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AN EXAMPLE OF A BOOLEAN-FREE TYPE CENTRAL LIMIT THEOREM

BY

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Abstract. We construct a product of Hilbert spaces and associated product of operators, which generalizes the boolean and the free products and provides a model for new independence. The related Central Limit Theorem is proved.

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1. INTRODUCTION

In classical probability random variables are real-valued functions, in particular they commute, and the notion of (their) independence is unique. In noncommutative probability random variables are (self-adjont) elements of a *-algebra with a given state, and they (usually) do not commute. Thus there is a need to study new notions of independence, which would allow one to drop the commutativity property. Such notions have been invented, the most fruitful being free [7], monotonic [6], and boolean [1] independences. In this framework one studies properties analogous to classical probability, like limit theorems, infinite divisibility, etc.

In this paper we investigate new construction of noncommutative random variables, which satisfy some condition of independence and for which we prove related Central Limit Theorems. The random variables are indexed by partially ordered sets and in special cases lead to boolean or free independence. By this we mean the situation where random variables are freely independent if their indexes are from a totally ordered set (i.e. where any two elements are comparable), and are boolean independent if the index set is totally disordered (i.e. where any two elements are incomparable).

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The paper is, in a sense, a continuation of our previous studies of independence of noncommutative random variables, which are indexed by partially ordered sets. There we investigated combination of the monotonic and boolean independences (which we called the *bm-independence*, cf. [5] and [8]–[10]).

Our noncommutative random variables are operators on a Hilbert space, which is a product (called the *bf-product*) of (given) Hilbert spaces, indexed by a partially ordered index set. For such random variables we formulate and prove related Central Limit Theorems (*bf-CLT*). When formulating a bf-CLT for random variables, which are indexed by a partially ordered set, one encounters the problem of reasonable formulation of it. In classical CLT one considers normalized partial sums of the form

$$S_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k,$$

where X_k are i.i.d. with mean zero and variance one. This formulation involves the total order of the index set $\mathbb{N}$ of positive integers, where the summation is over integers from the interval $[1, N] \subset \mathbb{N}$ and the normalization is by the square root of the number of elements.

In this paper we study one type of examples of partial orders, given by positive cones $\Pi = \Pi_d$ in Euclidean spaces $V = V_d$ ($d = 1, 2, 3, \ldots$), where $\mathbb{R}^d_+ = \Pi \subset V = \mathbb{R}^d$. In these spaces we define discrete lattices $I_d = \mathbb{N}^d \subset \Pi$, which play the role of $\mathbb{N}$ in the classical CLT.

Our main result is Theorem 3.1, the bf-CLT for $V = \mathbb{R}^d$. It states the convergence in moments of (normalized partial sums of) our random variables to a symmetric probability measure on $\mathbb{R}$ (the bf-CLT measure). Two crucial ingredients of the proof of the bf-CLT are the reduction to the bf-ordered non-crossing pair partitions (Section 3.1) and the observation linking the limit problem to homogeneous polynomials (Lemma 3.1).

2. CONSTRUCTION OF THE bf-FOCK SPACE

2.1. Partially ordered sets. Let us recall that a set $\mathcal{X}$ is *partially ordered* by a relation $\preceq$ if the relation is:

1. *reflexive*: $\xi \preceq \xi$ for all $\xi \in \mathcal{X}$,
2. *antisymmetric*: $\xi \preceq \eta$ and $\eta \preceq \xi$ imply $\xi = \eta$ for all $\xi, \eta \in \mathcal{X}$,
3. *transitive*: $\xi \preceq \eta$ and $\eta \preceq \rho$ imply $\xi \preceq \rho$ for every $\xi, \eta, \rho \in \mathcal{X}$.

In general, for two elements $\xi, \eta \in \mathcal{X}$ we shall write $\xi \prec \eta$ (or, equivalently, $\eta \succ \xi$) if $\xi \preceq \eta$ and $\xi \neq \eta$. Moreover, if the two elements are comparable (i.e. either $\xi \preceq \eta$ or $\eta \preceq \xi$), then we shall write $\xi \sim \eta$; otherwise, we shall write $\xi \nsim \eta$.

For a *chain* A in $\mathcal{X}$ (i.e. a totally ordered subset $A \subset \mathcal{X}$) and a $\xi \in \mathcal{X}$ we shall write $\xi \sim A$ if $\xi \sim \eta$ for all $\eta \in A$, i.e. if $A \cup \{\xi\}$ is again totally ordered. Otherwise we shall write $\xi \nsim A$.

Natural examples of partially ordered sets are defined by positive cones in vector spaces. Let us recall that a subset $\Pi \subset \mathcal{V}$ of a real or complex vector space

$\mathcal{V}$ is a *positive cone* if it is closed under addition (i.e. $u, v \in \Pi$ implies $u + v \in \Pi$) and under multiplication by positive scalars (i.e. $u \in \Pi$ and $\lambda > 0$ imply $\lambda u \in \Pi$). The partial order $\preceq_\Pi$ on the vector space $\mathcal{V}$ is defined by the positive cone Π in the following way: for $u, v \in \mathcal{V}$ we put $u \preceq_\Pi v$ if $v - u \in \Pi$.

