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CLASSICAL METHOD OF CONSTRUCTING A COMPLETE SET OF IRREDUCIBLE REPRESENTATIONS OF SEMIDIRECT PRODUCT OF A COMPACT GROUP WITH A FINITE GROUP

BY

TAKESHI HIRAI (KYOTO)

Abstract. Let $G = U \rtimes S$ be a group of semidirect product of U compact and S finite. For an irreducible representation (= IR) ρ of U, let $S([\rho])$ be the stationary subgroup in S of the equivalence class $[\rho] \in \widehat{U}$. Intertwining operators $J_\rho(s)$ $(s \in S([\rho]))$ between ρ and ${}^{s^{-1}}\rho$ gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation J'_ρ of a covering group $S([\rho])'$ of $S([\rho])$. Put $\pi^0 := \rho \cdot J'_\rho$, and take a spin representation π^1 of $S([\rho])$ corresponding to the factor set inverse to that of J_ρ, and put $\Pi(\pi^0, \pi^1) = \mathrm{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxdot \pi^1)$. We give a simple proof that $\Pi(\pi^0, \pi^1)$ is irreducible and that any IR of G is equivalent to some of $\Pi(\pi^0, \pi^1)$.

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1. INTRODUCTION

Let $G = U \rtimes S$ be a semidirect product group, with U compact and S finite. For an irreducible representation (= IR) ρ of U, let $S([\rho])$ be the stationary subgroup in S of the equivalence class $[\rho] \in \widehat{U}$. Intertwining operators $J_\rho(s)$ $(s \in S([\rho]))$, defined by $\rho\big(s(u)\big) = J_\rho(s)\,\rho(u)\,J_\rho(s)^{-1} (u \in U)$, gives in general a spin (= projective) representation of $S([\rho])$, which is lifted up to a linear representation J'_ρ of a certain covering group $S([\rho])'$ of $S([\rho])$ (cf. Lemma 2.2). Put $\pi^0 := \rho \cdot J'_\rho$. Take a spin representation π^1 of $S([\rho])$ corresponding to the factor set inverse to that of J_ρ; then the tensor product $\pi^0 \boxdot \pi^1$ is a non-spin IR of $U \rtimes S([\rho])$. Inducing $\pi^0 \boxdot \pi^1$ up to G, we get

$$\Pi(\pi^0, \pi^1) = \mathrm{Ind}^G_{U \rtimes S([\rho])}(\pi^0 \boxdot \pi^1).$$

We give a simple proof for the following theorem (cf. Theorem 5.1).

THEOREM 1.1. *Each* $\Pi(\pi^0, \pi^1)$ *is irreducible and the set of* $\Pi(\pi^0, \pi^1)$ *given above is complete in the sense that the dual* $\widehat{G}$ *of* G *has a complete set of representatives in it, or that any irreducible representation of* G *is equivalent to some of* $\Pi(\pi^0, \pi^1)$*'s.*

Our proof is elementary and needs only the least minimum on projective representations of groups, given in Section 2. (See [8] or [10] for more knowledge on general theory of such representations.) For proofs of irreducibility and completeness, we utilize fully characters of induced representations. Thus, our proof is completely independent of the results of Clifford in [1], with which we can give another proof (cf. [4], Section 3). It is also independent of Mackey's works [6], [7], in particular on construction of IRs of semidirect product groups.

2. PREPARATORY LEMMAS

2.1. Central extension. Let G' be a topological group and Z its closed central subgroup. Put $G := G'/Z$. Then the following sequence is exact:

$$1 \to Z \to G' \to G \to 1 \ \text{(exact)}, \tag{2.1}$$

and G' is called a *central extension* of G by Z. We call G' also a *covering group* of G, and G the *base group* of G'.

A representation Π of G' is said to be of *spin type* $\chi \in \widehat{Z}$ if $\Pi(z) = \chi(z)I$ $(z \in Z)$, where I denotes the identity operator. Denote by $\widehat{G'}$ the dual of G' consisting of equivalence classes $[\Pi]$ of unitary irreducible representations Π of G', and by $\widehat{G'}^{\chi}$ its subset consisting of $[\Pi]$ such that the spin type of Π is $\chi \in \widehat{Z}$. For a compact group H, denote by μ_H the normalized Haar measure on H such that $\mu_H(H) = 1$, and denote by $L^2(H)$ the Hilbert space of L^2-functions on H with respect to μ_H.

