiable on $\mathbb{R}^4 \setminus N$. Hence the restriction of f defined by (5.12) is differentiable and has a differentiable inverse.

The sets $N' \subseteq \mathbb{R}_+ \times (-\pi, \pi] \times \mathbb{R} \times \mathbb{R}$ and $N \subseteq \mathbb{R}^4$ are null sets with respect to the Lebesgue measure, and $P_X(N) = 0$. Let $x_0 \in \mathbb{R}^4 \setminus N$ be fixed arbitrarily and define the random variable $\widetilde{X} : \Omega \mapsto \mathbb{R}^4 \setminus N$ by

$$\widetilde{X}(\omega) = \begin{cases} X(\omega), & \omega \in \Omega \setminus X^{-1}(N), \\ x_0, & \omega \in X^{-1}(N). \end{cases} \tag{5.14}$$

It follows from Lemma 5.1 that $\widetilde{X}$ and X induce identical probability measures, i.e. $P_{\widetilde{X}}(A) = P_X(A)$ for all $A \in \mathcal{B}(\mathbb{R}^4)$. This means that

$$P_{\widetilde{X}}(A) = (2\pi)^{-2}|M|^{-1/2} \int_A \exp\left(-\frac{1}{2}x^T M^{-1} x\right) dx, \quad A \in \mathcal{B}(\mathbb{R}^4), \tag{5.15}$$

because X is a zero-mean Gaussian with covariance matrix M.

Define the random variable $Z : \Omega \mapsto \mathbb{R}^4$ by $Z = f^{-1}(\widetilde{X})$, which is well defined because $\widetilde{X}(\Omega) = \mathbb{R}^4 \setminus N$ and the restriction (5.12) is bijective. If we write $Z = (Z_1, Z_2, Z_3, Z_4)$ and $x_0 = (x_{0,1}, x_{0,2}, x_{0,3}, x_{0,4})$, then by (5.5), (5.13) and (5.14) we have

$$Z_4(\omega) = \begin{cases} Y(\omega), & \omega \in \Omega \setminus X^{-1}(N), \\ (x_{0,1}x_{0,2} - x_{0,3}x_{0,4})(x_{0,1}^2 + x_{0,3}^2)^{-1}, & \omega \in X^{-1}(N). \end{cases} \tag{5.16}$$

We claim that the random variables Y and Z_4 have identical probability measures, that is,

$$P_Y(A) = P_{Z_4}(A), \quad A \in \mathcal{B}(\mathbb{R}). \tag{5.17}$$

In fact, let $A \in \mathcal{B}(\mathbb{R})$. We decompose

$$Y^{-1}(A) = \big(Y^{-1}(A) \cap X^{-1}(N)\big) \cup \big(Y^{-1}(A) \setminus X^{-1}(N)\big).$$

Since $\mathbb{P}\big(Y^{-1}(A) \cap X^{-1}(N)\big) \leqslant \mathbb{P}\big(X^{-1}(N)\big) = P_X(N) = 0$ and, according to (5.16), $Y^{-1}(A) \setminus X^{-1}(N) = Z_4^{-1}(A) \setminus X^{-1}(N)$, we obtain

$$\begin{aligned} P_Y(A) &= \mathbb{P}\big(Y^{-1}(A)\big) = \mathbb{P}\big(Y^{-1}(A) \setminus X^{-1}(N)\big) = \mathbb{P}\big(Z_4^{-1}(A) \setminus X^{-1}(N)\big) \\ &= \mathbb{P}\big(Z_4^{-1}(A)\big) = P_{Z_4}(A), \end{aligned}$$

proving (5.17).

Denote the probability density functions for X, $\widetilde{X}$ and Z by p_X, $p_{\widetilde{X}}$ and p_Z, respectively. Then (5.15) implies

$$p_{\widetilde{X}}(x) = p_X(x) = (2\pi)^{-2}|M|^{-1/2} \exp\left(-\frac{1}{2}x^T M^{-1} x\right), \quad x \in \mathbb{R}^4.$$

For an arbitrary Borel set $A \in \mathcal{B}\big(\mathbb{R}_+ \times (-\pi, \pi] \times \mathbb{R} \times \mathbb{R}\big)$, we have

$$\begin{aligned} P_Z(A) &= \mathbb{P}(Z \in A) = \mathbb{P}\big(Z \in A \setminus (X \in N)\big) = \mathbb{P}\big(\widetilde{X} \in f(A) \setminus (X \in N)\big) \\ &= \mathbb{P}\big(X \in f(A) \setminus N\big) = \mathbb{P}\big(X \in f(A \setminus N')\big) \\ &= \int_{f(A\setminus N')} p_X(x) dx = \int_{A \setminus N'} p_X \circ f(y) \,|\det Df(y)|\, dy \\ &= \int_A p_X \circ f(y) \,|\det Df(y)|\, dy \\ &= \int_A p_X \circ f(y) \,|\det Df(y)|\, \chi_{[0,+\infty)}(y_1)\, \chi_{(-\pi,\pi]}(y_2)\, dy. \end{aligned}$$