In this paper we shall deal mostly with the case of positive cones in the Euclidean spaces $\mathcal{V} = \mathbb{R}^d$ with natural $\Pi = (\mathbb{R}_+)^d$. The other cases of the symmetric positive cones in Euclidean spaces, classified by Faraut and Korányi in [3], will be treated elsewhere.

2.2. bf-product of Hilbert spaces and bf-extensions of operators. Let $\mathcal{X}$ be a partially ordered set with the partial order $\preceq$, and let $\{\mathbf{H}_\xi = \mathbf{H}_\xi^0 \oplus \mathbb{C}\Omega : \xi \in \mathcal{X}\}$ be a family of Hilbert spaces, indexed by elements of $\mathcal{X}$, with a common unit vector Ω (orthogonal to each subspace $\mathbf{H}_\xi^0$).

DEFINITION 2.1 (bf-product). The *bf-product* of the family $\{\mathbf{H}_\xi : \xi \in \mathcal{X}\}$ is the Hilbert space $\mathcal{H}$ spanned by Ω and all simple tensors of the form $h_{\xi_1} \otimes h_{\xi_2} \otimes \ldots \otimes h_{\xi_m} \in \mathbf{H}_{\xi_1}^0 \otimes \mathbf{H}_{\xi_2}^0 \otimes \ldots \otimes \mathbf{H}_{\xi_m}^0$, where $\{\xi_1 \neq \xi_2 \neq \ldots \neq \xi_m\}$ is a chain, i.e. $\xi_i \sim \xi_j$ for all $1 \leqslant i, j \leqslant n$ and the scalar product is defined as

$$\langle h_{\xi_1} \otimes h_{\xi_2} \otimes \ldots \otimes h_{\xi_m} | h'_{\xi_1} \otimes h'_{\xi_2} \otimes \ldots \otimes h'_{\xi_n} \rangle = \delta_{m,n} \cdot \prod_{j=1}^{m} \langle h_{\xi_j} | h'_{\xi_j} \rangle_{\mathbf{H}_{\xi_j}}.$$

REMARK 2.1. This definition differs from the bm-product of Hilbert spaces defined in [8], where we allowed only decreasing indexes $\xi_1 \succ \xi_2 \succ \ldots \succ \xi_m$.

Let $\mathbf{A}_\xi \in B(\mathbf{H}_\xi)$ be a bounded operator on the Hilbert space $\mathbf{H}_\xi$ ($\xi \in \mathcal{X}$). We define its bf-extension onto $\mathcal{H}$ in the following manner.

DEFINITION 2.2 (bf-extension). For a $\xi \in \mathcal{X}$ the *bf-extension* operator $A_\xi \in B(\mathcal{H})$ of $\mathbf{A}_\xi \in B(\mathbf{H}_\xi)$ from $\mathbf{H}_\xi$ onto $\mathcal{H}$ is defined as follows:

$$A_\xi \Omega := \mathbf{A}_\xi \Omega = a_\xi \Omega + g_\xi, \qquad a_\xi = \langle \mathbf{A}_\xi \Omega, \Omega \rangle_{\mathbf{H}_\xi} \in \mathbb{C},\ g_\xi \in \mathbf{H}_\xi^0, \tag{2.1}$$

$$A_\xi h_\xi := \mathbf{A}_\xi h_\xi = a'_\xi \Omega + h'_\xi, \qquad a'_\xi \in \mathbb{C},\ h'_\xi \in \mathbf{H}_\xi^0, \tag{2.2}$$

$$A_\xi h_\eta := a_\xi \cdot h_\eta + g_\xi \otimes h_\eta \qquad \text{if } \xi \sim \eta \text{ and } \xi \neq \eta. \tag{2.3}$$

Moreover, for $\xi, \eta_1, \ldots, \eta_m \in \Pi$ and $h_{\eta_1} \in \mathbf{H}_{\eta_1}^0, \ldots, h_{\eta_m} \in \mathbf{H}_{\eta_m}^0$ we set

$$\begin{aligned} A_\xi(h_{\eta_1} \otimes \ldots \otimes h_{\eta_m}) &:= (\mathbf{A}_\xi h_{\eta_1}) \otimes h_{\eta_2} \otimes \ldots \otimes h_{\eta_m} \\ &= a'_\xi \cdot h_{\eta_2} \otimes \ldots \otimes h_{\eta_m} + h'_\xi \otimes h_{\eta_2} \otimes \ldots \otimes h_{\eta_m} \end{aligned} \tag{2.4}$$

if $\xi = \eta_1 \sim \{\eta_2, \ldots, \eta_m\}$;

$$\begin{aligned} A_\xi(h_{\eta_1} \otimes \ldots \otimes h_{\eta_m}) &:= (\mathbf{A}_\xi \Omega) \otimes h_{\eta_1} \otimes \ldots \otimes h_{\eta_m} \\ &= a_\xi \cdot h_{\eta_1} \otimes \ldots \otimes h_{\eta_m} + g_\xi \otimes h_{\eta_1} \otimes \ldots \otimes h_{\eta_m} \end{aligned} \tag{2.5}$$

if $\xi \neq \eta_1$ and $\xi \sim \{\eta_1, \ldots, \eta_m\}$; and

$$A_\xi(h_{\eta_1} \otimes \ldots \otimes h_{\eta_m}) := 0 \tag{2.6}$$

if $\xi \nsim \{\eta_1, \ldots, \eta_m\}$.