A function f on G' is said to be of *spin type* χ if $f(zg') = \chi(z)f(g')$ $(z \in Z, g' \in G')$. Assume G' is compact. For $\chi \in \widehat{Z}$, denote by $L^2(G'; \chi)$ the subspace of $L^2(G')$ consisting of all $f \in L^2(G')$ of spin type χ. Then $L^2(G')$ is an orthogonal direct sum of $L^2(G'; \chi)$ over $\chi \in \widehat{Z}$. A matrix element f of a representation Π of spin type χ is a spin function of the same type. Take a complete set of representatives $\Omega^{\chi} = \{\Pi\}$ of $\widehat{G'}^{\chi}$, and denote by $\mathcal{M}([\Pi])$ the space spanned by matrix elements of Π. Then $L^2(G'; \chi)$ is an orthogonal direct sum of $\mathcal{M}([\Pi])$ over $\Pi \in \Omega^{\chi}$.

LEMMA 2.1. (i) *Let* G' *be finite. Then for each* $\chi \in \widehat{Z}$

$$\sum_{[\Pi] \in \widehat{G'}^{\chi}} (\dim \Pi)^2 = \frac{1}{|Z|}\,|G'| = |G|.$$

(ii) *Let* G' *be infinite compact. Then for each* $\chi \in \widehat{Z}$ *the number of equivalence classes in* $\widehat{G'}^{\chi}$ *is infinite.*

P r o o f. The dimension of $\ell^2(G';\chi)$ is equal to the sum of $\dim\mathcal{M}([\Pi])$ over $[\Pi]\in\widehat{G'}^{\chi}$. Matrix elements of Π span $\mathcal{M}([\Pi])$, and $\dim\mathcal{M}([\Pi])=(\dim\Pi)^2$. This gives us $\dim\ell^2(G';\chi)=\sum_{[\Pi]\in\Omega^{\chi}}(\dim\Pi)^2$. On the other hand, a function f on G' belongs to $\ell^2(G';\chi)$ if and only if it satisfies $f(zg')=\chi(z)f(g')$ $(z\in Z,\ g'\in G')$. Hence $\dim\ell^2(G';\chi)=|G'|/|Z|=|G|$. ■

2.2. Cocycle and central extension. Let G be a topological group. A *projective* or *spin* representation π of G is defined as a map assigning to each $g\in G$ a linear map $\pi(g)$ on a vector space $V(\pi)$ which satisfies

$$\pi(g)\pi(h)=r_{g,h}\pi(gh)\quad(g,h\in G), \tag{2.2}$$

where $r_{g,h}\in\boldsymbol{C}^{\times}:=\{a\in\boldsymbol{C}\,;\,a\neq 0\}$. The function $r_{g,h}$ on $G\times G$ is called the *factor set* associated with π. If we replace $\pi(g)$ by its scalar multiple $\pi'(g):=\lambda_g\pi(g)$, then $\pi'(g)\pi'(h)=r'_{g,h}\pi'(gh)$ with $r'_{g,h}=(\lambda_g\lambda_h/\lambda_{gh})r_{g,h}$. When π is unitary, $r_{g,h}\in\boldsymbol{T}^1:=\{a\in\boldsymbol{C}\,;\,|a|=1\}$.

On the other hand, a function $r_{g,h}\in\boldsymbol{C}^{\times}$ $((g,h)\in G\times G)$ is called a *two-cocycle* of G (with values in $\boldsymbol{C}^{\times}$) if it satisfies

$$r_{g,h}r_{gh,k}=r_{g,hk}r_{h,k}\quad(g,h,k\in G). \tag{2.3}$$

Defining $r_{g,h}$ to be equivalent to $r'_{g,h}$ given above, we have the second cohomology group $H^2(G,\boldsymbol{C}^{\times})$ with multiplication product, called the *Schur multiplier* of G.

We assume that a cocycle $r_{g,h}$ is $\boldsymbol{T}^1$-valued, continuous, and normalized as $r_{e,e}=1$. Let Z be the closed subgroup of $\boldsymbol{T}^1$ generated by the set of values $r_{g,h}$. Then, since Z is closed, we have only the following two cases:

Case 1. $Z=\langle e^{2\pi i/n}\rangle\cong\boldsymbol{Z}_n$, a cyclic group of order n.

Case 2. $Z=\boldsymbol{T}^1$.

Starting with a cocycle $r_{g,h}$, we can define a central extension G' of G as follows.

