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ASYMPTOTIC RESULTS FOR RANDOM POLYNOMIALS ON THE UNIT CIRCLE

BY

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Abstract. In this paper we study the asymptotic behavior of the maximum magnitude of a complex random polynomial with i.i.d. uniformly distributed random roots on the unit circle. More specifically, let $\{n_k\}_{k=1}^{\infty}$ be an infinite sequence of positive integers and let $\{z_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. uniformly distributed random variables on the unit circle. The above pair of sequences determine a sequence of random polynomials $P_N(z) = \prod_{k=1}^{N} (z - z_k)^{n_k}$ with random roots on the unit circle and their corresponding multiplicities. In this work, we show that subject to a certain regularity condition on the sequence $\{n_k\}_{k=1}^{\infty}$, the log maximum magnitude of these polynomials scales as $s_N I^*$, where $s_N^2 = \sum_{k=1}^{N} n_k^2$ and I^* is a strictly positive random variable.

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1. INTRODUCTION

Random polynomials are ubiquitous in several areas of mathematics and have found several applications in diverse fields such as random matrix theory, representation theory and chaotic systems (see [9], [1], [13], [5], [14], [4]). The geometric structure of random polynomials is of significant interest as well. Constructing a random polynomial from its roots is a natural construction which can be expected to occur in a wide range of settings. For instance, consideration of the asymptotic behavior of the maximum magnitude allowed the authors to obtain a lower bound on the minimum singular value for random Vandermonde matrices (see [16] for more details).

In this work we continue the investigation of such polynomials, where we allow for non-constant multiplicity of the roots. We show that providing this sequence satisfies a simple sufficient condition, the limit distribution of the maximum magnitude on the unit circle is determined by a certain Gaussian process obtained from the Brownian bridge. Moreover, up to renormalization, it does not depend on the sequence itself.

Our construction for these random polynomials is as follows. Let $\{n_k\}_{k=1}^{\infty}$ be an infinite sequence of positive integers and let $\{z_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. uniformly distributed unit magnitude complex numbers. The above pair of sequences then determine a sequence of random polynomials $P_N(z) = \prod_{k=1}^{N} (z - z_k)^{n_k}$ with roots on the unit circle and their corresponding exponents. Our main result relies on a construction based on the Brownian bridge and enables us to conclude that

$$\frac{1}{s_N} \log \max\{|P_N(z)|^2 \colon |z| = 1\} \Rightarrow I^*, \tag{1.1}$$

where the expression on the left-hand side converges weakly to a positive random variable I^*, and $s_N^2 = \sum_{k=1}^{N} n_k^2$. We use $\Rightarrow$ to denote weak convergence (convergence in distribution) throughout the paper.

2. RANDOM POLYNOMIALS

2.1. Pointwise convergence and the Lindberg condition. Let P_N be as before and let $L_N(\psi)$ be defined as

$$L_N(\psi) := \log |P_N(e^{i\psi})|^2 = \sum_{k=1}^{N} n_k \log \Big(2\big(1 - \cos(\psi - \theta_k)\big)\Big),$$

where $\psi \in [0, 2\pi]$ and $\theta_k \doteq \arg z_k, \theta_k \in [0, 2\pi), k \in \mathbb{N}$, are the arguments of the roots. Moreover, define $s_N^2 := \sum_{k=1}^{N} n_k^2$ and $T_N(\psi) := L_N(\psi)/s_N$.

As explained in the Introduction, we are interested in the behavior of the maximum of $|P_N(z)|^2$ as N increases. Since $|P_N(z)|^2$ is a continuous function on the unit circle, it follows that there exists φ^* that attains its maximum. Let $T_N^* := T_N(\varphi^*)$ be the corresponding value of T_N. For later use, let $\Phi := \{\varphi_r \colon r \geqslant 0\}$ be the set of 2π times the dyadic rationals on the interval $[0, 1]$. Then it is clear that

$$\limsup_{r \to \infty} T_N(\varphi_r) = T_N^*.$$

The case $n_k = 1$ appears in connection with the asymptotic behavior of the minimum eigenvalue of random Vandermonde matrices (see [16] and [15] for more details). In this special case, weak convergence to the normal distribution,

$$T_N(\psi) \Rightarrow N(0, \sigma^2),$$

holds for every fixed ψ, where

$$\sigma^2 := \frac{1}{2\pi} \int_0^{2\pi} \log^2 \Big(2\big(1 - \cos(\psi)\big)\Big) d\psi \approx 3.292.$$

This is in fact a consequence of the central limit theorem since $\log\big(2\big(1-\cos(\psi)\big)\big)$ is square summable and $\int_0^{2\pi}\log\big(2\big(1-\cos(\psi)\big)\big)d\psi=0$.