The notation bf (abbreviation for *boolean-free*) is justified by the fact that, in particular cases, the extension operators are boolean independent (if $\mathcal{X}$ is totally disordered, i.e. every two elements are incomparable) or free independent (if $\mathcal{X}$ is totally ordered, i.e. every two elements are comparable). These properties require considering the vacuum state φ on $B(\mathcal{H})$ (bounded operators on $\mathcal{H}$), defined as

$$\varphi(A) := \langle A\Omega|\Omega\rangle.$$

It is immediate to see that if $\mathbf{A}_\xi$ is self-adjoint, then A_ξ is self-adjoint too.

THEOREM 2.1. *Let $\mathcal{H}$ be the bf-product of a family of Hilbert spaces $\{\mathbf{H}_\xi : \xi \in \mathcal{X}\}$ and let $\{\mathcal{A}_\xi : \xi \in \mathcal{X}\}$ be a family of algebras of bf-extension operators. Then the following holds*:

(B) *If the index set $\mathcal{X}$ is totally disordered, then the bf-extension operators $\{\mathcal{A}_\xi : \xi \in \mathcal{X}\}$ are boolean independent, i.e. they satisfy the condition*

$$\varphi(A_{\xi_1} \ldots A_{\xi_m}) = \varphi(A_{\xi_1}) \ldots \varphi(A_{\xi_m}) \tag{2.7}$$

for any $\xi_1, \ldots, \xi_m \in \mathcal{X}$ and any $A_{\xi_1} \in \mathcal{A}_{\xi_1}, \ldots, A_{\xi_m} \in \mathcal{A}_{\xi_m}$.

(F) *If the index set $\mathcal{X}$ is totally ordered, then the bf-extension operators $\{\mathcal{A}_\xi : \xi \in \mathcal{X}\}$ are freely independent, i.e. they satisfy the condition: for any $A_{\xi_1} \in \mathcal{A}_{\xi_1}, \ldots, A_{\xi_m} \in \mathcal{A}_{\xi_m}$, if $\varphi(A_\xi) = 0$ for all $\xi \in \mathcal{X}$, then*

$$\varphi(A_{\xi_1} A_{\xi_2} \ldots A_{\xi_m}) = 0 \quad \textit{for any } \xi_1 \neq \xi_2 \neq \ldots \neq \xi_m \in \mathcal{X}. \tag{2.8}$$

P r o o f. For the proof of the condition (B) let us assume that $\mathcal{X}$ is totally disordered. Observe that, for a given $\xi, \eta \in \mathcal{X}$, if $A_\eta\Omega = a_\eta\Omega + g_\eta$ with $g_\eta \in \mathbf{H}^0_\eta$, then $\xi \nsim \eta$ implies $A_\xi g_\eta = 0$, and thus $A_\xi A_\eta \Omega = a_\eta a_\xi \Omega + g_\xi$. Hence, by induction, one can easily show that

$$A_{\xi_1} \ldots A_{\xi_m} \Omega = a_{\xi_1} \ldots a_{\xi_m} \Omega + g_{\xi_1},$$

where $a_{\xi_j} := \varphi(A_{\xi_j}) = \langle A_{\xi_j}\Omega|\Omega\rangle$ and $A_{\xi_j}\Omega = a_{\xi_j}\Omega + g_{\xi_j}$ $(j = 1, \ldots, m)$. Therefore, using $g_{\xi_1} \perp \Omega$, we get

$$\varphi(A_{\xi_1} \ldots A_{\xi_m}) = \langle a_{\xi_1} \ldots a_{\xi_m}\Omega + g_{\xi_1}|\Omega\rangle = a_{\xi_1} \ldots a_{\xi_m} = \varphi(A_{\xi_1}) \ldots \varphi(A_{\xi_m}).$$

For the proof of the condition (F) let us assume that $\varphi(A_{\xi_j}) = 0$ for $1 \leqslant j \leqslant m$, and that $\mathcal{X}$ is totally ordered. To compute $\varphi(A_{\xi_1} \ldots A_{\xi_m})$ we analyse the construction of the vector $A_{\xi_1} \ldots A_{\xi_m}\Omega$. By the assumption we have $A_{\xi_j}\Omega = g_{\xi_j} \in$

$\mathbf{H}^0_{\xi_m}$ for all $j = 1, \dots, m$, so using $\xi_1 \sim \xi_2 \sim \dots \sim \xi_m$ and (2.5) one gets by induction

$$A_{\xi_j} \dots A_{\xi_m}\Omega = g_{\xi_j} \otimes \dots \otimes g_{\xi_m}$$

for all $1 \leqslant j \leqslant m$. Therefore,

$$A_{\xi_1} \dots A_{\xi_m}\Omega = g_{\xi_1} \otimes \dots \otimes g_{\xi_m} \perp \Omega,$$

which implies $\varphi(A_{\xi_1} \dots A_{\xi_m}) = 0$. ■

3. BOOLEAN-FREE TYPE CLT FOR EUCLIDEAN SPACE

In this section we consider the vector space $\mathcal{X} := \mathbb{R}^d$ and the positive cone $\Pi = \Pi_d := \{(a_1, \dots, a_d) \in \mathcal{V} : 0 \leqslant a_1, \dots, 0 \leqslant a_d\}$ for a given positive integer d. Then the partial order $\preceq$ on $\mathbb{R}^d$ is defined explicitly as follows: for $(a_1, \dots, a_d)$, $(b_1, \dots, b_d) \in \mathbb{R}^d$ we have $(a_1, \dots, a_d) \preceq (b_1, \dots, b_d)$ if and only if $a_1 \leqslant b_1, \dots,$ $a_d \leqslant b_d$.