In fact, the seventh equality above is the formula for changing variables in integrals ([9], Theorem 5.8). To justify its use, we need the fact that (5.12) is differentiable and has a differentiable inverse, which has been proved above. Inserting (5.8) and (5.9), after some computations, we obtain

$$\begin{aligned} p_Z(z) &= p_X\big(f(z)\big) \,|\det Df(z)|\, \chi_{[0,+\infty)}(z_1)\, \chi_{(-\pi,\pi]}(z_2) \\ &= \frac{1}{(2\pi)^2} |M|^{-1/2} z_1^2 \exp\left(-\frac{1}{2} f(z)^T M^{-1} f(z)\right) \chi_{[0,+\infty)}(z_1) \chi_{(-\pi,\pi]}(z_2) \\ &= \frac{1}{(2\pi)^2} |M|^{-1/2} z_1^2 \exp\Big(-|M|^{-1/2}\big(z_1^2(d + a z_4^2 - 2b z_4) + a z_3^2 - 2c z_1 z_3\big)/2\Big) \\ &\quad \times \chi_{[0,+\infty)}(z_1)\, \chi_{(-\pi,\pi]}(z_2). \end{aligned} \tag{5.18}$$

Using $\int_0^\infty x^2 \exp(-sx^2/2)dx = \sqrt{\pi/2}s^{-3/2}$, $s > 0$, and $|M| = (ad - c^2 - b^2)^2$, we obtain the marginal probability density for Z_4:

$$
\begin{aligned}
&p_{Z_4}(z_4) = \int_0^\infty \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} p_Z(z_1, z_2, z_3, z_4) dz_1 dz_2 dz_3 \\
&= (2\pi)^{-1}|M|^{-1/2} \int_0^\infty z_1^2 \exp\Big(-|M|^{-1/2}\left(z_1^2(d + az_4^2 - 2bz_4)/2\right)\Big) \\
&\quad \times \Big(\int_{-\infty}^{\infty} \exp\big(-|M|^{-1/2}(az_3^2 - 2cz_1z_3)/2\big)dz_3\Big)dz_1 \\
&= (2\pi a)^{-1/2}|M|^{-1/4} \int_0^\infty z_1^2 \exp\big(-|M|^{-1/2} z_1^2(d + az_4^2 - 2bz_4 - c^2/a)/2\big)dz_1 \\
&= \frac{a}{2}|M|^{1/2}\big((az_4 - b)^2 + |M|^{1/2}\big)^{-3/2}.
\end{aligned}
$$

Thus, the observation (5.17) now proves (5.6). Finally, $(d/dx)\big(x(x^2 + s)^{-1/2}\big) = s(x^2 + s)^{-3/2}$ gives

$$
\int_{-\infty}^{\infty} \frac{dx}{(x^2 + s)^{3/2}} = \frac{2}{s} \int_0^\infty \frac{d}{dx}\big(x(x^2 + s)^{-1/2}\big)dx = \frac{2}{s}, \qquad s > 0,
$$

so we have

$$
\begin{aligned}
\mathbb{E}Z_4 &= \frac{a}{2}|M|^{1/2} \int_{-\infty}^{\infty} \frac{zdz}{\big((az - b)^2 + |M|^{1/2}\big)^{3/2}} \\
&= \frac{|M|^{1/2}b}{2a} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + |M|^{1/2})^{3/2}} \\
&= \frac{b}{a}.
\end{aligned}
$$

Note that $p_{Z_4}(y)$ behaves like Cy^{-3} for large y. Therefore,

$$
\int_{\mathbb{R}} (y - \mathbb{E}Y_4)^2 p_{Z_4}(y)dy = +\infty,
$$

which means that the variance of Z_4, and therefore also that of Y, is infinite. This proves the proposition in the case $ad - c^2 - b^2 > 0$.

It remains to consider the case when $ad - c^2 - b^2 = 0$, i.e. M is not invertible. If $a = 0$, then $X_1 = 0$ a.s. and $X_3 = 0$ a.s. imply $Y = +\infty$ a.s. This proves the case (ii). Assume henceforth that $a > 0$. If $a > 0$ and $d = 0$, then $X_2 = 0$ a.s.; $X_4 = 0$ a.s. and $X_1^2 + X_3^2 > 0$ a.s. give $Y = 0$ a.s. Since $b^2 + c^2 = ad = 0$, we have $b = 0$, so the case (i) is proved if $d = 0$.

The rest of the proof is devoted to the case (i) with $ad - c^2 - b^2 = 0$, $a > 0$ and $d > 0$. We will employ a regularization technique. The characteristic polynomial of M is $\det(\lambda I - M) = \lambda^2 \big(\lambda - (a+d)\big)^2$. Let

$$U_{a+d} = \mathcal{N}\big(M - (a+d)I\big) \subseteq \mathbb{R}^4$$

denote the two-dimensional eigenspace for M corresponding to the nonzero eigenvalue $\lambda = a + d$. The probability measure P_X is a Gaussian which is supported on U_{a+d} and non-degenerate on this two-dimensional subspace. With N defined by (5.11), it can be verified that $N \cap U_{a+d} \subseteq W \subseteq \mathbb{R}^4$, where W is a linear subspace with $\dim W = 1$. This means that $P_X(N) = 0$. If $\widetilde{X}$ is defined by (5.14) for some $x_0 \in \mathbb{R}^4 \setminus N$, then Lemma 5.1 gives $P_X = P_{\widetilde{X}}$.

