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SCHWINGER–DYSON EQUATIONS: CLASSICAL AND QUANTUM

BY

JAMES A. MINGO* (KINGSTON) AND ROLAND SPEICHER** (SAARBRÜCKEN)

Abstract. In this note we want to have another look on Schwinger–Dyson equations for the eigenvalue distributions and the fluctuations of classical unitarily invariant random matrix models. We are exclusively dealing with one-matrix models, for which the situation is quite well understood. Our point is not to add any new results to this, but to have a more algebraic point of view on these results and to understand from this perspective the universality results for fluctuations of these random matrices. We will also consider corresponding non-commutative or "quantum" random matrix models and contrast the results for fluctuations and Schwinger–Dyson equations in the quantum case with the findings from the classical case.

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1. NOTATION AND PREREQUISITES

1.1. Free probability theory. For the basic notions and results about free probability theory we refer to the books [14] and [11]; in particular, we will follow the latter in regard of the definitions and fundamental results on free cumulants.

1.2. Non-commutative derivatives. In the sequel we will denote by ∂ and D the non-commutative and the cyclic derivative, respectively; see, for example, [13] for definitions and basic properties; note that in [13] the cyclic derivative is denoted by δ. We will only use these derivatives in the one-variable case; then, the cyclic derivative D coincides with usual differentiation. On the algebra $\mathbb{C}\langle x\rangle$ of polynomials in one variable x these derivatives are given by

$$\begin{aligned} D : \mathbb{C}\langle x\rangle &\to \mathbb{C}\langle x\rangle \\ x^n &\mapsto Dx^n := nx^{n-1} \end{aligned}$$

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and

$$\begin{aligned}\partial : \mathbb{C}\langle x\rangle &\to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \\ x^n &\mapsto \partial x^n := \sum_{k=0}^{n-1} x^k \otimes x^{n-k-1}.\end{aligned}$$

1.3. The Chebyshev polynomials. We will use the Chebyshev polynomials of first and second kind, for the interval $[-2,2]$. The ones orthogonal with respect to the semicircle (second kind) are denoted by S_n, the ones orthogonal with respect to the arc-sine distribution (first kind) by C_n; compare [7]. We have

$$C_0(x)=2, \quad C_1(x)=x, \quad C_2(x)=x^2-2, \quad C_3(x)=x^3-3x$$

and

$$xC_n(x)=C_{n+1}(x)+C_{n-1}(x) \quad (n>1);$$

and

$$S_0(x)=1, \quad S_1(x)=x, \quad S_2(x)=x^2-1, \quad S_3(x)=x^3-2x$$

and

$$xS_n(x)=S_{n+1}(x)+S_{n-1}(x) \quad (n>1).$$

One has, for $n \geqslant 0$, the following identities:

$$DC_n = nS_{n-1}, \quad \partial S_n = \sum_{k=0}^{n-1} S_k \otimes S_{n-k-1}.$$

Furthermore, $C_n = S_n - S_{n-2}$ (those are true for all $n \geqslant 0$, if we set $S_{-2}(x) = -1$ and $S_{-1}(x)=0$) and for $n,m \geqslant 0$

$$\begin{aligned}S_nS_m &= S_{n+m}+S_{n+m-2}+\ldots+S_{|n-m|},\\ C_nC_m &= C_{n+m}+C_{|n-m|}.\end{aligned}$$

These imply that we have for all $n,m \geqslant 0$

$$C_nS_m = \begin{cases} S_{n+m}+S_{m-n}, & n \leqslant m,\\ S_{n+m}, & n=m+1,\\ S_{n+m}-S_{n-m-2}, & n \geqslant m+2.\end{cases} \tag{1.1}$$

1.4. Non-commutative probability space of second order. A *second order non-commutative probability space* $(\mathcal{A},\varphi_1,\varphi_2)$ consists of a unital algebra $\mathcal{A}$, a tracial linear functional $\varphi_1 : \mathcal{A} \to \mathbb{C}$ with $\varphi(1)=1$ and a bilinear functional $\varphi_2 : \mathcal{A}\times\mathcal{A} \to \mathbb{C}$, which is symmetric in both arguments, i.e., $\varphi_2(a,b)=\varphi_2(b,a)$ for all $a,b \in \mathcal{A}$, tracial in each of its both arguments and which satisfies $\varphi_2(a,1) = 0 = \varphi_2(1,b)$ for all $a,b \in \mathcal{A}$. Compare [8] for more information.