In what follows we derive a simple sufficient condition for the asymptotic normality of the random variable T_N. We use the notation $\mathbb{E}\,(X;A):=\mathbb{E}\,(X\mathbb{1}_A)$, where X is a random variable and A is a Borel set. Let X_k be a sequence of independent zero mean and variance σ_k^2 random variables. We say that the sequence satisfies the *Lindberg condition* [8] if and only if

$$\lim_{N\to\infty}\frac{1}{s_N^2}\sum_{k=1}^{N}\mathbb{E}(X_k^2\,;\,|X_k|\geqslant\epsilon s_N)=0 \tag{2.1}$$

for every $\epsilon>0$. If $X_k=\sigma_k Y_k$ then this condition becomes

$$\lim_{N\to\infty}\frac{1}{s_N^2}\sum_{k=1}^{N}\sigma_k^2\mathbb{E}\left(Y^2\,;\,|Y|\geqslant\frac{\epsilon s_N}{\sigma_k}\right)=0. \tag{2.2}$$

For our purposes we only focus on the case where the random variables Y_k are i.i.d. according to the distribution of $Y=\log\big(2\big(1-\cos(2\pi U)\big)\big)$ and where U is a uniform random variable on $[0,1]$. The main result of this section is the following theorem.

THEOREM 2.1 (Lindberg exponent). *If*

$$\lim_{N\to\infty}\sum_{k=1}^{N}\exp\left(-\frac{\epsilon s_N}{n_k}\right)=0 \tag{2.3}$$

holds for every $\epsilon>0$, *then*

$$T_N(\psi)=\frac{L_N(\psi)}{s_N}\Rightarrow N(0,\sigma^2). \tag{2.4}$$

Proof. By definition, $Y=\log\big(2\big(1-\cos(2\pi U)\big)\big)$, where U is uniform on $[0,1]$, from which it follows that $Y\leqslant\log\big(2\big(1-\cos(\pi)\big)\big)=\log 4$, and hence the moment generating function exists for all $t>0$.

Additionally, since s_N, n_k and ϵ are positive, it follows that

$$\mathbb{E}\left(Y^2;Y\geqslant\frac{s_N\epsilon}{n_k}\right)\leqslant\log^2(4)\mathbb{P}\left(Y\geqslant\frac{s_N\epsilon}{n_k}\right), \tag{2.5}$$

and applying Markov's inequality with $t>0$ we obtain

$$\mathbb{P}\left(Y\geqslant\frac{s_N\epsilon}{n_k}\right)\leqslant 4^t\exp\left(-\frac{ts_N\epsilon}{n_k}\right). \tag{2.6}$$

We thus obtain the following bound for the upper Lindberg condition:

$$\begin{aligned}\frac{1}{s_N^2}\sum_{k=1}^{N} n_k^2\mathbb{E}\left(Y^2; Y \geqslant \frac{s_N\epsilon}{n_k}\right) &\leqslant \frac{\log^2(4)4^t}{s_N^2}\sum_{k=1}^{N} n_k^2 \exp\left(-\frac{ts_N\epsilon}{n_k}\right)\\ &\leqslant \log^2(4)4^t\sum_{k=1}^{N}\exp\left(-\frac{ts_N\epsilon}{n_k}\right).\end{aligned}$$

The sum on the right-hand side goes to zero as this is the condition we assumed holds and because t and ϵ are arbitrary and fixed. We now turn to the lower Lindberg condition, where $Y \leqslant -s_N\epsilon/n_k$. Let $\xi_{N,k} := \epsilon s_N/n_k$, and θ be in the interval $[-\pi/3, \pi/3]$. Let us use the following two basic inequalities:

$$\cos(\theta) \leqslant 1-\theta^2/4 \tag{2.7}$$

for $0 \leqslant |\theta| \leqslant \pi/2$, and

$$\log^2\Big(2\big(1-\cos(\theta)\big)\Big) \leqslant \log^2(\theta^2/2)$$

for $0 \leqslant |\theta| \leqslant \pi/3$. Since we may take $\theta = 2\pi U - \pi$ and $Y = \log\big(2\big(1-\cos(\theta)\big)\big)$ we see that

$$\begin{aligned}\mathbb{E}(Y^2; Y \leqslant -\xi_{N,k}) &\leqslant \mathbb{E}\big(\log^2(\theta^2/2)\,; Y \leqslant -\xi_{N,k}\big)\\ &\leqslant \mathbb{E}\big(\log^2(\theta^2/2)\,; E_{N,k}\big),\end{aligned}$$

where $E_{N,k} = \{|\theta| \leqslant \sqrt{2}\exp(-\xi_{N,k}/2)\}$.