3.1. bf-ordered pair partitions. For further considerations we need some combinatorial preparation regarding partitions with blocks indexed by elements of $\mathcal{X}$. We shall use the notation $\mathcal{NC}(2n)$ (resp. $\mathcal{NC}_2(2n)$) for the set of all (resp. pair) non-crossing partitions of the set $\{1, 2, \dots, 2n\}$. Recall that a block $B = \{a < b\}$ $\in \mathcal{V} \in \mathcal{NC}_2(2n)$ is called *outer* if there is no other block $B' = \{a' < b'\} \in \mathcal{V}$ such that $a' < a < b < b'$; otherwise the block is *inner*. Thus, a partition $\mathcal{V} \in \mathcal{NC}_2(2n)$ has *only one outer block* if and only if it contains the block $\{1, 2n\}$.

DEFINITION 3.1. We say that a sequence of k blocks $(B_{i_1}, \dots, B_{i_k})$ (with $B_{i_j} = \{a_j, b_j\}$ for $1 \leqslant j \leqslant k$) of a partition $\mathcal{V} \in \mathcal{NC}_2(2n)$ is a *maximal descending sequence* if the following holds. For every $1 \leqslant j \leqslant k-1$ we have:

(i) $a_j < a_{j+1} < b_{j+1} < b_j$;

(ii) there is no other block $B = \{c, d\} \in \mathcal{V}$ such that $a_j < c < a_{j+1} < b_{j+1} < d < b_j$;

(iii) there is no other block $B = \{c, d\} \in \mathcal{V}$ such that $c < a_j < a_{j+1} < b_{j+1} < b_j < d$ or $a_j < a_{j+1} < c < d < b_{j+1} < b_j$.

For a pair of blocks $(B_{i_j}, B_{i_{j+1}})$ which satisfy the conditions (i) and (ii) we shall use the notation $B_{i_j} \mapsto B_{i_{j+1}}$; (iii) ensures maximality.

It will be convenient to order the blocks of a non-crossing partition by their 'left legs' (i.e. the minimal elements) defined as $\min(B_j) = a_{j,1}$ when $B_j = \{a_{j,1} < \dots < a_{j,s_j}\}$. The block B_1 will denote the one that contains 1. In general, the notation $\mathcal{V} = (B_1, \dots, B_k)$ will mean that $\min(B_j) < \min(B_{j+1})$ for $1 \leqslant j \leqslant k-1$.

DEFINITION 3.2. We say that a sequence $(\xi_1, \xi_2, \dots, \xi_{2n})$ of elements from $\mathcal{X}$ *is associated with a partition* $\mathcal{V} = (B_1, \dots, B_k) \in \mathcal{NC}(2n)$ if for every $1 \leqslant j$ $\leqslant k$ there exists the *label* $\xi(j) \in \{\xi_1, \xi_2, \dots, \xi_{2n}\}$ such that $B_j := \{1 \leqslant s \leqslant 2n :$

$\xi_s = \xi(j)\}$ (we shall denote this by $B_j \vdash \xi(j)$ or $\xi(j) \dashv B_j$). In other words, the blocks are *labelled* by the elements.

DEFINITION 3.3 (bf-ordered partitions). A sequence $(\xi_1, \ldots, \xi_{2n})$ of elements in $\mathcal{X}$ *establishes a bf-order* on associated partition $\mathcal{V} = (B_1, \ldots, B_n) \in \mathcal{NC}_2(2n)$ if for every maximal descending sequence $B_{i_1} \mapsto \ldots \mapsto B_{i_k}$ of blocks, the associated labels $\xi(1) \dashv B_{i_1}, \ldots, \xi(k) \dashv B_{i_k}$ form a chain, i.e. $\xi(s) \sim \xi(t)$ for every $1 \leqslant s, t \leqslant k$. In particular, every element ξ_j in the sequence has to appear exactly twice.

3.2. bf-Central Limit Theorem. Consider the family $\{\mathbf{J}_\xi : \xi \in \mathbf{I}_d\}$ of subsets of $\mathbf{I}_d := \mathbb{N}_0^d$ (where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) with $\xi := (a_1, a_2, \ldots, a_d) \in \mathbf{I}_d$, and defined by $\mathbf{J}_\xi := [0, \xi] \cap \mathbb{N}_0 = \{\xi' \in \mathbb{N}_0^d : \xi' \preceq \xi\}$. This family is increasing ($\xi' \preceq \xi$ implies $\mathbf{J}_{\xi'} \subset \mathbf{J}_\xi$), and satisfies $\bigcup_{\xi \in \mathbf{I}_d} \mathbf{J}_\xi = \mathbb{N}_0^d$. Thus it can play the same role in $\mathbb{R}^d$ as $\mathbb{N}$ plays in $\mathbb{R}$ in the classical CLT (where one considers the summation $(1/\sqrt{M}) \sum_{m \leqslant M} X_m$ for i.i.d. $X_1, X_2, \ldots$). In our formulation of bf-CLT we shall replace this by

$$S_\xi := \frac{1}{\sqrt{|\mathbf{J}_\xi|}} \sum_{\xi' \preceq \xi} A_{\xi'} = \frac{1}{\sqrt{|\mathbf{J}_\xi|}} \sum_{\xi' \in \mathbf{J}_\xi} A_{\xi'}$$

for the family of bf-extension operators $\{A_\xi : \xi \in \mathbf{I}_d\}$, and we shall consider the limit $\xi \to \infty$ in the sense that $a_1, \ldots, a_d \to \infty$.