2. SCHWINGER–DYSON EQUATIONS FOR CLASSICAL UNITARILY INVARIANT ENSEMBLES

We will be interested in unitarily invariant random matrices; the most prominent class of random matrices of this type is given by a density of the following form. We consider Hermitian $N \times N$-random matrices $A = (a_{ij})_{i,j=1}^N$ equipped with the probability measure

$$d\mu_N(A) = \frac{1}{Z_N} \exp\{-N\mathrm{Tr}[P(A)]\}dA, \tag{2.1}$$

where

$$dA = \prod_{1 \leqslant i < j \leqslant N} d\,\mathrm{Re}\, a_{ij}\, d\,\mathrm{Im}\, a_{ij} \prod_{i=1}^N da_{ii}.$$

Here, P is a polynomial in one variable, which we will address in the following as "potential", and Z_N is a normalization constant to make (2.1) into a probability distribution.

At least formally, it is quite easy to see that the asymptotic eigenvalue distribution and fluctuations of these ensembles satisfy in the large N-limit the following so-called Schwinger–Dyson equations (see [4], Chapter 8), also called the method of equation of motion or the loop equation in [3], Chapter 6. We will ignore all analytic questions and just work in the algebraic setting; thus we take our non-commutative probability space $\mathcal{A} = \mathbb{C}\langle x \rangle$ as the polynomials in one variable x.

DEFINITION 2.1. Let $(\mathbb{C}\langle x\rangle, \varphi_1, \varphi_2)$ be a non-commutative probability space of second order and $V \in \mathbb{C}\langle x\rangle$ a polynomial in x. We put $\xi := DV(x) \in \mathbb{C}\langle x\rangle$. We say that φ_1 satisfies the *first order Schwinger–Dyson equations* for the potential V if we have for all $p(x) \in \mathbb{C}\langle x\rangle$

$$\varphi_1\big(\xi p(x)\big) = \varphi_1 \otimes \varphi_1\big(\partial p(x)\big) \tag{2.2}$$

(i.e., ξ is the conjugate variable for x). If we have in addition for all $p(x), q(x) \in \mathbb{C}\langle x\rangle$

$$\begin{aligned}\varphi_2\big(\xi p(x), q(x)\big) &\\ = \varphi_2\Big([\varphi_1 \otimes \mathrm{id} + \mathrm{id} \otimes \varphi_1]\big(\partial p(x)\big), q(x)\Big) &+ \varphi_1\big(p(x)Dq(x)\big),\end{aligned} \tag{2.3}$$

then (φ_1, φ_2) satisfies the *second order Schwinger–Dyson equations*.

Corresponding analogues exist also for the case of several matrices, but since we have nothing substantial to say about the multivariate case we will stick in the following to the one-matrix case. Existence and uniqueness of the solution of these equations (under positivity requirements for φ_1) are well-studied in the one-matrix case, and are one of the main problems in random matrix theory for the case of several variables; for some positive results in the latter case see [5].

We will in the following ignore the uniqueness question and present a solution to the Schwinger–Dyson equations for the one-matrix case.

THEOREM 2.1. *For a given $V \in \mathbb{C}\langle x\rangle$, we decompose DV with respect to the Chebyshev polynomials of the first kind*:

$$\xi = DV(x) = \sum_{n\geqslant 0} \alpha_n C_n(x).$$

Assume that we have normalized V in such a way that $\alpha_0 = 0$ and $\alpha_1 = 1$. We define on $\mathbb{C}\langle x\rangle$ a φ_1 by

$$\varphi_1\big(S_n(x)\big) := \alpha_{n+1} \quad (n \geqslant 0)$$

(*note that we need $\varphi_1(1) = \alpha_1 = 1$ for this*) *and a φ_2 by*

$$\varphi_2\big(C_n(x), C_m(x)\big) := n\delta_{nm} \quad (n, m \geqslant 0).$$

Then φ_1 and φ_2 satisfy the first and second order Schwinger–Dyson equations for the potential V.

The prescriptions above provide well-defined and unique φ_1 and φ_2, because both $\{S_n \mid n \geqslant 0\}$ and $\{C_n \mid n \geqslant 0\}$ are linear bases of $\mathbb{C}\langle x\rangle$.