Since $\int \log^2(x)dx = x\log^2(x) - 2x\log(x) + 2x$, we make the substitution $\phi = \theta/\sqrt{2}$ and determine the above expectation to be

$$\mathbb{E}\left(\log^2(\phi^2); |\phi| \leqslant \delta\right) = \frac{8}{\pi\sqrt{2}}h(\delta), \tag{2.8}$$

where $h(\delta) := \delta\log^2(\delta) - 2\delta\log(\delta) + 2\delta$ and $\delta := \exp(-\xi_{N,k}/2)$. Rewriting this expression we obtain

$$\frac{4\sqrt{2}}{\pi}\exp(-\epsilon s_N/2n_k)\left(\frac{\epsilon^2 s_N^2}{4n_k^2} + \frac{\epsilon s_N}{n_k} + 2\right).$$

Summing over k and dividing by s_N^2 we obtain

$$\frac{4\sqrt{2}}{\pi}\sum_{k=1}^{N}\exp(-\epsilon s_N/2n_k)\left(\frac{\epsilon^2}{4} + \frac{\epsilon n_k}{s_N} + 2\frac{n_k^2}{s_N^2}\right) \to 0.$$

Now applying the classical Lindberg's theorem [3] we complete the proof. ∎

The condition is easily verified to hold when $n_k = k^p$ for $p \geqslant 0$. Similarly, it is easy to show that this condition fails if $n_k = 2^k$.

2.2. Finite dimensional limits. The result in Section 2.1 was for the marginal distribution of $T_N(\psi)$ for a fixed value ψ. However, we would like to consider the weak limit for the sequence $T_{N,r}$, where $T_{N,r} := T_N(\varphi_r)$. It is well known that weak convergence in the sequence space $\mathbb{R}^\infty$, with metric

$$\rho(\mathbf{x},\mathbf{y}) := \sum_{r=0}^{\infty} \frac{|x_r - y_r|}{1 + |x_r - y_r|} 2^{-r},$$

is entailed by weak convergence of the finite-dimensional distributions. For this reason, it is important to understand the joint distribution of s such variables. We first focus on the case $s = 2$. Define the covariance function

$$K(\theta) := (2\pi)^{-1} \int_0^{2\pi} \log\Big(2\big(1 - \cos(\psi)\big)\Big) \log\Big(2\big(1 - \cos(\psi + \theta)\big)\Big) d\psi \tag{2.9}$$

for $\theta \in [0, 2\pi]$. A plot of this function is shown in Figure 1.

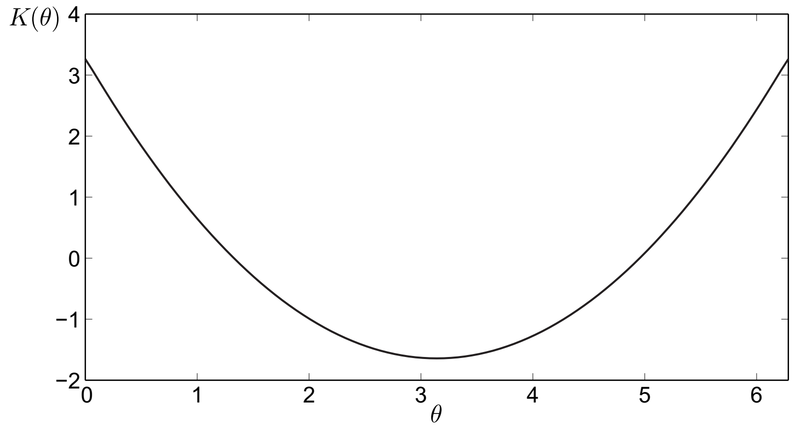

FIGURE 1. Graph of the covariance function $K(\theta)$

Given t and $u \in \mathbb{R}$ define

$$\begin{aligned} V_N &:= tT_N(\varphi_1) + uT_N(\varphi_2) \\ &= \frac{t}{s_N} \sum_{k=1}^{N} n_k \log|e^{i\varphi_1} - e^{i\theta_k}|^2 + \frac{u}{s_N} \sum_{k=1}^{N} n_k \log|e^{i\varphi_2} - e^{i\theta_k}|^2 \\ &= \frac{1}{s_N} \sum_{k=1}^{N} n_k V_{k,N}, \end{aligned}$$

which is a weighted sum of zero mean i.i.d. random variables with variance

$$\mathbb{E}(V_{k,N}^2) = (t^2 + u^2)\sigma^2 + 2tuK(|\varphi_1 - \varphi_2|).$$

In what follows we drop the indices and write $V = tW + uZ$, where W is defined as $W = \log|e^{i\varphi_1} - e^{i\theta}|$. We assume that $t \neq 0$ and $u \neq 0$ since otherwise there will be nothing to show. It is not difficult to see that

$$\begin{aligned} V^2 &= t^2W^2 + 2tuWZ + u^2Z^2 \\ &\leqslant t^2W^2 + |tu|(W^2 + Z^2) + u^2Z^2. \end{aligned}$$