THEOREM 3.1 (bf-CLT for bf-extensions). *Let $\{A_\xi : \xi \in \mathbf{I}_d\}$ be a family of self-adjoint bf-extension operators on the bf-product $\mathcal{H}$ and let φ be the vacuum state. Assume that $\varphi(A_\xi) = 0$ and $\varphi(A_\xi^2) = 1$ for every $\xi \in \mathbf{I}_d$. Then, for every $n \in \mathbb{N}_0$, $\lim_{\xi \to \infty} \varphi\big((S_\xi)^{2n+1}\big) = 0$. Moreover, there exists a sequence $\big(g_n(d)\big)_{n=0}^{\infty}$ of real numbers such that for $n \in \mathbb{N}_0$ there exists the limit*

$$g_n(d) := \lim_{\xi \to \infty} \varphi\big((S_\xi)^{2n}\big) = \lim_{\xi \to \infty} \varphi\left(\left[\frac{1}{\sqrt{|\mathbf{J}_\xi|}} \sum_{\xi' \in \mathbf{J}_\xi} A_{\xi'}\right]^{2n}\right). \tag{3.1}$$

The sequence is the (even) moment sequence of a symmetric probability measure μ_d on $\mathbb{R}$:

$$g_n(d) = \int_{-\infty}^{+\infty} x^{2n}\, \mu_d(dx), \qquad 0 = \int_{-\infty}^{+\infty} x^{2n+1}\, \mu_d(dx)$$

for $n = 0, 1, 2, \ldots$

Proof. For the proof of the theorem we first use standard quantitative arguments (see [5] and [10]) to reduce the statement to summation related to noncrossing pair partitions with bf-order. Namely,

$$g_n(d) = \lim_{\xi \to \infty} \frac{1}{|\mathbf{J}_\xi|^n} \sum_{\mathtt{bfNC}_2^n(\mathbf{J}_\xi)} \varphi(A_{\xi_1'} A_{\xi_2'} \ldots A_{\xi_{2n}'}),$$

where the summation is over all $\mathtt{bfNC}_2^n(\mathbf{J}_\xi)$ sequences $(\xi'_1, \xi'_2, \dots, \xi'_{2n})$, i.e. sequences of elements $\xi'_1, \xi'_2, \dots, \xi'_{2n} \in \mathbf{J}_\xi$, which are associated with partitions from $\mathcal{NC}_2(2n)$ on which they establish bf-order. Under these conditions and by the assumptions we have $\varphi(A_{\xi'_1}A_{\xi'_2}\dots A_{\xi'_{2n}}) = 1$ since then there is the product formula

$$\varphi(A_{\xi'_1}A_{\xi'_2}\dots A_{\xi'_{2n}}) = \prod_{j=1}^{n} \varphi(A^2_{\xi'_{s_j}}) = 1,$$

where $\{\xi'_{s_1}, \dots, \xi'_{s_n}\} = \{\xi'_1, \dots, \xi'_{2n}\}$. Hence proving (3.1) is reduced to showing that, for each $n \in \mathbb{N}$, there exists the limit

$$g_n(d) = \lim_{\xi\to\infty} \frac{|\mathtt{bfNC}_2^n(\mathbf{J}_\xi)|}{|\mathbf{J}_\xi|^n}. \tag{3.2}$$

Thus we need to estimate the numbers $d_n(\xi) := |\mathtt{bfNC}_2^n(\mathbf{J}_\xi)|$ for $\xi \to \infty$.

We split the proof into several lemmas, and, for simplicity of the notation, shall present the proofs for the case $d = 2$. This does not restrict the full generality of the theorem (see Remark 3.2).

We first describe the behaviour of the number of 'nested' partitions. With this aid, we estimate the cardinality of the set of sequences with one outer block. Finally, we look on the behaviour of $d_n(\xi)$ as $\xi \to \infty$.

Below we shall frequently use the following technical result (cf. Faulhaber's formula in [2]):

$$\sum_{c=0}^{y} (y-c)^k c^m = \gamma(k,m) y^{k+m+1} + O\big(y^{k+m}\big). \tag{3.3}$$

Indeed, by Faulhaber's formula we have

$$\begin{aligned}
&\sum_{c=0}^{y} (y-c)^k c^m = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y^{k-j} \sum_{c=0}^{y} c^{m+j} \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j y^{k-j} \frac{1}{m+j+1} \sum_{i=1}^{m+j+1} (-1)^{\delta_{i,m+j}} \binom{m+j+1}{i} B_{m+j+1-i} y^i \\
&= \Big[\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{m+j+1} \Big] y^{m+k+1} + \frac{1}{2} \sum_{j=0}^{k} \binom{k}{j} (-1)^j y^{k+m} \\
&\quad + \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{1}{m+j+1} \sum_{i=1}^{m+j-1} \binom{m+j+1}{i} B_{m+j+1-i}\, y^{k-j+i} \\
&= \gamma(k,m) y^{k+m+1} + O(y^{m+k}),
\end{aligned}$$

where B_k's are Bernoulli numbers, and

$$\gamma(k,m) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{m+j+1}.$$

DEFINITION 3.4. Let π belong to $\mathtt{bfNC}_2^n(\mathbf{J}_\rho)$ and be of the form

$$\pi = (\xi_1, \xi_2, \ldots, \xi_t, \pi', \xi_t, \ldots, \xi_2, \xi_1), \tag{3.4}$$