Note also the crucial fact that φ_2 does not depend on V. Actually, our definition of φ_2 is in essence just a reformulation of the universality of the asymptotic fluctuations for the random matrix ensemble given by (2.1). In the physical literature this observation goes at least back to Politzer [12], culminating in the paper of Ambjørn et al. [1], whereas a proof on the mathematical level of rigour is due to Johansson [6]. The above theorem arouse out of our attempts to understand this universality result. Actually, it can (and should) also be seen as a streamlined algebraic proof of this universality result.

Our original motivation in this context was to look for multivariate versions of this result. As will be seen from the following proof, the result relies crucially on various algebraic properties of the Chebyshev polynomials, for which no multivariate version exists. Thus it should be clear that the universality result is a genuine one-dimensional phenomenon. Actually, in [8] we have shown, by using the machinery of second order freeness, that for one of the most canonical families of several random matrices the fluctuations depend indeed on the potential V.

P r o o f. Consider the first order. We have to show that

$$\varphi_1\big(\xi p(x)\big) = \varphi_1 \otimes \varphi_1\big(\partial p(x)\big)$$

for all $p(x) \in \mathbb{C}\langle x\rangle$. By linearity, it suffices to treat the cases $p(x) = S_m(x)$ for all $m \geqslant 0$. So fix such an m. Thus we have to show

$$\sum_{n\geqslant 0} \alpha_n \varphi_1\big(C_n(x)S_m(x)\big) = \varphi_1 \otimes \varphi_1\big(\partial S_m(x)\big).$$

For the left-hand side we have

$$\begin{aligned}\sum_n \alpha_n \varphi_1(C_n S_m) &= \sum_{n\leqslant m} \alpha_n\big(\varphi_1(S_{n+m}) + \varphi_1(S_{m-n})\big) + \alpha_{m+1}\varphi(S_{2m+1})\\ &\quad + \sum_{n\geqslant m+2} \alpha_n\big(\varphi_1(S_{n+m}) - \varphi_1(S_{n-m-2})\big)\\ &= \sum_{n\leqslant m} \alpha_n(\alpha_{n+m+1} + \alpha_{m-n+1}) + \alpha_{m+1}\alpha_{2m+2}\\ &\quad + \sum_{n\geqslant m+2} \alpha_n(\alpha_{n+m+1} - \alpha_{n-m-1})\\ &= \sum_n \alpha_n\alpha_{n+m+1} - \sum_{n\geqslant m+2}\alpha_n\alpha_{n-m-1} + \sum_{n\leqslant m}\alpha_n\alpha_{m-n+1}.\end{aligned}$$

But the first two sums cancel as the summation in n starts at $n = 1$ (since $\alpha_0 = 0$), and thus we remain with exactly the same as in

$$\varphi_1 \otimes \varphi_1\big(\partial S_m(x)\big) = \sum_{k=0}^{m-1} \varphi_1(S_k)\varphi_1(S_{m-k-1}) = \sum_{k=0}^{m-1} \alpha_{k+1}\alpha_{m-k}.$$

Now consider the second order. For this we have to show that

$$\begin{aligned}&\varphi_2\big(\xi p(x), q(x)\big)\\ &\qquad = \varphi_2\Big([\varphi_1 \otimes \mathrm{id} + \mathrm{id} \otimes \varphi_1]\big(\partial p(x)\big), q(x)\Big) + \varphi_1\big(p(x)\cdot Dq(x)\big)\end{aligned}$$

for all p and q. Again, by linearity, it is enough to show this for $p = C_m$ and $q = C_k$, for arbitrary $m, k \geqslant 0$. Thus we have to show

$$\begin{aligned}&\sum_{n\geqslant 0} \alpha_n \varphi_2(C_n C_m, C_k)\\ &\qquad = \varphi_2\big([\varphi_1 \otimes \mathrm{id} + \mathrm{id}\otimes\varphi_1](\partial C_m), C_k\big) + \varphi_1(C_m k S_{k-1}).\end{aligned} \tag{2.4}$$

We have (note that we set $S_{-2} = -1$ and $S_{-1} = 0$)

$$\begin{aligned}\partial C_m &= \partial(S_m - S_{m-2})\\ &= \sum_{l=0}^{m-1} S_l \otimes S_{m-l-1} - \sum_{l=0}^{m-3} S_l \otimes S_{m-2-l-1}\\ &= \sum_{l=0}^{m-1} S_l \otimes \tilde{C}_{m-l-1},\end{aligned}$$

where $\tilde{C}_r = C_r$ for $r \geqslant 1$ and $\tilde{C}_0 = 1 = S_0$. Thus we have

$$\varphi_1 \otimes \mathrm{id}(\partial C_m) = \sum_{l=0}^{m-1} \varphi_1(S_l)\tilde{C}_{m-l-1} = \sum_{l=0}^{m-1} \alpha_{l+1}\tilde{C}_{m-l-1}.$$