LEMMA 2.2. *Let G be a topological group, $r_{g,h}$ $(g,h\in G)$ a cocycle, and let $Z\subset\boldsymbol{T}^1$ be the closed subgroup generated by the set of values $r_{g,h}$.*

(i) *Introduce in the set $Z\times G$ the following product rule*:

$$(z,g)(z',h):=(zz'r_{g,h},\,gh)\quad(z,z'\in Z,\ g,h\in G). \tag{2.4}$$

Then we get a central extension G' of G by Z as in (2.1).

(ii) *Let π be a projective* (*or spin*) *representation of G whose factor set is $r_{g,h}$* (*resp. $r_{g,h}^{-1}$*)*. Then it can be lifted up to a linear representation π' of G' acting on the same representation space $V(\pi)$ in such a way that $\pi'\big((z,g)\big):=z\,\pi(g)$* (*resp. $z^{-1}\pi(g)$*) *for $(z,g)\in G'$.*

We say that the central extension G' in (i) is *associated* with the cocycle $r_{g,h}$, and that the representation π' in (ii) is called a *spin* representation of G' (and also a *spin* representation of G). When we apply this lemma later in Section 3, it is for a finite group such as $S([\rho])$, and so the central subgroup Z is finite in that case (cf. Section 3).

2.3. Central character of G' and spin representation of G. Let G' be a central extension of G by a closed central subgroup Z as in (2.1). Take a section $s : G \to \mathcal{S}_G \subset G'$ for the canonical homomorphism $G' \to G$. Then, for $g, h \in G$, we have $s(g)s(h) = z_{g,h}s(gh)$ with a $z_{g,h} \in Z$.

LEMMA 2.3. *For a representation Π of G' of spin type $\chi \in \widehat{Z}$, put $\pi(g) := \Pi\big(s(g)\big)$ $(g \in G)$. Then π is a spin representation of G with factor set $t_{g,h} = \chi(z_{g,h})$.*

LEMMA 2.4. *Let G be a compact group, and $r_{g,h}$ a continuous cocycle of G with values in $\boldsymbol{T}^1$. Take a central extension G' of G associated with $r_{g,h}$. Then there exist unitary IRs of G' of a certain spin type χ (resp. χ_0) such that $\chi(z_{g,h}) = r_{g,h}$ (resp. $\chi_0(z_{g,h}) = r_{g,h}^{-1}$).*

For the existence of such a π, we apply Lemma 2.1.

Let π and π'' be spin representations of G with factor sets $r_{g,h}$ and $r''_{g,h}$, respectively. Then the tensor product $\pi \otimes \pi''$ is a spin representation with factor set $r_{g,h}r''_{g,h}$. Therefore, if $r''_{g,h}$ is the inverse of $r_{g,h}$, that is, $r''_{g,h} = r_{g,h}^{-1}$ $(g, h \in G)$, then $\pi \otimes \pi''$ is non-spin or is reduced from G' to a linear representation of the base group G.

3. CASE OF SEMIDIRECT PRODUCT GROUPS

Let G be a compact group of semidirect product type $G = U \rtimes S$, where U is a compact group, normal in G, and S is a finite group. Here the action of $s \in S$ on $u \in U$ is denoted by $s(u)$.

Take an IR ρ of U and consider its equivalence class $[\rho] \in \widehat{U}$. Every $s \in S$ acts on ρ as ${}^s\rho(u) := \rho\big(s^{-1}(u)\big)$ $(u \in U)$, and on equivalence classes as $[\rho] \to [{}^s\rho]$. Denote by $\widehat{U}/S$ the set of S-orbits in the dual $\widehat{U}$ of U. Take a stationary subgroup $S([\rho])$ of $[\rho]$ in S, that is, $S([\rho]) = \{s \in S\,;\, {}^s\rho \cong \rho\}$. Put $H := U \rtimes S([\rho])$. For $s \in S([\rho])$, we determine explicitly an intertwining operator $J_\rho(s)$ as

$$\rho\big(s(u)\big) = J_\rho(s)\,\rho(u)\,J_\rho(s)^{-1} \quad (u \in U). \tag{3.1}$$