$\pi' \in \mathtt{bfNC}_2^{n-t}(\mathbf{J}_\rho)$. Then we call π' a $\mathtt{bfNC}$-*subsequence nested at the depth t under $\xi_1, \ldots, \xi_t$*. Given a sequence $\rho \succ \xi_1 \succ \ldots \succ \xi_t \succ 0$, the number of all $\mathtt{bfNC}_2^{n-t}(\mathbf{J}_\rho)$-subsequences nested at the depth t under $\xi_1, \ldots, \xi_t$ will be denoted by $c_n^t[\rho](\xi_1 \succ \ldots \succ \xi_t)$.

REMARK 3.1. *If we omit the assumption about the decreasing ordering of elements $\xi_1, \ldots, \xi_t$, then the number of all $\mathtt{bfNC}_2^{n-t}(\mathbf{J}_\rho)$-subsequences nested at the depth t under $\xi_1, \ldots, \xi_t$ equals $t! \cdot c_n^t[\rho](\xi_1 \succ \ldots \succ \xi_t)$.*

DEFINITION 3.5. Let us denote by $\mathtt{bfNC}_{2,1}^n(\mathbf{J}_\rho)$ the collection of all sequences in $\mathtt{bfNC}_2^n(\mathbf{J}_\rho)$ such that the associated non-crossing pair partition, on which the sequence establishes bf-order, has exactly one outer block. Let $B_\rho^1(n) = B_\rho^1(n, d) := |\mathtt{bfNC}_{2,1}^n(\mathbf{J}_\rho)|$ denote the cardinality of the set.

LEMMA 3.1. *Let n be a fixed integer. For any $t < n$ we have*

$$c_n^t[\rho](\xi_1 \succ \ldots \succ \xi_t) = P_{n,t}(v_0, v_1, \ldots, v_t) + p_{n,t}(x_0, y_0, x_1, \ldots, x_t, y_t), \tag{3.5}$$

where

$$P_{n,t}(v_0, v_1, \ldots, v_t) = \sum_{k_0+\ldots+k_t=n-t} \alpha_{k_0,\ldots,k_d}^{n,t} v_0^{k_0} \ldots v_t^{k_t}$$

is a homogeneous polynomial in $t+1$ variables of degree $n-t$ and

$$v_j = \mathrm{vol}([\xi_j, \xi_{j+1}]), \quad j = 0, 1, \ldots, t.$$

Moreover, $p_{n,t}$ is a polynomial in $2t+2$ variables $x_j = a_j - a_{j+1}$, $y_j = b_j - b_{j+1}$ of degree at most $2(n-t-1)$, where $\xi_0 = \rho$, $\xi_{t+1} = 0$, $\xi_i = (a_i, b_i)$ for $i = 0, \ldots, t$.

Proof. We proceed by the inverse induction on t. For $t = n-1$ the only $\mathtt{bfNC}_2^1(\mathbf{J}_\rho)$-subsequence will be related to the partition $\pi' = \{\eta, \eta\}$ with η comparable to all ξ_i's. Thus

$$\begin{aligned} c_n^{n-1}[\rho](\xi_1 \succ \ldots \succ \xi_{n-1}) &= |\{\eta : \exists j\ \xi_{j+1} \prec \eta \prec \xi_j\}| = \sum_{j=0}^{d} \sum_{\eta:\xi_{j+1}\prec\eta\prec\xi_j} 1 \\ &= \sum_{j=0}^{d} (a_j - a_{j+1})(b_j - b_{j+1}) = v_0 + \ldots + v_d \end{aligned}$$

is homogeneous of degree one, and $p_{n,n-1} = 0$.

Let us now assume that for any $(t+1)$-elements ordered sequence in $[0, \rho]$ the number $c_n^t[\rho](\xi_1 \succ \ldots \succ \xi_{t+1})$ is of the form (3.5).

Let $\xi_1 \succ \ldots \succ \xi_t$, π be of the form (3.4). Then π' itself has $n-t$ blocks and p outer blocks $B_1, \ldots, B_p$. Each such outer block induces a subpartition $\sigma_i \in \mathtt{bfNC}^{m_i}_{2,1}(\mathbf{J}_\rho)$. Let us focus on one such subpartition σ (for convenience we omit the index i), consisting of $m \leqslant n-t$ blocks, and denote the label of its outer block by η. Since $\eta \sim \{\xi_1, \ldots, \xi_t\}$, there exists j ($0 \leqslant j \leqslant t$) such that $\xi_{j+1} \prec \eta \prec \xi_j$. Thus, by the inductive assumption, the number of bf-subsequences nested at the depth $t+1$ under $\xi_1, \ldots, \xi_t, \eta$ satisfies (3.5). This means that there are polynomials $P_{m+t,t+1}$ homogeneous, in $t+2$ variables, of degree $m-1$, and an appropriate polynomial $p_{m+t,t+1}$ of smaller order such that