However, the triangle inequality implies that if $|V| \geqslant \epsilon s_N/n_k$ then either $|W| \geqslant \epsilon s_N/2|t|n_k$ or the corresponding inequality for Z holds (or both). However, we have already demonstrated that

$$\frac{1}{s_N^2}\sum_{k=1}^{N} n_k^2 \mathbb{E}\left(W^2; |W| \geqslant \frac{\epsilon s_N}{2|t|n_k}\right) \to 0$$

as $N \to \infty$. As far as the terms involving Z^2 are concerned we only need to show the above in the case where

$$|Z| \leqslant \frac{\epsilon s_N}{2|u|n_k}.$$

That is, we wish to show that

$$\frac{1}{s_N^2}\sum_{k=1}^{N} n_k^2 \mathbb{E}\left(Z^2; |W| \geqslant \frac{\epsilon s_N}{2|t|n_k}, |Z| \leqslant \frac{\epsilon s_N}{2|u|n_k}\right) \to 0,$$

but the above is smaller than

$$\frac{\epsilon^2}{4u^2}\sum_{k=1}^{N} \mathbb{P}\left(|W| \geqslant \frac{\epsilon s_N}{2|t|n_k}\right). \tag{2.10}$$

Using Markov's inequality as in (2.6) we see that

$$\mathbb{P}\left(W \geqslant \frac{\epsilon s_N}{2|t|n_k}\right) \leqslant 4^{\eta} \exp\left(-\eta\frac{\epsilon s_N}{2|t|n_k}\right).$$

Applying (2.7) and using the condition that $\theta \sim U[0, 2\pi]$ we obtain

$$\mathbb{P}\left(W \leqslant -\frac{\epsilon s_N}{2|t|n_k}\right) \leqslant \frac{2}{\pi} \exp\left(-\frac{\epsilon s_N}{4|t|n_k}\right).$$

It follows that (2.10) tends to zero as $N \to \infty$.

Thus by an extension of the arguments given in the proof of Theorem 2.1 it can be shown that condition (2.3) is sufficient for the Lindberg condition to hold in respect of the random variables $V_{N,k}$. Therefore, $\big(T_N(\varphi_1), T_N(\varphi_2)\big) \Rightarrow N(0, \Sigma)$ with $\Sigma_{11} = \Sigma_{22} = \sigma^2$ and $\Sigma_{12} = \Sigma_{21} = K(|\varphi_1 - \varphi_2|)$ as an application of the Cramér–Wold device [2].

The same arguments go through in the case of three or more variables. Therefore, the following result holds.

THEOREM 2.2. *Suppose the sequence* $\{n_k\}_{k=1}^{\infty}$ *satisfies* (2.3). *Let* $(\varphi_1, \ldots, \varphi_s)$ *be* s *numbers on the interval* $[0, 2\pi]$ *and let* $\big(T_N(\varphi_1), \ldots, T_N(\varphi_s)\big)$ *be the corresponding random vector. Then*,

$$\big(T_N(\varphi_1), \ldots, T_N(\varphi_s)\big) \Rightarrow N(0, \Sigma_s),$$

i.e., *asymptotically joint normal with covariance determined by*

$$\Sigma_{k,\ell} = K(|\varphi_k - \varphi_\ell|),$$

where

$$K(\theta) := (2\pi)^{-1} \int_0^{2\pi} \log\Big(2\big(1-\cos(\psi)\big)\Big) \log\Big(2\big(1-\cos(\psi+\theta)\big)\Big) d\psi.$$

For each $N \in \mathbb{N}$ define the random sequence $\mathbf{T}_N$ to be

$$\mathbf{T}_N \doteq \{T_{n,r}\}_{r=0}^{\infty}. \tag{2.11}$$

Then the conclusion is that the limit distribution of $\mathbf{T}_N$ exists and is independent of the sequence $\{n_k\}_{k=1}^{\infty}$ provided (2.3) holds. In what follows we work with the sequence $n_k = 1$ for all k so that $s_N^2 = N$.

3. ASYMPTOTIC DISTRIBUTION OF T_N^*

The following is an alternative way to construct the limit distribution of the random sequence $T_{N,r}$. Consider a realization of the Brownian bridge W^o on $[0, 2\pi]$ (which satisfies $W^o(0) = W^o(2\pi) = 0$). A φ shift of the Brownian bridge is defined as

$$W_\varphi^o(\theta) := \begin{cases} W^o(\varphi+\theta) - W^o(\varphi), & \theta \in [0, 2\pi - \varphi], \\ W^o(\varphi+\theta-2\pi) - W^o(\varphi), & \theta \in [2\pi-\varphi, 2\pi]. \end{cases}$$

In addition, define the function $I : [0, 2\pi] \to \mathbb{R}$ by

$$I_\varphi := \int_0^{2\pi} W_\varphi^o(\theta) \frac{\sin(\theta)}{1-\cos(\theta)} d\theta$$

for $\varphi \in [0, 2\pi]$. Figure 2 shows us a realization of I_φ. The following lemma shows that I is a well defined quantity almost surely.