Hence

$$\text{(2.5)} \quad \varphi_2\big([\varphi_1 \otimes \mathrm{id} + \mathrm{id} \otimes \varphi_1](\partial C_m), C_k\big) \\ = 2\sum_{l=0}^{m-1} \alpha_{l+1}\varphi_2(\tilde{C}_{m-l-1}, C_k) = \begin{cases} 2\alpha_{m-k}k, & k \leqslant m, \\ 0, & k > m. \end{cases}$$

Next, using the formula (1.1) for $C_m S_{k-1}$ we have

$$\varphi_1(C_m k S_{k-1}) = \begin{cases} \alpha_{m+k} + \alpha_{k-m}, & m \leqslant k-1, \\ \alpha_{m+k}, & m = k, \\ \alpha_{m+k} - \alpha_{m-k}, & m > k. \end{cases}$$

If we add this to the right-hand side of (2.5) we see that the right-hand side of (2.4) is $k(\alpha_{m+k} + \alpha_{|m-k|})$. Finally, let us check the left-hand side of (2.4):

$$\sum_{n \geqslant 0} \alpha_n \varphi_2(C_m C_n, C_k) = \sum_{n \geqslant 1} \alpha_n \{\varphi_2(C_{m+n}, C_k) + \varphi_2(C_{|m-n|}, C_k)\} \\ = \begin{cases} k(\alpha_{m+k} + \alpha_{k-m}), & m < k, \\ k(\alpha_{m+k} + \alpha_{m-k}), & m \geqslant k, \end{cases} \\ = k(\alpha_{m+k} + \alpha_{|m-k|}).$$

Thus both sides of (2.4) equal $k(\alpha_{m+k} + \alpha_{|m-k|})$ as claimed. ■

3. QUANTUM MATRIX MODELS

Now we want to consider non-commutative (or "quantum") analogues of our classical random matrix models; i.e., we consider matrices where the entries are not commutative random variables, but in general non-commutative ones. We want to address the question about fluctuations in such a context.

The essential property of the classical ensemble (2.1) is the invariance under unitary conjugation, i.e., the joint distribution of the entries of $A = (a_{ij})_{i,j=1}^N$ does not change if we go over to the conjugated matrix $B := UAU^*$ for any $N \times N$ unitary matrix U. We will now look on analogues of this for quantum $N \times N$ matrices $A = (a_{ij})_{i,j=1}^N$ (where the entries a_{ij} come from some non-commutative probability space $(\mathcal{A}, \varphi)$), but where we ask not just for invariance under conjugation by classical unitary matrices, but – in line with the idea that one should also replace classical symmetries by corresponding quantum symmetries in a non-commutative context – for the stronger corresponding invariance under the action of the quantum unitary group U_N^+. By [2], a big class of such invariant matrices are given by the requirement that A is free from $M_N(\mathbb{C})$. Another characterization of this is as follows: the matrix A is R-cyclic (in the sense of [10]) and the non-vanishing cumulants of its entries depend only on the length of the cumulant. A way to construct such quantum random matrices is by compressing some random variable a with free matrix units; compare Lecture 14 in [11].

Recall that a matrix $A = (a_{ij})_{i,j=1}^N \in M_N(\mathcal{A})$ is *R-cyclic* if for every n we have $\kappa_n(a_{i(1)j(1)}, \ldots, a_{i(n)j(n)}) = 0$ unless $j(1) = i(2), \ldots, j(n) = i(1)$ (see [11], Lecture 20). Suppose we have a family of matrices $\{A_1, \ldots, A_s\}$, where we write $A_k = (a_{ij}^{(k)})_{i,j=1}^N$. The family is *R-cyclic* if for every n and for every $r(1), \ldots, r(n)$ we have $\kappa_n(a_{i(1)j(1)}^{(r(1))}, \ldots, a_{i(n)j(n)}^{(r(n))}) = 0$ unless $j(1) = i(2), \ldots, j(n) = i(1)$. In [10], Theorem 4.3, it was shown that matrices from the algebra generated by an R-cyclic family are themselves R-cyclic (see also [11], Exercise 20.23).