$$\begin{aligned}
&c_n^{t+1}[\rho](\xi_1 \succ \ldots \succ \xi_j \succ \eta \succ \xi_{j+1} \succ \ldots \succ \xi_t)\\
&= \sum_{k_0+\ldots+k_t=m-1, k_j=k_j^{(1)}+k_j^{(2)}} \alpha^{m+t,t+1}_{k_0,\ldots,k_d} v_0^{k_0} \ldots (v_j^{(1)})^{k_j^{(1)}} (v_j^{(2)})^{k_j^{(2)}} \ldots v_t^{k_t}\\
&\quad + p_{m+t,t+1}(x_0, y_0, \ldots, y_{j-1}, a_j - c, b_j - d, c - a_{j+1}, d - b_{j+1}, x_{j+1}, \ldots, y_t),
\end{aligned}$$

where v_j denotes the volume of the interval $[\xi_{j+1}, \xi_j]$ and $v_j^{(1)} = \mathrm{vol}[\eta, \xi_j]$, and $v_j^{(2)} = \mathrm{vol}[\xi_{j+1}, \eta]$.

Now, to compute all possible subsequences nested under $\xi_1, \ldots, \xi_t$ related to the partition σ, we need to sum over all possible points η lying in $[\xi_{j+1}, \xi_j]$, and then sum over all j's. We shall first focus on the leading term (i.e. the one related to P). Note that when applying the summation over $\eta \in [\xi_{j+1}, \xi_j]$ to it, the only term which depends on η is $(v_j^{(1)})^{k_j^{(1)}} (v_j^{(2)})^{k_j^{(2)}}$. That is why, using (3.3) and writing $k_j^{(1)} = k$ and $k_j^{(1)} = l$, we first compute

$$\begin{aligned}
&\sum_{\eta \in [\xi_{j+1}, \xi_j]} (v_j^{(1)})^{k_j^{(1)}} (v_j^{(2)})^{k_j^{(2)}}\\
&= \sum_{c=a_{j+1}}^{a_j} (c - a_j)^k (a_{j+1} - c)^l \sum_{d=b_{j+1}}^{b_j} (d - b_j)^k (b_{j+1} - d)^l\\
&= \sum_{c=0}^{a_j - a_{j+1}} c^k (a_{j+1} - a_j - c)^l \sum_{d=0}^{b_j - b_{j+1}} d^k (b_{j+1} - b_{j+1} - d)^l\\
&= \Big(\gamma(k,l)(a_{j+1} - a_j)^{k+l+1} + O\big((a_{j+1} - a_j)^{k+l}\big)\Big)\\
&\quad \times \Big(\gamma(k,l)(b_{j+1} - b_j)^{k+l+1} + O\big((b_{j+1} - b_j)^{k+l}\big)\Big)\\
&= \gamma(k_j^{(1)}, k_j^{(2)})^2 \cdot v_j^{k_j^{(1)}+k_j^{(2)}+1} + O\big((a_{j+1} - a_j)^{k_j^{(1)}+k_j^{(2)}+1} (b_{j+1} - b_j)^{k_j^{(1)}+k_j^{(2)}}\big)\\
&\quad + O\big((a_{j+1} - a_j)^{k_j^{(1)}+k_j^{(2)}} (b_{j+1} - b_j)^{k_j^{(1)}+k_j^{(2)}+1}\big).
\end{aligned}$$

We see that the leading term of the result is proportional to the volume of the interval $[\xi_{j+1}, \xi_j]$. This means that the leading term of

$$\sum_{\eta\in[\xi_{j+1},\xi_j]} c_n^{t+1}[\rho](\xi_1 \succ \ldots \succ \xi_j \succ \eta \succ \xi_{j+1} \succ \ldots \succ \xi_t)$$

will be a homogeneous polynomial in variables $v_0, \ldots, v_j, \ldots, v_t$ of degree $k_0 + \ldots + k_{j-1} + k_j^{(1)} + k_j^{(2)} + 1 + k_{j+1} + \ldots + k_t = m$, which does not depend on j. Thus summing over all $j = 0, \ldots, t$ we get a sum of polynomials in the same variables, homogeneous of the same degree m. The result will be of the same type. Finally, we need to consider all blocks $B_1, \ldots, B_p$. Let P_i denote the homogeneous polynomial in variables $v_0, \ldots, v_j, \ldots, v_t$ of degree m_i, corresponding to the block B_i. Since the choice of the points labelling blocks in different subpartitions $\sigma_1, \ldots, \sigma_p$ can be regarded as independent (formally, the points should not repeat, but such a situation is negligible), the number of subsequences nested under fixed $\xi_1, \ldots, \xi_t$ is the product of P_i's, which is again a homogeneous polynomial in variables $v_0, \ldots, v_j, \ldots, v_t$ of degree $\sum_{i=1}^{p} m_i = n - t$.

By the same technique we can show that the remaining terms are of type $(a_j - a_{j+1})^k (b_j - b_{j+1})^l$ with $k + l < n - t$ and $j = 0, \ldots, t$. ∎

The next lemma estimates $B_\xi^1(n, d)$.