Let $n > 0$ be an integer. Define

$$X^{(n)} = (X_1, X_2 + X_2'/\sqrt{n}, X_3, X_4 + X_4'/\sqrt{n}),$$

where X_2', X_4' are Gaussian zero-mean, unit-variance random variables, pairwise independent of each other and of X_1, X_2, X_3 and X_4. Then $X^{(n)} : \Omega \mapsto \mathbb{R}^4$ is a Gaussian random variable with covariance matrix

$$M_n = \begin{pmatrix} a & b & 0 & c \\ b & d + n^{-1} & c & 0 \\ 0 & c & a & -b \\ c & 0 & -b & d + n^{-1} \end{pmatrix},$$

the determinant of which is $|M_n| = a^2/n^2 > 0$, so M_n is invertible. For the characteristic functions of $X^{(n)}$ and X, denoted by $\phi_{X^{(n)}}$ and ϕ_X, respectively, we have

$$\begin{aligned}
\phi_{X^{(n)}}(\xi) &= \mathbb{E}\big(\exp(i\xi^T X^{(n)})\big) = \exp\left(-\frac{1}{2}\xi^T M_n \xi\right) \\
&= \exp\left(-\frac{1}{2}\xi^T M \xi - \frac{1}{2n}(\xi_2^2 + \xi_4^2)\right) \\
&\to \exp\left(-\frac{1}{2}\xi^T M \xi\right) = \phi_X(\xi), \qquad n \to +\infty,\ \xi \in \mathbb{R}^4.
\end{aligned}$$

This is equivalent to the weak convergence of probability measures

$$P_{X^{(n)}} \to P_X \qquad \text{as } n \to +\infty$$

([1], Theorem 7.6), which means that $\int h(x) P_{X^{(n)}}(dx) \to \int h(x) P_X(dx)$ for all bounded and continuous functions h defined on $\mathbb{R}^4$.

If we define the random variable $\widetilde{X}^{(n)}$ by

$$\widetilde{X}^{(n)}(\omega) = \begin{cases} X^{(n)}(\omega), & \omega \in \Omega \setminus (X^{(n)})^{-1}(N), \\ x_0, & \omega \in (X^{(n)})^{-1}(N), \end{cases}$$

with N determined by (5.11) and $x_0 \in \mathbb{R}^4 \setminus N$ fixed for all $n > 0$, then Lemma 5.1 again gives $P_{X^{(n)}} = P_{\widetilde{X}^{(n)}}$ for all $n > 0$. Hence $P_{\widetilde{X}^{(n)}} \to P_{\widetilde{X}}$ weakly. Let us define $Z = f^{-1}(\widetilde{X})$ and $Z^{(n)} = f^{-1}(\widetilde{X}^{(n)})$, where $Z = (Z_1, Z_2, Z_3, Z_4)$ and $Z^{(n)} = (Z_1^{(n)}, Z_2^{(n)}, Z_3^{(n)}, Z_4^{(n)})$. Then we have $P_Y = P_{Z_4}$ as in the first part of the proof. Since f^{-1} is continuous on the range spaces of $\widetilde{X}$ and $\widetilde{X}^{(n)}$, and $P_{\widetilde{X}^{(n)}} \to P_{\widetilde{X}}$ weakly, we may conclude that $P_{Z^{(n)}} \to P_Z$ weakly ([1], Section 1.5). The weak convergence $P_{Z^{(n)}} \to P_Z$ is equivalent to the limit of characteristic functions ([1], Theorem 7.6)

$$\phi_Z(\xi) = \lim_{n\to\infty} \phi_{Z^{(n)}}(\xi) \quad \forall \xi \in \mathbb{R}^4. \tag{5.19}$$

In the following we will compute $\phi_{Z^{(n)}}$ using the probability density evaluated in (5.18) for invertible covariance matrix M. Since $|M_n|^{1/2} = a/n$, we obtain from (5.18) the pdf of $Z^{(n)}$ as

(5.20)

$$p_{Z^{(n)}}(z) = \frac{n}{a(2\pi)^2} z_1^2 \exp\left(-\frac{n}{2a}\left[z_1^2\left(d + \frac{1}{n} + az_4^2 - 2bz_4\right) + az_3^2 - 2cz_1z_3\right]\right)$$
$$\times \chi_{[0,+\infty)}(z_1)\, \chi_{(-\pi,\pi]}(z_2).$$

It depends trivially on z_2, so we may concentrate on the $\mathbb{R}^3$-valued random variable $U^{(n)} = (Z_1^{(n)}, Z_3^{(n)}, Z_4^{(n)})$. The marginal probability density for $U^{(n)}$ is

$$\begin{aligned}
&p_{U^{(n)}}(z_1, z_3, z_4)\\
&= \int_{-\pi}^{\pi} p_{Z^{(n)}}(z_1, z_2, z_3, z_4)\, dz_2\\
&= \frac{n}{2\pi a}\, z_1^2 \exp\Big(-\frac{n}{2a}\Big[z_1^2\Big(d + \frac{1}{n} + az_4^2 - 2bz_4\Big) + az_3^2 - 2cz_1z_3\Big]\Big)\chi_{[0,+\infty)}(z_1).
\end{aligned}$$