So let us in the following fix a selfadjoint random variable a and denote by $\kappa_n := \kappa_n(a, \ldots, a)$ the free cumulants of a. Then, for each $N \in \mathbf{N}$, we define a quantum random matrix $A = (a_{ij})_{i,j=1}^N$ by prescribing the free cumulants of the entries as follows: the cyclic cumulants of the matrix entries are given by

$$\kappa_n(a_{i(1)i(2)}, \ldots, a_{i(n)i(1)}) = \frac{1}{N^{n-1}} \kappa_n(a, \ldots, a), \tag{3.1}$$

all other cumulants being zero.

We are interested in calculating, for $N \to \infty$, cumulants of traces of powers of A. Fix $n \geqslant 1$ and $k(1), \ldots, k(n) \geqslant 1$. Let $k = k(1) + \ldots + k(n)$. We have

$$\begin{aligned} &\kappa_n\big(\mathrm{Tr}(A^{k(1)}), \ldots, \mathrm{Tr}(A^{k(n)})\big) \\ &= \sum_{i(1),\ldots,i(k)=1}^N \kappa_n\big(a_{i(1)i(2)} \cdots a_{i(k_1)i(1)}, a_{i(k_1+1)i(k_1+2)} \cdots a_{i(k_1+k_2)i(k_1+1)}, \ldots, \\ &\qquad a_{i(k_1+\ldots+k_{n-1}+1)i(k_1+\ldots+k_{n-1}+2)} \cdots a_{i(k_1+\ldots+k_n)i(k_1+\ldots+k_{n-1}+1)}\big). \end{aligned} \tag{3.2}$$

Now since A is R-cyclic, the family $\{A^{k(1)}, \ldots, A^{k(n)}\}$ is an R-cyclic family; so we know that only cyclic cumulants in these powers are different from zero. This means that in the sum above only terms with $i(1) = i(k_1 + 1) = \ldots = i(k_1 + \ldots + k_{n-1} + 1)$ can be different from zero.

Next we use the formula for cumulants with products as entries (see [11], Lecture 11) and write

$$\begin{aligned} &\kappa_n\big(a_{i(1)i(2)} \cdots a_{i(k_1)i(1)}, a_{i(k_1+1)i(k_1+2)} \cdots a_{i(k_1+k_2)i(k_1+1)}, \ldots, \\ &\qquad a_{i(k_1+\ldots+k_{n-1}+1)i(k_1+\ldots+k_{n-1}+2)} \cdots a_{i(k_1+\ldots+k_n)i(k_1+\ldots+k_{n-1}+1)}\big) \end{aligned}$$

as

$$\begin{aligned} &\sum_\pi \kappa_\pi\big(a_{i(1)i(2)}, \ldots, a_{i(k_1)i(1)}, \ldots, \\ &\qquad a_{i(k_1+\ldots+k_{n-1}+1)i(k_1+\ldots+k_{n-1}+2)}, \ldots, a_{i(k_1+\ldots+k_n)i(k_1+\ldots+k_{n-1}+1)}\big), \end{aligned}$$

where the sum runs over all $\pi \in NC\big(k(1) + \ldots + k(n)\big)$ which have the property that they connect the blocks of

$$\tau = \big\{\big(1, \ldots, k(1)\big), \ldots, \big(k(1) + \ldots + k(n-1) + 1, \ldots, k(1) + \ldots + k(n)\big)\big\}.$$

In the language of [11], Definition 9.15, this means that $\pi \vee \tau = 1_k$.

For such a π to make a non-zero contribution some relations on the indices must be satisfied. Let us work out what this means. Recall that there is an embedding of $NC(k)$ into S_k the symmetric group on $[k]$; namely, put the elements of the blocks of $\pi \in NC(k)$ in increasing order and regard them as the cycles of permutation (see, e.g., [11], Remark 23.24).