Then it is determined up to a non-zero scalar factor. Hence we have a projective representation $S([\rho]) \ni s \mapsto J_\rho(s)$. Let $\alpha_{s,t}$ be its factor set given as

$$J_\rho(s)J_\rho(t) = \alpha_{s,t}\,J_\rho(st) \quad (s, t \in S([\rho])).$$

Let $S([\rho])'$ be a central extension of $S([\rho])$ associated with the cocycle $\alpha_{s,t}$: $1 \to Z \to S([\rho])' \xrightarrow{\Phi_S} S([\rho]) \to 1$ (exact), where Φ_S denotes the canonical homomorphism. Then, by Lemma 2.4, J_ρ can be lifted up to a linear representation J'_ρ of $S([\rho])'$. Put $H' := U \rtimes S([\rho])'$ with the action $s'(u) := s(u)$, $s' \in S([\rho])'$, $s = \Phi_S(s')$. Put also

$$\pi^0\big((u, s')\big) := \rho(u) \cdot J'_\rho(s') \quad (u \in U,\ s' \in S([\rho])').$$

Then $\pi^0 = \rho \cdot J'_\rho$ is an IR of H'. Take an IR π^1 of $S([\rho])'$ and consider it as a representation of H' through the homomorphism $H' \to S([\rho])' \cong H'/U$, and consider the inner tensor product $\pi := \pi^0 \boxdot \pi^1$ as a representation of H'. Let the factor set of π^1, viewed as a spin representation of the base group $S([\rho])$, be $\beta_{s,t}$; then that of π is $\alpha_{s,t}\,\beta_{s,t}$.

To get an IR of G, we pick up π^1 with the factor set $\beta_{s,t} = \alpha_{s,t}^{-1}$ (this is possible by Lemmas 2.1 and 2.4). Then π becomes a linear representation of the base group $H = U \rtimes S([\rho])$. Thus we obtain a representation of G by inducing it up as

$$\Pi(\pi^0, \pi^1) := \mathrm{Ind}_H^G\, \pi = \mathrm{Ind}_H^G(\pi^0 \boxdot \pi^1). \tag{3.2}$$

LEMMA 3.1. *Let ρ be an IR of U, and J'_ρ of $S([\rho])'$ and $\pi^0 = \rho \cdot J'_\rho$ of $H' = U \rtimes S([\rho])'$ be as above. Let π^1 and π^1_o be IRs of $S([\rho])'$, mutually inequivalent, with the factor set inverse to that of J_ρ. Then $\pi = \pi^0 \boxdot \pi^1$ and $\pi_o := \pi^0 \boxdot \pi^1_o$ are irreducible and mutually inequivalent as representations of $H = U \rtimes S([\rho])$.*

4. CHARACTER AND IRREDUCIBILITY OF $\Pi(\pi^0, \pi^1)$

4.1. Character of $\Pi(\pi^0, \pi^1)$. Put $\Pi = \Pi(\pi^0, \pi^1)$, and let χ_Π be the character of Π. Since $\Pi = \mathrm{Ind}_H^G \pi$, we have the following expression of χ_Π from the general formula for induced representations:

$$\chi_\Pi(g) = \int_{H\backslash G} \chi_\pi(kgk^{-1})\, d\nu_{H\backslash G}(\dot{k}). \tag{4.1}$$

Here the character χ_π of π is extended from H to G by putting 0 outside H, and $\nu_{H\backslash G}$ is the invariant measure on $H\backslash G$ giving mass one to each point, and $\dot{k} = Hk$. Since $H\backslash G \cong S([\rho])\backslash S$ is finite, (4.1) can be rewritten, using the normalized Haar measure μ_G on G, as

$$\chi_\Pi(g) = |H\backslash G| \int_G \chi_\pi(kgk^{-1})\, d\mu_G(k). \tag{4.2}$$

Note that, for $(u, s) \in H = U \rtimes S([\rho])$, $\chi_\pi\big((u, s)\big) = \chi_{\pi^0}\big((u, s')\big)\, \chi_{\pi^1}(s')$, with a preimage $s' \in S([\rho])'$ of s: $s = \Phi_S(s')$.

4.2. Irreducibility of $\Pi(\pi^0, \pi^1)$.

THEOREM 4.1. *Let $G = U \rtimes S$ with U compact and S finite. Then the induced representation $\Pi(\pi^0, \pi^1) = \mathrm{Ind}_H^G(\pi^0 \boxdot \pi^1)$ of G in* (3.2) *is irreducible.*

To prove this, we use the following lemma.