Thus, the characteristic function of $U^{(n)}$ is, with $(\xi_1, \xi_3, \xi_4) \in \mathbb{R}^3$,

$$\begin{aligned}
&\phi_{U^{(n)}}(\xi) = \phi_{U^{(n)}}(\xi_1, \xi_3, \xi_4) = \mathbb{E}\Big(\exp\big(i(\xi_1, \xi_3, \xi_4)^T U^{(n)}\big)\Big)\\
&= \iiint_{\mathbb{R}^3} p_{U^{(n)}}(z_1, z_3, z_4)\, \exp\big(i(z_1\xi_1 + z_3\xi_3 + z_4\xi_4)\big)\, dz_1 dz_3 dz_4\\
&= \frac{n}{2\pi a} \int_0^{+\infty} z_1^2 \exp\Big(-\frac{n}{2a} z_1^2\left(d + \frac{1}{n}\right) + iz_1\xi_1\Big)\\
&\qquad \times \Big(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\Big[-\frac{n}{2} z_1^2\Big(z_4^2 - \frac{2b}{a} z_4\Big) + iz_4\xi_4\\
&\qquad\qquad - \frac{n}{2}\Big(z_3^2 - \frac{2c}{a} z_1 z_3\Big) + iz_3\xi_3\Big] dz_3 dz_4\Big) dz_1\\
&= a^{-1} \exp\left(i\xi_4 \frac{b}{a} - \frac{\xi_3^2}{2n}\right) \int_0^{+\infty} z_1 \exp\Big[-\frac{z_1^2}{2a} + iz_1\Big(\xi_1 + \xi_3 \frac{c}{a}\Big) - \frac{\xi_4^2}{2nz_1^2}\Big] dz_1.
\end{aligned}$$

Set $U=(Z_1,Z_3,Z_4)$. It follows from (5.19) and the dominated convergence that for $(\xi_1,\xi_3,\xi_4)\in\mathbb{R}^3$

(5.21)
$$\begin{aligned}\phi_U(\xi_1,\xi_3,\xi_4)&=\phi_Z(\xi_1,0,\xi_3,\xi_4)=\lim_{n\to\infty}\phi_{Z^{(n)}}(\xi_1,0,\xi_3,\xi_4)\\&=\lim_{n\to\infty}\phi_{U^{(n)}}(\xi_1,\xi_3,\xi_4)\\&=a^{-1}\exp\left(i\xi_4\frac{b}{a}\right)\int_0^{+\infty}z_1\exp\left[-\frac{z_1^2}{2a}+iz_1\left(\xi_1+\xi_3\frac{c}{a}\right)\right]dz_1.\end{aligned}$$

From (5.21) we may conclude that the characteristic function for Z_4 is

$$\begin{aligned}\phi_{Z_4}(\xi_4)=\phi_U(0,0,\xi_4)&=a^{-1}\exp\left(i\xi_4\frac{b}{a}\right)\int_0^{+\infty}z_1\exp\left(-\frac{z_1^2}{2a}\right)dz_1\\&=\exp\left(i\xi_4\frac{b}{a}\right),\qquad\xi_4\in\mathbb{R}.\end{aligned}$$

This implies that the probability measure for Z_4 is $P_{Z_4}=\delta_{b/a}$. Finally, the equalities $P_Y=P_{Z_4}=\delta_{b/a}$ give $Y=b/a$ a.s. ■

REMARK 5.1. If z is WSS, then $r_x(t,s)=\rho_x(t-s)$, $t,s\in\mathbb{R}$, for some even function ρ_x, which implies $\partial\rho_x(0)=0$ and, consequently, $c=\partial_1 r_x(t,t)=0$ for all $t\in\mathbb{R}$. In this case the matrix M is block-diagonal and this special case of Proposition 5.1 was proved by Miller [17], Theorem 9.3.

As a corollary to Proposition 5.1, we obtain the following result for Gaussian stochastic processes. To formulate it we need to introduce a partition of the time axis into two disjoint sets, depending on the covariance function r_z for a given process z. The partition is

(5.22)
$$T:=\left\{t\in\mathbb{R}:\ r_x(t,t)\,\partial_1\partial_2 r_x(t,t)-\left(\partial_1 r_{yx}(t,t)\right)^2-\left(\partial_1 r_x(t,t)\right)^2=0\right\},$$

and, consequently,

$$\mathbb{R}\setminus T=\left\{t\in\mathbb{R}:\ r_x(t,t)\,\partial_1\partial_2 r_x(t,t)-\left(\partial_1 r_{yx}(t,t)\right)^2-\left(\partial_1 r_x(t,t)\right)^2>0\right\},$$

since $r_x(t,t)\,\partial_1\partial_2 r_x(t,t)-\left(\partial_1 r_{yx}(t,t)\right)^2-\left(\partial_1 r_x(t,t)\right)^2\geqslant 0$ holds for all $t\in\mathbb{R}$ due to (5.2) and (5.3). It is clear that $T\subseteq\mathbb{R}$ is closed under the assumptions of Corollary 5.1, since T is the inverse image of $\{0\}$ of a continuous function. We further subdivide $T=T'\cup T''$, where $T''\subseteq\mathbb{R}$ is closed as a disjoint union of the measurable sets defined by

$$\begin{aligned}T'&:=\{t\in T: r_x(t,t)>0\},\\T''&:=\{t\in T: r_x(t,t)=0\}.\end{aligned}$$