Suppose $(j_1, \ldots, j_r)$ is a block of π; then the corresponding factor of κ_π is

$$\kappa_r\big(a_{i(j_1)i(j_1+1)}, \ldots, a_{i(j_r)i(j_r+1)}\big).$$

In order for this cumulant to be different from zero we must have

$$i(j_1 + 1) = i(j_2), i(j_2 + 1) = i(j_3), \ldots, i(j_r + 1) = i(j_1).$$

Let $\gamma \in S_k$ be the permutation with the single cycle $(1, \ldots, k)$. Then our relation on i can be expressed as

$$i(j_k) = i(j_{k-1} + 1) = i\big(\gamma(j_{k-1})\big) = i\Big(\gamma\big(\pi^{-1}(j_k)\big)\Big)$$

or as $i = i \circ \gamma\pi^{-1}$. An important fact of the embedding of $NC(k)$ into S_k is that the Kreweras complement of π, $K(\pi) = \pi^{-1}\gamma$. What we have here is the 'other' Kreweras complement $\gamma\pi^{-1}$ which is the conjugation of $K(\pi)$ by γ (see [11], Exercise 9.23 (1)).

Thus in order for

$$\kappa_\pi\big(a_{i(1)i(2)}, \ldots, a_{i(k_1)i(1)}, \ldots, a_{i(k_1+\ldots+k_{n-1}+1)i(k_1+\ldots+k_{n-1}+2)}, \ldots, a_{i(k_1+\ldots+k_n)i(k_1+\ldots+k_{n-1}+1)}\big) \neq 0 \tag{3.3}$$

we must have that i is constant on the cycles of $\gamma\pi^{-1}$. This is true for any $\pi \in NC(k)$. Let us now consider what happens when we add the condition $\pi \vee \tau = 1_k$. According to Lemma 14 in [9], $\pi \vee \tau = 1_k$ if and only if each point of the set $\{k(1), k(1) + k(2), \ldots, k(1) + \ldots + k(n)\}$ lies in a different block of $K(\pi)$; af-

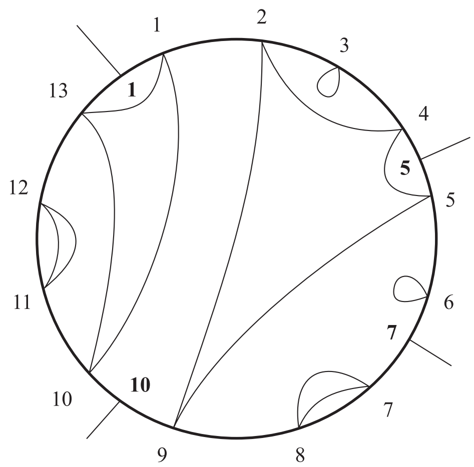

FIGURE 1. In this example $k(1) = 4$, $k(2) = 2$, $k(3) = 3$, and $k(4) = 4$. The partition $\pi = \{(1,10,13)(2,4,5,9)(3)(6)(7,8)(11,12)\}$; the 'other' Kreweras complement of π is $\gamma\pi^{-1} = \{(\mathbf{1})(2,\mathbf{10})(3,4)(\mathbf{5})(6,\mathbf{7},9)(8)(11,13)(12)\}$. Note that since $\pi \vee \tau = 1_{13}$, each of the points of $\{1,5,7,10\}$ is in a separate block of $\gamma\pi^{-1}$. There are $d = \#(\gamma\pi^{-1}) - n + 1 = 5$ degrees of freedom in $i(1), \ldots, i(13)$, namely $i(1) = i(2) = i(5) = i(6) = i(7) = i(9) = i(10)$, $i(3) = i(4)$, $i(11) = i(13)$, $i(8)$, and $i(12)$; i.e. we join the blocks of $\gamma\pi^{-1}$ containing a bold number and the rest remain as they are.

ter conjugation by γ this condition becomes that each point of $\{1, k(1)+1, \ldots, k(1)+\ldots+k(n-1)+1\}$ is in a separate cycle of $\gamma\pi^{-1}$. Now recall that we had earlier observed that R-cyclicity forced us to have $i(1) = i(k_1+1) = \ldots = i(k_1+\ldots+k_{n-1}+1)$ in order for the corresponding term of (3.2) to be different from zero.