LEMMA 4.1. *Let ρ_o be an IR of U, and define an IR $\pi^0_o := \rho_o \cdot J'_{\rho_o}$ of $H'_o := U \rtimes S([\rho_o])'$ in a similar way to $\pi^0 = \rho \cdot J'_\rho$ of $H' = U \rtimes S([\rho])'$. Assume that ρ_o is not equivalent to ρ. Then, for any $s' \in S([\rho])'$, $s'' \in S([\rho_o])'$,*

$$\int_U \chi_{\pi^0}\big((u, s')\big)\, \overline{\chi_{\pi^0_o}\big((u, s'')\big)}\, d\mu_U(u) = 0. \tag{4.3}$$

P r o o f. Note that the character $\chi_{\pi^0}\big((u,s')\big) = \mathrm{tr}\big(\rho(u)J'_\rho(s')\big)$ is, as a function in $u \in U$, a linear combination of matrix elements of ρ. Similarly, $\chi_{\pi^0_o}\big((u,s'')\big) = \mathrm{tr}\big(\rho_o(u)J'_{\rho_o}(s'')\big)$ is a linear combination of matrix elements of ρ_o. On the other hand, any matrix element of ρ is orthogonal in $L^2(U)$ to any such one of ρ_o. Hence the assertion of the lemma follows. ■

P r o o f o f T h e o r e m 4.1. Put $\Pi = \Pi(\pi^0, \pi^1)$. Note that Π is irreducible if and only if

$$\|\chi_\Pi\|^2 = \int_G |\chi_\Pi(g)|^2\, d\mu_G(g) = 1. \tag{4.4}$$

Therefore, it is enough for us to calculate the integral $\int_G |\chi_\Pi(g)|^2\, d\mu_G(g)$. It is equal to

$$\begin{aligned}|H\backslash G|^2 \int_G \iint_{G\times G} &\chi_\pi(k_1 g k_1^{-1})\, \overline{\chi_\pi(k_2 g k_2^{-1})}\, d\mu_G(k_1)\, d\mu_G(k_2)\, d\mu_G(g)\\ &= |H\backslash G|^2 \int_G\int_G \chi_\pi(g)\, \overline{\chi_\pi(kgk^{-1})}\, d\mu_G(k)\, d\mu_G(g)\\ &= \int_H \int_{H\backslash G} \chi_\pi(h)\, \overline{\chi_\pi(khk^{-1})}\, d\nu_{H\backslash G}(\dot{k})\, d\mu_H(h) =: I_\pi \ \text{(put)}.\end{aligned}$$

Take a complete set of representatives of $H\backslash G \cong S([\rho])\backslash S$ as $\{s_q \in S\,;\, q \in Q\}$ with $s_{q_0} = e$. Then

$$I_\pi = \sum_{q\in Q} \int_H \chi_\pi(h)\, \overline{\chi_\pi(s_q h s_q^{-1})}\, d\mu_H(h). \tag{4.5}$$

On the other hand, for $h' = (u, s') \in H' = U \rtimes S([\rho])'$ with $h = (u,s)$, $s = \Phi_S(s')$, we have $s_q h s_q^{-1} = (s_q u s_q^{-1}, s_q s s_q^{-1})$. Hence

$$\begin{aligned}\chi_\pi(h) &= \mathrm{tr}\big(\rho(u)J'_\rho(s')\big)\cdot \chi_{\pi^1}(s'),\\ \chi_\pi(s_q h s_q^{-1}) &= \mathrm{tr}\big(\rho(s_q u s_q^{-1})J'_\rho(s'_q s' {s'_q}^{-1})\big)\cdot \chi_{\pi^1}(s'_q s' {s'_q}^{-1}),\end{aligned} \tag{4.6}$$

where $s'_q \in S([\rho])'$ is a preimage of s_q: $\Phi_S(s'_q) = s_q$. For any s_q, $q \neq q_0$, since $s_q \notin S([\rho])$, IR ${}^{s_q^{-1}}\rho$ is not equivalent to ρ, where $({}^{s_q^{-1}}\rho)(u) = \rho(s_q u s_q^{-1})$. Therefore, taking into account $d\mu_H(h) = d\mu_U(u)\, d\mu_{S([\rho])}(s)$ for $h = (u,s) \in U \rtimes S([\rho])$, and the function form in (4.6), we can apply Lemma 4.1 to the integral term in (4.5) for $q \neq q_0$, and get $= 0$. Thus, applying Lemma 3.1 to the case of $q = q_0$, we obtain, as desired,