COROLLARY 5.1. *Suppose that* $z(t) = x(t) + iy(t) = |z(t)|e^{i\varphi(t)}$, $t \in \mathbb{R}$, *is a zero-mean proper Gaussian stochastic process which is differentiable according to Definition* 3.1. *Let the instantaneous frequency stochastic process* $\dot\varphi(t)$ *be defined by* (4.3) *for all* $t \in \mathbb{R}$. *Fix* $t \in \mathbb{R}$, *define* a, b, c, d *by* (5.2) *and the matrix* M *by* (5.1). *Then* $\dot\varphi(t)$ *has the pdf*

$$p_{\dot\varphi(t)}(y) = \begin{cases} (a/2)|M|^{1/2}\left((ay-b)^2 + |M|^{1/2}\right)^{-3/2} & \textit{if } t \in \mathbb{R}\setminus T, \\ \delta_{b/a}(y) & \textit{if } t \in T', \end{cases} \tag{5.23}$$

and if $t \in T''$, *then* $\dot\varphi(t) = +\infty$ *a.s. Consequently,* $\dot\varphi(t)$ *has mean*

$$\mathbb{E}\dot\varphi(t) = \begin{cases} b/a & \textit{if } t \in \mathbb{R}\setminus T'', \\ +\infty & \textit{if } t \in T'', \end{cases}$$

and variance

$$\mathbb{E}\big(\dot\varphi(t) - \mathbb{E}\dot\varphi(t)\big)^2 = \begin{cases} +\infty & \textit{if } t \in \mathbb{R}\setminus T, \\ 0 & \textit{if } t \in T', \\ \textit{undefined} & \textit{if } t \in T''. \end{cases}$$

Finally, we restrict ourselves to harmonizable processes whose spectral measure satisfies (3.4). This assumption admits a connection to the Wigner spectrum as follows.

THEOREM 5.1. *Suppose that* $z(t) = x(t) + iy(t) = |z(t)|e^{i\varphi(t)}$, $t \in \mathbb{R}$, *is a proper Gaussian harmonizable stochastic process whose spectral measure satisfies* (3.4). *Let the instantaneous frequency stochastic process* $\dot\varphi(t)$ *be defined by* (4.3) *for all* $t \in \mathbb{R}$. *Fix* $t \in \mathbb{R}$. *Then* $\dot\varphi(t)$ *has the pdf* (5.23) *where* a, b, c, d *are defined by* (5.2) *and the matrix* M *by* (5.1). *The process* $\dot\varphi(t)$ *has mean*

$$\mathbb{E}\dot\varphi(t) = \begin{cases} \dfrac{\int_{\xi\in\mathbb{R}} \xi\, \widetilde{W}_z(t, d\xi)}{\int_{\xi\in\mathbb{R}} \widetilde{W}_z(t, d\xi)} & \textit{if } t \in \mathbb{R}\setminus T'', \\ +\infty & \textit{if } t \in T'', \end{cases}$$

and variance

$$\mathbb{E}\big(\dot\varphi(t) - \mathbb{E}\dot\varphi(t)\big)^2 = \begin{cases} +\infty & \textit{if } t \in \mathbb{R}\setminus T, \\ 0 & \textit{if } t \in T', \\ \textit{undefined} & \textit{if } t \in T''. \end{cases}$$

P r o o f. First we note that (3.3), (3.6) and (3.10) give

$$\begin{aligned} \mathbb{E}|z(t)|^2 = r_z(t,t) &= \iint_{\mathbb{R}^2} e^{it(\xi-\eta)} m_z(d\xi, d\eta) = \iint_{\mathbb{R}^2} e^{it\eta} m_z \circ \kappa(d\xi, d\eta) \\ &= \frac{1}{2\pi} \int_{\xi\in\mathbb{R}} \widetilde{W}_z(t, d\xi). \end{aligned} \tag{5.24}$$

We further observe that (3.1), (5.24) and (5.2) give

$$a = r_x(t,t) = \frac{1}{2}\,\mathrm{Re}\, r_z(t,t) = \frac{1}{2}\, r_z(t,t) = \frac{1}{4\pi} \int\limits_{\xi\in\mathbb{R}} \widetilde{W}_z(t, d\xi).$$

Next, from the requirement (3.4) it follows that (3.9) may be differentiated with respect to τ under the integral:

$$\frac{\partial}{\partial\tau}\left(r_z \circ \kappa\right)(t,\tau) = \iint\limits_{\mathbb{R}^2} i\xi e^{i(\tau\xi + t\eta)} m_z \circ \kappa(d\xi, d\eta).$$