LEMMA 3.2. *For any* $n \in \mathbb{N}$ *there exists a constant* $L_n > 0$, *independent of* ρ, *such that*

$$B_\rho^1(n) = |\mathtt{bfNC}_{2,1}^n(\mathbf{J}_\rho)| = L_n \cdot |\mathbf{J}_\rho|^n + O(|\mathbf{J}_\rho|^{n-1}).$$

P r o o f. For any fixed sequence in $\mathtt{bfNC}_{2,1}^n(\mathbf{J}_\rho)$ let $\xi = (a, b)$ denote the label of the outer block in the sequence. By the bf-ordering property we must have $\xi \sim \rho = (M, N)$. Applying Lemma 3.1 to the case of $t = 1$, we see that

$$\begin{aligned}
B_\rho^1(n) &= \sum_{\xi\sim\rho} c_{n-1}^1[\rho](\xi) \\
&= \sum_{a=0}^{M}\sum_{b=0}^{N} \big[P_{n,1}\big(ab, (M-a)(N-b)\big) + p_{n,1}\big(ab, (M-a)(N-b)\big)\big] \\
&= \sum_{a=0}^{M}\sum_{b=0}^{N}\sum_{k_0+k_1=n-1} \alpha_{k_0,k_1}^{n,1}(ab)^{k_0}[(M-a)(N-b)]^{k_1} \\
&\quad + \sum_{a=0}^{M}\sum_{b=0}^{N}\sum_{l_0+l_1<n-1} \beta_{l_0,l_1}^{n,1}(ab)^{l_0}[(M-a)(N-b)]^{l_1} \\
&= \sum_{m=0}^{n-1} \alpha_{m,n-1-m}^{n,1}\gamma(m, n-m-1)^2 (MN)^n + O\big((MN)^{n-1}\big),
\end{aligned}$$

which shows that

$$B_{\rho}^{1}(n) = L_n|\mathbf{J}_{\rho}|^n + O(|\mathbf{J}_{\rho}|^{n-1}), \quad L_n = \sum_{m=0}^{n-1} \alpha_{m,n-m-1}^{n,1}\gamma(m, n-m-1)^2,$$

and the proof is completed. ■

Now we are ready to estimate the number of $\mathtt{bfNC}$-sequences.

LEMMA 3.3. *For any $n \in \mathbb{N}$ there exists a constant $K_n > 0$, independent of ρ, such that*

$$|\mathtt{bfNC}_2^n(\mathbf{J}_{\rho})| = K_n \cdot |\mathbf{J}_{\rho}|^n + O(|\mathbf{J}_{\rho}|^{n-1}).$$

Proof. Let us consider $\pi = \{\xi_1, \ldots, \xi_{2n}\} \in \mathtt{bfNC}_2^n(\mathbf{J}_{\rho})$ associated with a non-crossing pair partition of n blocks. Consider the block $\{1, 2k\}$ labelled with $\xi = \xi_1 = \xi_{2k}$. The bf-ordered subpartition $\pi' = \{\xi_1, \ldots, \xi_{2k}\}$ has exactly one outer block, i.e. $\pi' \in \mathtt{bfNC}_{2,1}^k(\mathbf{J}_{\rho})$. The remaining part $\{\xi_{2k+1}, \ldots, \xi_{2n}\}$ belongs to $\mathtt{bfNC}_2^{n-k}(\mathbf{J}_{\rho})$ and its ordering can be chosen independently of π'. That is why

$$|\mathtt{bfNC}_2^n(\mathbf{J}_{\rho})| = \sum_{k=1}^{n} |\mathtt{bfNC}_2^{n-k}(\mathbf{J}_{\rho})| B_{\rho}^{1}(k) = \Big(\sum_{k=1}^{n} K_{n-k}L_k\Big) \cdot |\mathbf{J}_{\rho}|^n + O(|\mathbf{J}_{\rho}|^{n-1}),$$

by Lemma 3.2 and the inductive assumption. ■

The application of Lemma 3.3 completes the proof of Theorem 3.1, because (cf. (3.2))

$$g_n(d) = \lim_{\rho\to\infty} \frac{|\mathtt{bfNC}_2^n(\mathbf{J}_{\rho})|}{|\mathbf{J}_{\rho}|^n} = K_n = \sum_{k=1}^{n} K_{n-k}L_k < \infty. \; ■$$

REMARK 3.2. *The same idea of proof applies to bf-CLT in higher dimensional space $\mathbb{R}^d$. The difference is that the variable v_j, $j = 0, \ldots, d$, in Lemma 3.1 is the volume of the d-dimensional interval $[\xi_{j+1}, \xi_j]$. Also all possible coordinates of differences $\xi_j - \xi_{j+1}$ will appear in the polynomial $p_{n,t}$. Consequently, the d-th power of $\gamma(k, d)$ will appear in the constant L_k, but the limit g_n remains finite for any $n \in \mathbb{N}$.*

REMARK 3.3. *The recurrence for the sequence $(g_n)_{n\in\mathbb{N}}$ is given by*

$$g_0 = 1, \quad g_n = \sum_{k=1}^{n} g_{n-k}L_k, \; n > 1,$$

where L_k is a linear combination of the coefficients in the polynomial $P_{n,1}$ from Lemma 3.1, and thus is difficult to compute. The first values of the sequence, for $d = 2$, are

$$g_0 = g_1 = 1, \; g_2 = \frac{3}{2}, \; g_3 = \frac{22}{9}, \; g_4 = \frac{599}{144}.$$

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