$$I_\pi = \int_H \chi_\pi(h)\, \overline{\chi_\pi(h)}\, d\mu_H(h) \ = \ 1. \ ■$$

4.3. Orthogonality of characters χ_Π. For $G = U \rtimes S$, let $\{\rho_i\,;\ \text{IR of } U,\ i \in I_{U,S}\}$ be a complete set of representatives for $\widehat{U}/S$, and for each $i \in I_{U,S}$ let $\{\pi^1_{i,j}\,;\ j \in J_i\}$ be a complete set of representatives of equivalence classes of IRs of $S([\rho_i])'$ with factor set inverse to that of J_{ρ_i}. Put $H_i := U \rtimes S([\rho_i])$, $H'_i := U \rtimes S([\rho_i])'$, $\pi_{i,j} = \pi^0_i \boxdot \pi^1_{i,j}$, and $\Pi_{i,j} = \Pi(\pi^0_i, \pi^1_{i,j}) = \mathrm{Ind}^G_{H_i}\pi_{i,j}$. Define a set of IRs of G as

$$\Omega(G) := \{\Pi_{i,j} := \Pi(\pi^0_i, \pi^1_{i,j})\,;\ i \in I_{U,S},\ j \in J_i\}. \tag{4.7}$$

THEOREM 4.2. *For characters* $\chi_{\Pi_{i,j}}$, $\Pi_{i,j} \in \Omega(G)$, *the following orthogonality relations in* $L^2(G)$ *hold true*:

$$\big\langle \chi_{\Pi_{i,j}}, \chi_{\Pi_{i',j'}} \big\rangle_{L^2(G)} = \begin{cases} 1 & \textit{if } (i,j) = (i',j'), \\ 0 & \textit{if } (i,j) \neq (i',j'). \end{cases} \tag{4.8}$$

P r o o f. The case of $(i,j) = (i',j')$ has been shown in the proof of Theorem 4.1. Assume $(i,j) \neq (i',j')$, and put $I^{i,j}_{i',j'} := \big\langle \chi_{\Pi_{i,j}}, \chi_{\Pi_{i',j'}} \big\rangle_{L^2(G)}$. Then, as in the proof of Theorem 4.1, we have

$$I^{i,j}_{i',j'} = |H_i\backslash G| \cdot |H_{i'}\backslash G| \int_G \int_G \chi_{\pi_{i,j}}(g)\, \overline{\chi_{\pi_{i',j'}}(kgk^{-1})}\, d\mu_G(k)\, d\mu_G(g).$$

Take a complete set of representatives of $H_{i'}\backslash G \cong S([\rho_{i'}])\backslash S$ as $\{s_q \in S\,;\ q \in Q\}$ with $s_{q_0} = e$. Then

$$I^{i,j}_{i',j'} = \sum_{q \in Q} \int_{H_i} \chi_{\pi_{i,j}}(h)\, \overline{\chi_{\pi_{i',j'}}(s_q h s_q^{-1})}\, d\mu_{H_i}(h). \tag{4.9}$$

(1) For $i = i'$, according to the reasoning in the paragraph following the formula (4.5), by Lemma 4.1 we have to consider only the term for $q = q_0$: $I^{i,j}_{i,j'} = \int_{H_i} \chi_{\pi_{i,j}}(h)\, \overline{\chi_{\pi_{i,j'}}(h)}\, d\mu_{H_i}(h)$. By Lemma 3.1, we know that $\pi_{i,j}$ is irreducible, and that $\pi_{i,j} \not\cong \pi_{i,j'}$. Therefore $I^{i,j}_{i,j'} = \delta_{j,j'}$, as desired.

(2) Assume that $i \neq i'$. Since S-orbits of $[\rho_i]$ and $[\rho_{i'}]$ are different, there exists no s_q such that ${}^{s_q^{-1}}\rho_{i'}$ is equivalent to ρ_i. In the sum over $q \in Q$ in (4.9), note that $d\mu_{H_i}(h) = d\mu_U(u)\, d\mu_{S[\rho_i]}(s)$ for $h = (u,s)$, $u \in U$, $s \in S([\rho_i])$, and apply Lemma 4.1 again. Then we see that the integral for any q is equal to zero. So $I^{i,j}_{i,j'} = 0$, as desired. ■

COROLLARY 4.1. *The set* $\Omega(G)$ *of IRs of G defined in* (4.7) *consists of mutually inequivalent IRs.*

5. COMPLETENESS OF THE SET $\Omega(G)$ OF IRs

Let us prove that the set $\Omega(G)$ in (4.7) of IRs $\Pi_{i,j} = \Pi(\pi_i^0, \pi_{i,j}^1)$ is complete, or that our method of induced representations gives essentially all IRs of G.