Using (3.10) we obtain

$$\frac{\partial}{\partial\tau}\left(r_z \circ \kappa\right)(t,0) = i \iint\limits_{\mathbb{R}^2} \xi\, e^{it\eta} m_z \circ \kappa(d\xi, d\eta) = \frac{i}{2\pi} \int\limits_{\xi\in\mathbb{R}} \xi\, \widetilde{W}_z(t, d\xi). \tag{5.25}$$

On the other hand, (3.1) implies

$$\partial_1 r_x(t,t) = \partial_2 r_x(t,t) \quad \text{and} \quad \partial_2 r_{yx}(t,t) = -\partial_1 r_{yx}(t,t).$$

Hence we get

(5.26)

$$\begin{aligned} \frac{\partial}{\partial\tau}\left(r_z \circ \kappa\right)(t,\tau)\Big|_{\tau=0} &= 2\frac{\partial}{\partial\tau}\left(r_x \circ \kappa + i r_{yx} \circ \kappa\right)(t,\tau)\Big|_{\tau=0} \\ &= \partial_1 r_x(t,t) - \partial_2 r_x(t,t) + i\big(\partial_1 r_{yx}(t,t) - \partial_2 r_{yx}(t,t)\big) \\ &= 2i\partial_1 r_{yx}(t,t). \end{aligned}$$

Combining (5.25), (5.26) and (5.2), we have proved that

$$4\pi b = \int\limits_{\xi\in\mathbb{R}} \xi\, \widetilde{W}_z(t, d\xi).$$

The result now follows from Corollary 5.1, which in particular gives

$$\mathbb{E}\dot{\varphi}(t) = \frac{b}{a} = \frac{\int\limits_{\xi\in\mathbb{R}} \xi\, \widetilde{W}_z(t, d\xi)}{\int\limits_{\xi\in\mathbb{R}} \widetilde{W}_z(t, d\xi)} \tag{5.27}$$

provided $t \in \mathbb{R} \setminus T''$. ∎

6. CASES OF CONSTANTLY ZERO OR CONSTANTLY INFINITE VARIANCE IF

The process $\dot{\varphi}(t)$ exhibits completely different behavior on $t \in T' = T \setminus T''$ (where it is equal to b/a with probability one), and $t \in \mathbb{R} \setminus T$ (where it has infinite variance with mean value b/a). It is therefore of interest to investigate questions like necessary or sufficient conditions for $T = \emptyset$ or $T = \mathbb{R}$. We restrict ourselves to harmonizable processes whose spectral measure satisfies (3.4). First we look at WSS processes.

PROPOSITION 6.1. *Suppose that z is nonzero, satisfies the requirements of Theorem 5.1, and z is WSS with covariance function $r_z(t,s) = \rho_z(t-s)$ for $t, s \in \mathbb{R}$. Then we have $T'' = \emptyset$, and either $T = \emptyset$ or $T = T' = \mathbb{R}$. In the latter case $\rho_z = 2\rho_x + 2i\rho_{yx}$, where $\rho_x(t) = \alpha\cos(\beta t)$, $t \in \mathbb{R}$, $\alpha > 0$ and $\beta \geqslant 0$.*

Proof. The assumption that z is nonzero means that $\rho_x(0) > 0$. As observed in Remark 5.1, we have $c = \partial_1 r_x(t,t) = 0$ for all $t \in \mathbb{R}$ if z is WSS. Since $\partial_1\partial_2 r_x(t,s) = -\partial^2\rho_x(t-s)$, we have

$$\begin{aligned} T &= \{t \in \mathbb{R} :\ r_x(t,t)\,\partial_1\partial_2 r_x(t,t) = \big(\partial_1 r_{yx}(t,t)\big)^2\} \\ &= \{t \in \mathbb{R} : -\rho_x(0)\,\partial^2\rho_x(0) = \big(\partial\rho_{yx}(0)\big)^2\}. \end{aligned}$$

It is thus clear that either $T = \mathbb{R}$ or $T = \emptyset$, depending on whether

$$-\rho_x(0)\,\partial^2\rho_x(0) = \big(\partial\rho_{yx}(0)\big)^2 \quad \text{or} \quad -\rho_x(0)\,\partial^2\rho_x(0) > \big(\partial\rho_{yx}(0)\big)^2$$

holds. If $T = \emptyset$, then, of course, $T'' = \emptyset$. Thus suppose that $T = \mathbb{R}$, i.e.