Let us summarize our calculation. In order for (3.3) to hold we require: i is constant on the cycles of $\gamma\pi^{-1}$; each point of $\{1, k(1)+1, \ldots, k(1)+\ldots+k(n-1)+1\}$ is in a separate cycle of $\gamma\pi^{-1}$; and i is constant on the union of the cycles of $\gamma\pi^{-1}$ containing the points of $\{1, k(1)+1, \ldots, k(1)+\ldots+k(n-1)+1\}$. This leaves $\#(\gamma\pi^{-1}) - n + 1$ cycles on which we can arbitrarily choose values of i (recall that $\#(\gamma\pi^{-1})$ denotes the number of cycles of $\gamma\pi^{-1}$). Thus the number of choices for i is N^d, where $d = \#(\gamma\pi^{-1}) - n + 1$. (See Figure 1.) So if we sum for a fixed such π over all free indices (each choice of them will give the same contribution, because the cyclic cumulants of the a_{ij} do not depend on the actual choice of the indices) then we get altogether for such a π the contribution

$$\begin{aligned} N^d \prod_{V\in\pi} \frac{\kappa_{|V|}}{N^{|V|-1}} &= N^{d+\#(\pi)-|V_1|-\ldots-|V_{\#(\pi)}|} \prod_{V\in\pi} \kappa_{|V|} \\ &= N^{d+\#(\pi)-k}\kappa_\pi(a,\ldots,a) \\ &= N^{-n+2}\kappa_\pi(a,\ldots,a), \end{aligned}$$

where $d + \#(\pi) - k = \#(\gamma\pi^{-1}) + \#(\pi) - n - k + 1 = -n + 2$ because $\#(\pi) + \#(\gamma\pi^{-1}) = k + 1$ (see [11], Exercise 9.23).

But carrying out the sum over the π is now the same as calculating cumulants of powers of a. So finally we get the simple result

$$\kappa_n\big(\mathrm{Tr}(A^{k(1)}), \dots, \mathrm{Tr}(A^{k(n)})\big) = N^{-n+2}\kappa_n(a^{k(1)}, \dots, a^{k(n)}). \tag{3.4}$$

One should note that, compared to the case of classical random matrices, there are no subleading orders. Thus the limit $N \to \infty$ does not produce any new feature and contains essentially the same information as the random variable a, i.e., the case $N = 1$. In this sense, these quantum random matrices are less interesting from the point of view of fluctuations than their classical counterparts. Still, let us elaborate a bit on what happens with respect to fluctuations.

First, it is clear from (3.4) that all cumulants of higher order than two go to zero, and thus each centered trace of a power of A goes to a semicircular element. The covariance between two such traces of powers is (actually for any N) given by

$$\kappa_2\big(\mathrm{Tr}(A^p), \mathrm{Tr}(A^q)\big) = \kappa_2(a^p, a^q).$$

Since those fluctuations depend on the distribution of a, we do not have universality for the fluctuations in the quantum case.

Let us finally also check whether there is some kind of analogue of the Schwinger–Dyson equations. We put

$$\varphi_1\big(p(x)\big) := \lim_{N\to\infty} \kappa_1\Big(\mathrm{Tr}\big(p(A)\big)\Big) = \kappa_1\big(p(a)\big) = \varphi\big(p(a)\big)$$

and

$$\varphi_2\big(p(x), q(x)\big) := \lim_{N\to\infty} \kappa_2\Big(\mathrm{Tr}\big(p(A)\big), \mathrm{Tr}\big(q(A)\big)\Big) = \kappa_2\big(p(a), q(a)\big).$$

Since φ_1 captures just the information about the distribution of a, the first order equation is nothing else but the definition of the conjugate variable ξ for a, namely for this we just have the equation

$$\varphi_1\big(\xi p(x)\big) = \varphi\big(\xi p(a)\big) = \varphi \otimes \varphi\big(\partial p(a)\big) = \varphi_1 \otimes \varphi_1\big(\partial p(x)\big).$$

For the second order we have

$$\varphi_2\big(\xi p(x), q(x)\big) = \kappa_2\big(\xi p(a), q(a)\big),$$

which yields, by using again the formula for free cumulants with products as arguments, the following kind of linear analogue of (2.3):

$$\begin{aligned} &\varphi_2\big(\xi p(x), q(x)\big) \\ &\qquad = \varphi_2\Big(\varphi \otimes \mathrm{id}\big(\partial p(x)\big), q(x)\Big) + \varphi_1 \otimes \varphi_1\big(p(x) \otimes 1 \cdot \partial q(x)\big). \end{aligned} \tag{3.5}$$

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Queen's University
Department of Mathematics and Statistics
Kingston, Ontario
K7L 3N6, Canada
E-mail: mingo@mast.queensu.ca

Universität des Saarlandes
FR 6.1−Mathematik
Postfach 15 11 50
D-66123 Saarbrücken, Germany
E-mail: speicher@math.uni-sb.de

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