THEOREM 5.1. *Let $G = U \rtimes S$ be such that U is compact and S is finite. Let $\Omega(G)$ be the set of IRs of G defined in* (4.7)*. Then $\Omega(G)$ gives a complete set of representatives of the dual $\widehat{G}$.*

For the proof, first note that

$$\dim \Pi_{i,j} = \dim \pi_{i,j} \cdot |H_i \backslash G| = \dim \rho_i \cdot \dim \pi_{i,j}^1 \cdot |S([\rho_i]) \backslash S|.$$

Recall that $\{\pi_{i,j}^1,\, j \in J_i\}$ is a complete set of representatives of spin IRs of $S([\rho_i])'$, viewed from the base group $S([\rho_i])$, of a fixed factor set (the inverse of that of J_ρ). Then we have, by Lemma 2.1 (i), $\sum_{j\in J_i} (\dim \pi_{i,j}^1)^2 = |S([\rho_i])|$, whence

$$\sum_{j\in J_i} (\dim \Pi_{i,j})^2 = (\dim \rho_i)^2 \cdot |S([\rho_i]) \backslash S| \times |S|. \tag{5.1}$$

5.1. Proof in the case where G is finite. Assume that $G = U \rtimes S$ is finite. By Theorem 4.2, to prove the completeness, it is enough to establish the equality

$$\sum_{\Pi_{i,j}\in\Omega(G)} (\dim \Pi_{i,j})^2 = |G|. \tag{5.2}$$

Note that $(\dim \rho_i)^2 \cdot |S([\rho_i]) \backslash S|$ is equal to the sum of $(\dim \rho)^2$ over $[\rho]$ in the S-orbit of $[\rho_i]$. Since $\{\rho_i\,;\, I_{U,S}\}$ is a complete set of representatives of $\widehat{U}/S$, we have

$$\sum_{i\in I_{U,S}} (\dim \rho_i)^2 \cdot |S([\rho_i]) \backslash S| = \sum_{[\rho]\in\widehat{U}} (\dim \rho)^2 = |U|.$$

By (5.1), this gives the desired equality (5.2), because $|U| \cdot |S| = |G|$.

5.2. Proof in the case where G is compact. Let $G = U \rtimes S$ with U compact and S finite. In this case, to prove the completeness of the set $\Omega(G)$ of $\Pi_{i,j}$'s, first we give the following lemma, which corresponds to Lemma 2.1 for G finite.

LEMMA 5.1. *Let ρ be an IR of U. Then the number of equivalence classes $[\Pi] \in \widehat{G}$ of IRs Π of G such that $\Pi|_U$ contains ρ, or $\Pi|_U \supset \rho$, is finite, and*

$$\sum_{[\Pi]\in\widehat{G}:\, \Pi|_U\supset\rho} (\dim \Pi)^2 = (\dim \rho)^2 \cdot |S([\rho]) \backslash S| \cdot |S|. \tag{5.3}$$

P r o o f. Denote by $\mathcal{M}_\rho(G)$ the space spanned by matrix elements of $\mathrm{Ind}_U^G \rho$. Then it is a direct sum of spaces $\mathcal{M}(\Pi)$ spanned by matrices of Π, which appear in $\mathrm{Ind}_U^G \rho$ or $\left[\mathrm{Ind}_U^G \rho : \Pi\right] > 0$. By the Frobenius reciprocity law we have $\left[\Pi|_U : \rho\right] =$

$\big[\mathrm{Ind}_U^G\rho : \Pi\big]$, and the last condition is equivalent to $\big[\Pi|_U : \rho\big] > 0$, that is, $\Pi|_U \supset \rho$. Hence we obtain

$$\dim \mathcal{M}_\rho(G) = \sum_{[\Pi]\in\widehat{G}:\,\Pi|_U\supset\rho} (\dim \Pi)^2. \tag{5.4}$$

On the other hand, the space $V(\mathrm{Ind}_U^G\rho)$ is spanned by $V(\rho)$-valued functions f on G such that $f(ug) = \rho(u)f(g)$ $(u \in U,\ g \in G)$. Therefore f corresponds one-to-one to $\varphi := f|_S$ in $\mathcal{F}\big(S; V(\rho)\big)$, the space of $V(\rho)$-valued L^2-functions on S for which the norm is $\|\varphi\|^2 = \int_S \|\varphi(s)\|^2_{V(\rho)} d\mu_S(s)$, where $\|\cdot\|_{V(\rho)}$ denotes the norm in $V(\rho)$. Denote by Π_ρ the realization of $\mathrm{Ind}_U^G\rho$ on $\mathcal{F}\big(S; V(\rho)\big)$. Note that, for $s \in S$ and $g_0 = (u_0, s_0) \in U \rtimes S$, we have $sg_0 = (e, s)g_0 = (su_0s^{-1}, ss_0)$, and so