$$-\rho_x(0)\,\partial^2\rho_x(0) = \big(\partial\rho_{yx}(0)\big)^2.$$

This means that

$$\begin{aligned} \mathbb{E}x(t)x(t)\,\mathbb{E}\dot{x}(t)\dot{x}(t) &= \big(\mathbb{E}\dot{y}(t)x(t)\big)^2 \leqslant \mathbb{E}x(t)x(t)\,\mathbb{E}\dot{y}(t)\dot{y}(t) \\ &= \mathbb{E}x(t)x(t)\,\mathbb{E}\dot{x}(t)\dot{x}(t), \quad t \in \mathbb{R}, \end{aligned}$$

where the inequality follows from the Cauchy–Schwarz inequality, and where

$$\mathbb{E}\dot{x}(t)^2 = \mathbb{E}\dot{y}(t)^2 = \partial_1\partial_2 r_x(t,t) \quad \forall t \in \mathbb{R},$$

since $r_x = r_y$. Therefore, we have equality in the Cauchy–Schwarz inequality, i.e.,

$$\mathbb{E}x(t)^2\,\mathbb{E}\dot{y}(t)^2 = \big(\mathbb{E}\dot{y}(t)x(t)\big)^2 \quad \forall t \in \mathbb{R}.$$

This means that, for any $t \in \mathbb{R}$, $\dot{y}(t)$ equals a real multiple of $x(t)$ as a member of $L_0^2(\Omega)$, i.e., $\dot{y}(t) = c(t)x(t)$, where $c(t) \in \mathbb{R}$ for $t \in \mathbb{R}$. This gives $r_{\dot{y}}(t,s) = \partial_1\partial_2 r_y(t,s) = -\partial^2\rho_x(t-s) = c(t)c(s)\rho_x(t-s) \Rightarrow c(t) = \pm\sqrt{-\partial^2\rho_x(0)/\rho_x(0)}$. Thus $c(t)$ is constant. We obtain the differential equation $-\partial^2\rho_x(t) = c^2\rho_x(t)$,

$t \in \mathbb{R}$, with solution $\rho_x(t) = \alpha\cos(ct) = \alpha\cos(|c|t)$ (since ρ_x is even), where $\alpha = \rho_x(0) = \mathbb{E}x(t)^2 > 0$, and $\beta = |c| \geqslant 0$. Finally, $T = T'$, that is, $T'' = \emptyset$, since $\rho_x(0) > 0$. ∎

EXAMPLE 6.1. Consider the process $z(t) = X_1 e^{it\xi} + X_2 e^{it\eta}$, $t \in \mathbb{R}$, where $\xi, \eta \in \mathbb{R}$, $\xi \neq \eta$, and X_1, X_2 are proper, independent zero-mean Gaussians. Thus we have $\mathbb{E}X_1\overline{X_2} = \mathbb{E}X_1X_2 = \mathbb{E}X_1^2 = \mathbb{E}X_2^2 = 0$. The process z is proper, and z is WSS because the covariance function is

$$r_z(t,s) = \mathbb{E}|X_1|^2 e^{i\xi(t-s)} + \mathbb{E}|X_2|^2 e^{i\eta(t-s)}, \quad t,s \in \mathbb{R}.$$

The real part of r_z is

$$\begin{aligned}\operatorname{Re} r_z(t,s) &= 2r_x(t,s) = 2\rho_x(t-s)\\ &= \mathbb{E}|X_1|^2\cos\big(\xi(t-s)\big) + \mathbb{E}|X_2|^2\cos\big(\eta(t-s)\big).\end{aligned}$$

From Proposition 6.1 we may conclude that $T = \emptyset$, that is, the IF process $\dot{\varphi}(t)$ has infinite variance for all $t \in \mathbb{R}$. So a linear combination of pure exponential functions, with independent proper Gaussian weights, has an infinite-variance IF process everywhere unless it consists of a single term. In the latter case it follows from Remark 4.1 that its IF process is deterministic (with variance zero for all time instants).

Next we study the class of harmonizable processes, which is larger than the class of mean-square continuous WSS processes. By the following three examples we show that $T = \mathbb{R}$ or $T = \emptyset$ may occur for harmonizable processes that are not WSS. We do not know whether $\emptyset \subsetneq T \subsetneq \mathbb{R}$ may occur.

EXAMPLE 6.2. Let $z = x + iy$, where x and y are independent Gaussian real-valued processes with identical *locally stationary* ([8], Section 1.2.2; [22]) covariance function

$$r_x(t,s) = r_y(t,s) = \exp\left[-2a\left(\frac{t+s}{2}\right)^2 - \frac{b}{2}(t-s)^2\right], \quad t,s \in \mathbb{R}.$$

This is a covariance function if $b \geqslant a \geqslant 0$ (see [22]). We compute $\partial_1\partial_2 r_x(t,t) = \big((b-a) + 4a^2t^2\big)\exp(-2at^2)$ and $\partial_1 r_x(t,t) = -2at\exp(-2at^2)$. This gives

$$\begin{aligned}r_x(t,t)\,\partial_1\partial_2 r_x(t,t) - \big(\partial_1 r_x(t,t)\big)^2 &= (b - a + 4a^2t^2 - 4a^2t^2)\exp(-4at^2)\\ &= (b-a)\exp(-4at^2) > 0, \quad t \in \mathbb{R},\end{aligned}$$

if $b > a$. Thus, by (5.22), $T = \emptyset$ in this example.

EXAMPLE 6.3. Let $g \in C^1(\mathbb{R})$ be real-valued and the Fourier transform of a bounded measure, and let again x and y be independent Gaussian real-valued processes with equal covariance function $r_x(t,s) = r_y(t,s) = g(t)g(s)$, $t,s \in \mathbb{R}$.