LEMMA 3.1. *Given a realization of the Brownian bridge* W^o, *then a.s. the following integral exists for all* $\varphi \in [0, 2\pi)$:

$$|I_\varphi| = \left| \int_0^{2\pi} W_\varphi^o \frac{\sin(\psi)}{1-\cos(\psi)} d\psi \right| < \infty.$$

In addition, *the function* $\varphi \mapsto I_\varphi$ *is continuous.*

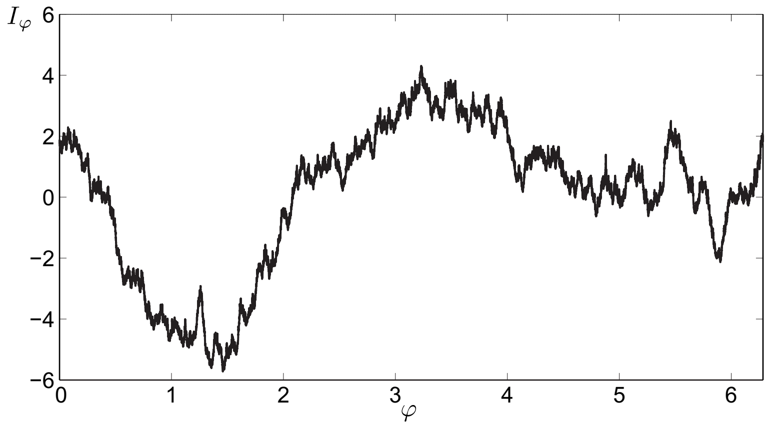

FIGURE 2. I_φ for a realization of the Brownian bridge

P r o o f. When $\varphi = 0$ we write the above integral as I. The Lévy global modulus of continuity tells us that for standard Brownian motion B on $[0, 2\pi)$

$$\lim_{\delta\to 0} \limsup_{0\leqslant t\leqslant 2\pi-\delta} \frac{|B(t+\delta) - B(t)|}{w(\delta)} = 1,$$

where $w(\delta) = \sqrt{2\delta \log \delta^{-1}}$ (see [10] for a proof of this result). Also, since W^o is, by definition,

$$W^o(\psi) = B(\psi) - \frac{\psi}{2\pi} B(2\pi),$$

our argument is the same no matter which value of φ is chosen because the Lévy modulus applies to the entire sample path. We therefore set $\varphi = 0$. By definition of the Lévy modulus, there exists $\delta_2 > 0$ such that

$$\frac{|B(t+\delta) - B(t)|}{w(\delta)} \leqslant 2$$

for all $t \in [0, 2\pi - \delta]$ and $0 < \delta \leqslant \delta_2$ almost surely. Define

$$a(\delta) := |W^o(\psi+\delta) - W^o(\psi)| \leqslant 2w(\delta) + \frac{|B(2\pi)|}{2\pi}\delta. \tag{3.1}$$

We may therefore split the integral as follows:

$$\begin{aligned} I = &\int_{\delta_2}^{2\pi-\delta_2} W^o(\psi) \frac{\sin(\psi)}{1-\cos(\psi)} d\psi + \int_0^{\delta_2} W^o(\psi) \frac{\sin(\psi)}{1-\cos(\psi)} d\psi \\ &+ \int_{2\pi-\delta_2}^{2\pi} W^o(\psi) \frac{\sin(\psi)}{1-\cos(\psi)} d\psi. \end{aligned} \tag{3.2}$$

The first integral is finite being the integral of a continuous function over the interval $[\delta_2, 2\pi - \delta_2]$. We may further suppose that δ_2 has been chosen so that $\left|\psi \frac{\sin(\psi)}{1-\cos(\psi)}\right| \leqslant 4$ for $0 < \psi < \delta_2$ with the corresponding inequality in a similar neighborhood of 2π. By choice of δ_2 we obtain

$$\left|\int_0^{\delta_2} W^o(\psi)\frac{\sin(\psi)}{1-\cos(\psi)}d\psi\right| \leqslant 8\int_0^{\delta_2} \frac{a(\psi)}{\psi}d\psi = O(\delta_2^{1/3})$$

for sufficiently small δ_2. The same argument applies to the last integral. Since $w(\delta_2)$ gives a uniform bound the result holds for all $\varphi \in [0, 2\pi)$. Continuity in φ follows by a similar argument,