$$\Pi_\rho(g_0)\varphi(s) = \rho(su_0s^{-1})\big(\varphi(ss_0)\big). \tag{5.5}$$

The space $\mathcal{F}\big(S; V(\rho)\big)$ is spanned by functions of the form $\varphi_{v,\psi}(s) := v \cdot \psi(s)$ $(s \in S)$, where $v \in V(\rho)$, $\psi \in L^2(S)$. Take $\varphi_1, \varphi_2 \in \mathcal{F}\big(S; V(\rho)\big)$ as $\varphi_i(s) = v_i \cdot \psi_i(s)$ $(s \in S)$ with $v_i \in V(\rho)$ and $\psi_i \in L^2(S)$. Calculate the matrix element for Π_ρ as

$$\begin{aligned}\big\langle \Pi_\rho(g_0)\varphi_1, \varphi_2\big\rangle &= \int_S \big\langle \Pi_\rho(g_0)\varphi_1(s), \varphi_2(s)\big\rangle_{V(\rho)}\, d\mu_S(s)\\ &= \int_S \big\langle \rho(su_0s^{-1})v_1, v_2\big\rangle_{V(\rho)}\, \psi_1(ss_0)\,\overline{\psi_2(s)}\,\mu_S(s) =: F(g_0)\ \text{(put)}.\end{aligned}$$

For $t \in S$, denote by δ_t the delta functions on S given as $\delta_t(s) = 1$ or 0 according to $s = t$ or not. Put $\psi_i = \delta_{t_i}$ for $t_i \in S$. Then

$$F(g_0) = |S|^{-1} \cdot \big\langle \rho(t_2u_0t_2^{-1})v_1, v_2\big\rangle_{V(\rho)} \cdot \delta_{t_2^{-1}t_1}(s_0). \tag{5.6}$$

Here, the second factor, as a function in $u_0 \in U$, spans the space $\mathcal{M}([{}^{t_2^{-1}}\rho])$ of matrix elements of an IR ${}^{t_2^{-1}}\rho$ of U. The third factor, as a function in $s_0 \in S$, i.e. $\delta_t(s_0)$ with $t = t_2^{-1}t_1$, spans the space $\mathcal{F}(S)$ of all functions on S. Thus we obtain

$$\begin{aligned}\dim \mathcal{M}_\rho(G) &= \sum_{\text{different } [{}^s\rho]} \dim \mathcal{M}([{}^s\rho]) \cdot \dim \mathcal{F}(S)\\ &= \sum_{[{}^s\rho]} (\dim {}^s\rho)^2 \cdot |S| = (\dim \rho)^2 \cdot |S([\rho])\backslash S| \cdot |S|.\end{aligned} \tag{5.7}$$

From (5.4) and (5.7) we obtain the desired equality (5.3). ■

Applying Lemma 5.1 we see that the completeness of the set $\Omega(G)$ of $\Pi_{i,j}$ is equivalent to the following equality, given already in (5.1): for each $i \in I_{U,S}$,

$$\sum_{j\in J_i} (\dim \Pi_{i,j})^2 = (\dim \rho_i)^2 \cdot |S([\rho_i])\backslash S| \cdot |S|. \ ■$$

REMARK 5.1. In his Chicago lecture note [6], Mackey discussed construction of irreducible representation or factor representations of semidirect product groups $U \rtimes S$ mainly in a locally compact case. However, an explicit statement such as Theorem 1.1 or Theorem 5.1 in the present paper cannot be found for the case where U is compact and S is finite. It seems that his discussion is beyond this classical case (cf. also [7]).

Acknowledgments. The main result of this paper gives a general background of works [2], [3], and [5], in which we construct all spin IRs of generalized symmetric groups of certain spin types, using their semidirect product structures (and calculate spin characters and their limits). On these subjects, I had opportunities to give talks twice in Workshops "Non-Commutative Harmonic Analysis" at Będlewo organized by Professor M. Bożejko and peoples of Wrocław University, in August 2007 and August 2009. Each time I enjoyed warm hospitality in which I could think about a proof of the completeness of the set of IRs $\Pi(\pi^0, \pi^1)$, yet not knowing the work of Clifford [1]. I would like to thank them very much for these precious situations.

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22-8 Nakazaichi-Cho, Iwakura
Sakyo-Ku, Kyoto 606-0027, Japan
E-mail: hirai.takeshi.24e@st.kyoto-u.ac.jp

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