If $z = x + iy$, then $r_z(t,s) = 2g(t)g(s)$, $t,s \in \mathbb{R}$. We have

$$r_x(t,t)\,\partial_1\partial_2 r_x(t,t) - \big(\partial_1 r_x(t,t)\big)^2 = g(t)^2\dot g(t)^2 - \big(\dot g(t)g(t)\big)^2 = 0, \quad t \in \mathbb{R},$$

which means that $T=\mathbb{R}$, where T is defined by (5.22). In this example $|r_z(t,s)|^2 = r_z(t,t)r_z(s,s)$, i.e. we have equality in the Cauchy–Schwarz inequality for all $t,s \in \mathbb{R}$. Thus there exists $s_0 \in \mathbb{R}$ such that $z(t) = c(t)z(s_0)$, where $c(t)$ is a deterministic function. In fact, $c(t)$ is a multiple of $g(t)$. Thus we are in the situation of Remark 4.1.

EXAMPLE 6.4. This final example is a generalization of Example 6.1. Let $z(t) = X_1 e^{it\xi} + X_2 e^{it\eta}$, $t \in \mathbb{R}$, where $\xi,\eta \in \mathbb{R}$, $\xi \neq \eta$ and X_1, X_2 are jointly proper, zero-mean, unit-variance Gaussians, i.e. $\mathbb{E}|X_1|^2 = \mathbb{E}|X_2|^2 = 1$ and $\mathbb{E}X_1X_2 = \mathbb{E}X_1^2 = \mathbb{E}X_2^2 = 0$. In contrast to Example 6.1, we allow a nonzero correlation between X_1 and X_2, $\mathbb{E}X_1\overline{X_2} := c \neq 0$. We assume $|c| < 1$, since otherwise we have equality $|\mathbb{E}X_1\overline{X_2}|^2 = \mathbb{E}|X_1|^2\mathbb{E}|X_2|^2$ in the Cauchy–Schwarz inequality, which implies that X_2 is a complex multiple of X_1, reducing the problem to that of Remark 4.1.

The process z is proper, harmonizable but not WSS, because the covariance function is $r_z(t,s) = e^{i\xi(t-s)} + e^{i\eta(t-s)} + ce^{i(\xi t-\eta s)} + \overline{c}e^{i(\eta t-\xi s)}$, $t,s \in \mathbb{R}$. The terms in the decomposition $r_z = 2r_x + 2ir_{yx}$ are

$$\begin{aligned}
&2r_x(t,s) = \cos\big(\xi(t-s)\big) + \cos\big(\eta(t-s)\big)\\
&+ \operatorname{Re} c\big(\cos(\xi t-\eta s) + \cos(\eta t-\xi s)\big) + \operatorname{Im} c\big(\sin(\eta t-\xi s) - \sin(\xi t-\eta s)\big),\\
&2r_{yx}(t,s) = \sin\big(\xi(t-s)\big) + \sin\big(\eta(t-s)\big)\\
&+ \operatorname{Re} c\big(\sin(\xi t-\eta s) + \sin(\eta t-\xi s)\big) + \operatorname{Im} c\big(\cos(\xi t-\eta s) - \cos(\eta t-\xi s)\big).
\end{aligned}$$

Straightforward computations give

$$\begin{aligned}
2r_x(t,t) &= 2\big[1 + \operatorname{Re} c\cos\big(t(\xi-\eta)\big) - \operatorname{Im} c\sin\big(t(\xi-\eta)\big)\big],\\
2\partial_1\partial_2 r_x(t,t) &= \xi^2 + \eta^2 + 2\xi\eta\big[\operatorname{Re} c\cos\big(t(\xi-\eta)\big) - \operatorname{Im} c\sin\big(t(\xi-\eta)\big)\big],\\
2\partial_1 r_x(t,t) &= (\eta-\xi)\big[\operatorname{Re} c\sin\big(t(\xi-\eta)\big) + \operatorname{Im} c\cos\big(t(\xi-\eta)\big)\big],\\
2\partial_1 r_{yx}(t,t) &= (\eta+\xi)\big[1 + \operatorname{Re} c\cos\big(t(\xi-\eta)\big) - \operatorname{Im} c\sin\big(t(\xi-\eta)\big)\big].
\end{aligned}$$

This yields

$$4\big[r_x(t,t)\,\partial_1\partial_2 r_x(t,t) - \big(\partial_1 r_{yx}(t,t)\big)^2 - \big(\partial_1 r_x(t,t)\big)^2\big] = (\xi-\eta)^2(1-|c|^2) > 0$$

for all $t \in \mathbb{R}$. This means that $T = \emptyset$. We conclude that a linear combination of two exponential functions, with correlated proper Gaussian weights of equal power, is a nonstationary process whose IF process $\dot\varphi$ has infinite variance everywhere.

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Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Torino (TO), Italy
E-mail: patrik.wahlberg@unito.it

Signal and System Theory Group
Dept. of Electrical Engineering and
Information Technology (EIM-E)
Universität Paderborn
33098 Paderborn, Germany
E-mail: peter.schreier@upb.de

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