$$\begin{aligned}
(3.3) \qquad \left|I_\varphi - I_{\tilde\varphi}\right| \leqslant & \left|\int_\delta^{2\pi-\delta} (W^o_\varphi - W^o_{\tilde\varphi})\frac{\sin(\psi)}{1-\cos(\psi)}d\psi\right| \\
& + \int_0^\delta |W^o_\varphi(\psi)|\left|\frac{\sin(\psi)}{1-\cos(\psi)}\right|d\psi \\
& + \int_{2\pi-\delta}^{2\pi} |W^o_\varphi(\psi)|\left|\frac{\sin(\psi)}{1-\cos(\psi)}\right|d\psi \\
& + \int_0^\delta |W^o_{\tilde\varphi}(\psi)|\left|\frac{\sin(\psi)}{1-\cos(\psi)}\right|d\psi \\
& + \int_{2\pi-\delta}^{2\pi} |W^o_{\tilde\varphi}(\psi)|\left|\frac{\sin(\psi)}{1-\cos(\psi)}\right|d\psi.
\end{aligned}$$

Provided that $0 < \delta < \delta_2$, the tail integrals are all at most $O(\delta^{1/3})$ as before. We bound the first integral by two positive integrals, to obtain

$$\begin{aligned}
(3.4) \quad & \left|\int_\delta^{2\pi-\delta} (W^o_\varphi - W^o_{\tilde\varphi})\frac{\sin(\psi)}{1-\cos(\psi)}d\psi\right| \\
& \leqslant 2\sup|W^o_\varphi(\psi) - W^o_{\tilde\varphi}(\psi)|\int_\delta^\pi \frac{\sin(\psi)}{1-\cos(\psi)}d\psi \\
& \leqslant 2\sup|W^o_\varphi(\psi) - W^o_{\tilde\varphi}(\psi)|\big[\log\big(1-\cos(\psi)\big)\big]_\delta^\pi \\
& \leqslant 6a(\delta)\Big(\log 2 - \log\big(1-\cos(\delta)\big)\Big)
\end{aligned}$$

provided $|\varphi - \tilde\varphi| < \delta$. Finally, since

$$a(\delta)\Big(\log(2) - \log\big(1-\cos(\delta)\big)\Big) \to 0$$

as $\delta \to 0$, we finish the proof. ∎

Let $\Phi = \{\varphi_r : r \geqslant 0\}$ be the sequence described in Section 2.1 and let $\mathbf{I} = \{I_r\}_{r=0}^{\infty}$ be the sequence defined as $I_r := I_{\varphi_r}$. Since the function I_φ is continuous on the interval $[0, 2\pi]$, there exists a value φ^* determining the maximum value of I_φ, which we denote by I^*. Since Φ is dense on the unit circle, it follows that

$$I^* := \sup\{I_r : r \in \mathbb{N}\} \tag{3.5}$$

and its distribution is determined via the infinite sequence I_r. We now derive one more lemma for use later on.

LEMMA 3.2. *Let Y be a function in $D[0, 2\pi]$. Then Y is Lebesgue measurable, and its integral exists,*

$$\int_0^{2\pi} Y(s)ds < \infty. \tag{3.6}$$

Furthermore, let Y_n be a sequence of functions in $D[0, 2\pi]$ such that $Y_n \to Y$ in D (i.e., with respect to the Skorokhod topology). Then

$$\int_0^{2\pi} Y_n(s)ds \to \int_0^{2\pi} Y(s)ds. \tag{3.7}$$

Proof. The existence of the integral follows from Lemma 1, page 110 of [2], and the subsequent discussion which shows that functions in D on a closed bounded interval are both Lebesgue measurable and bounded. The former follows from the fact that they can be uniformly approximated by simple functions, a direct consequence of Lemma 1 and the latter also.

Convergence follows from the Lebesgue dominated convergence theorem. This holds since the sequence Y_n is uniformly bounded by a constant, so the sequence is dominated. Second, Y is continuous a.e. with pointwise convergence holding at points of continuity, as a consequence of convergence in D (see [2]). ■

Remember that, for each $N \in \mathbb{N}$, $\mathbf{T}_N$ is the random sequence defined as $\mathbf{T}_N = \{T_{N,r}\}_{r=0}^{\infty}$, see (2.11). We now proceed to prove the following theorem:

THEOREM 3.1. *The sequence $\mathbf{T}_N$ converges in distribution to the sequence $\mathbf{I}$,*

$$\mathbf{T}_N \Rightarrow \mathbf{I}, \tag{3.8}$$

as $N \to \infty$.

Proof. In order to do so we use Theorem 4.2 of [2]. This states that if there is a metric space $\mathcal{S}$ with metric ρ and sequences $\mathbf{T}_{N,\epsilon}$, $\mathbf{I}_\epsilon$ and $\mathbf{T}_N$ all lying in $\mathcal{S}$ such that the conditions

$$\mathbf{T}_{N,\epsilon} \Rightarrow \mathbf{I}_\epsilon \quad \text{and} \quad \mathbf{I}_\epsilon \Rightarrow \mathbf{I} \tag{3.9}$$

hold together with the further condition that, given arbitrary $\eta > 0$,

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{P}\big(\rho(\mathbf{T}_{N,\epsilon},\mathbf{T}_N)\geqslant\eta\big)=0. \tag{3.10}$$

Then we infer that $\mathbf{T}_N \Rightarrow \mathbf{I}$. First, we define $\mathbf{I}_\epsilon$ using a realization of the Brownian bridge as follows:

$$I_{r,\epsilon} := \Big[W^o_{\varphi_r}(\psi)\log\Big(2\big(1-\cos(\psi)\big)\Big)\Big]_\epsilon^{2\pi-\epsilon} - \int\limits_\epsilon^{2\pi-\epsilon} W^o_{\varphi_r}(\psi)\frac{\sin(\psi)}{1-\cos(\psi)}d\psi.$$

The definition of the other sequence is more involved, and so we defer it for a moment. We have shown that the limit integrals exist a.s., and so we only need to prove that the first term converges to zero. Since $\log\big(2\big(1-\cos(\psi)\big)\big) = O\big(|\log(\epsilon)|\big)$ when ϵ is small, and in a neighborhood of 0 and 2π we may invoke the Lévy modulus of continuity, wrapped around at 2π, to see that this term is

$$O\big(|a(\epsilon)\log\epsilon|\big)\to 0.$$

Hence, coordinate convergence of the integrals holds so that

$$\left|I_{r,\epsilon} + \int\limits_\epsilon^{2\pi-\epsilon} W^o_{\varphi_r}(\psi)\frac{\sin(\psi)}{1-\cos(\psi)}d\psi\right| \Rightarrow 0$$

and it follows that $\mathbf{I}_\epsilon \Rightarrow \mathbf{I}$ as $\epsilon\to 0$, since the sign of the integral is immaterial. We have thus demonstrated the second condition of (3.9). Next, we proceed by rewriting $T_N(\varphi_r)$ in terms of the empirical distribution function $F_N : [0,2\pi]\to[0,1]$ determined by

$$F_N(\psi) := \frac{\#\{\theta_q : 0\leqslant\theta_q\leqslant\psi\}}{N}.$$

By definition of $F_N(\psi)$ and the Lebesgue–Stieltjes integral we see that

$$\begin{aligned} T_N(\varphi_r) &= \sqrt{N}\int\limits_0^{2\pi}\log\Big(2\big(1-\cos(\varphi_r-\psi)\big)\Big)dF_N(\psi)\\ &= \sqrt{N}\int\limits_0^{2\pi}\log\Big(2\big(1-\cos(\tilde\psi)\big)\Big)dF_{N,\varphi_r}(\tilde\psi), \end{aligned}$$

where the change of variables, $\tilde\psi = \psi-\varphi$, has been made. Given $\varphi\in(0,2\pi)$ for $\psi\in[0,2\pi)$ we define $F_{N,\varphi}(\psi)$ as the cyclically translated empirical distribution function of F_N by

$$F_{N,\varphi}(\psi) = \begin{cases} \dfrac{\#\{\varphi\leqslant\theta_q\leqslant\varphi+\psi\}}{N} & \text{if } 0\leqslant\psi\leqslant 2\pi-\varphi,\\ F_{N,\varphi}(2\pi-\varphi) + \dfrac{\#\{0<\theta_q\leqslant\psi-2\pi+\varphi\}}{N} & \text{if } 2\pi-\varphi<\psi<2\pi. \end{cases}$$

To define the sequence $T_{N,\epsilon}(\varphi_r)$ we split the integral into two parts as in $\int_\epsilon^{2\pi-\epsilon}$ and $\int_0^\epsilon + \int_{2\pi-\epsilon}^{2\pi}$ and then use integration by parts on the first part, which yields the expression

$$(3.11) \quad T_{N,\epsilon}(\varphi_r) := \sqrt{N}\left(\left[\left(F_{N,\varphi_r}(\psi) - \frac{\psi}{2\pi}\right)\log\Big(2\big(1-\cos(\psi)\big)\Big)\right]_\epsilon^{2\pi-\epsilon}\right) - \sqrt{N}\int\limits_\epsilon^{2\pi-\epsilon}\left(F_{N,\varphi_r}(\psi) - \frac{\psi}{2\pi}\right)\frac{\sin(\psi)}{1-\cos(\psi)}d\psi.$$

For later use we make the following definition:

$$W_{N,\varphi}(\psi) := \sqrt{N}\left(F_{N,\varphi}(\psi) - \frac{\psi}{2\pi}\right).$$

The last term in (3.11) is not quite equal to the original sum, since

$$\int\limits_0^{2\pi}\log\Big(2\big(1-\cos(\psi)\big)\Big)d\psi = 0$$

so that the ψ terms do not give zero but rather cancel with μ_ϵ to be defined in a moment. In the remainder we express it as a sum, noting that we must include the mean, which is, by symmetry,

$$(3.12) \quad \mu_\epsilon := \frac{2}{2\pi}\int\limits_0^\epsilon \log\Big(2\big(1-\cos(\psi)\big)\Big)d\psi = \frac{2}{\pi}\big(\epsilon\log(\epsilon) - \epsilon + o(\epsilon)\big).$$

Define $S_\epsilon(\varphi) := \{\theta_q : \theta_q \in [\varphi-\epsilon, \varphi+\epsilon]\}$, and hence the sum can be written as

$$(3.13) \quad Z_{N,\epsilon}(\varphi_r) := \frac{1}{\sqrt{N}}\sum_{\theta_q\in S_\epsilon(\varphi_r)}\log\Big(2\big(1-\cos(\varphi_r-\theta_q)\big)\Big) - \sqrt{N}\mu_\epsilon.$$

Denote the corresponding sequence by $\mathbf{Z}_{N,\epsilon}$. Taking expectations we thus find that

$$\mathbb{E}\big(Z_{N,\epsilon}(\varphi_r)\big) = \frac{\sqrt{N}}{2\pi}\int\limits_{-\epsilon}^{\epsilon}\log\Big(2\big(1-\cos(\psi)\big)\Big)d\psi - \sqrt{N}\mu_\epsilon = 0$$

is a sequence of random variables with zero mean. We finally write

$$(3.14) \quad \mathbf{T}_N = \mathbf{T}_{N,\epsilon} + \mathbf{Z}_{N,\epsilon}.$$

We now proceed to demonstrate the first condition of (3.9), i.e., $\mathbf{T}_{N,\epsilon} \Rightarrow \mathbf{I}_\epsilon$. The random variable $T_{N,\epsilon}(\varphi_r)$ is a functional of an empirical distribution and therefore of a process lying in $D[0, 2\pi]$. Define the random sequence J_ϵ for $f \in D[0, 2\pi]$, where $f(0) = f(2\pi) = 0$, with the component term

(3.15)
$$J_{\epsilon,r}(f) = \int_\epsilon^{2\pi-\epsilon} f_{\varphi_r}(\psi)\frac{\sin(\psi)}{1-\cos(\psi)}d\psi - \Big[f_{\varphi_r}(\psi)\log\Big(2\big(1-\cos(\psi)\big)\Big)\Big]_\epsilon^{2\pi-\epsilon}.$$

It is well known that $W_{N,0} \Rightarrow W^o$ in D, which implies that $W_{N,\varphi_r} \Rightarrow W^o_{\varphi_r}$ as $N \to \infty$ for all r. The result follows on showing that J_ϵ defines a measurable mapping $J_\epsilon : D[0, 2\pi] \to \mathbb{R}^\infty$ in $D[0, 2\pi]$. Since

$$-J_{\epsilon,r}(W_{N,0}) = T_{N,\epsilon}(\varphi_r),$$

we may therefore apply Theorem 5.1, Corollary 1 of [2], which states that if $W_{N,0} \Rightarrow W^o$, then $J_\epsilon(W_{N,0}) \Rightarrow J_\epsilon(W^o)$ (and hence $\mathbf{T}_{N,\epsilon} \Rightarrow \mathbf{I}_\epsilon$) provided that we also verify

$$\mathbb{P}(W^o \in D_{J_\epsilon}) = 0, \tag{3.16}$$

where D_{J_ϵ} is the discontinuity set. To deal with the measurability question we first observe that the coordinate maps are measurable, and since $\sin(\psi)/\big(1-\cos(\psi)\big)$ is continuous in $[\epsilon, 2\pi-\epsilon]$, it follows by Lemma 3.2 that $J_{\epsilon,r}$ is measurable for each r, and hence so is the sequence mapping J_ϵ. For the second part, (3.7) shows that the sequence of integrals converges with respect to ρ, leaving only the final term. However, since the limit W^o is almost surely continuous, it follows that

$$f^{(n)}_{\varphi_r}(\epsilon) \to W^o_{\varphi_r}(\epsilon),$$
$$f^{(n)}_{\varphi_r}(2\pi-\epsilon) \to W^o_{\varphi_r}(2\pi-\epsilon)$$

for each r if $f^{(n)} \to W^o$ in $D[0, 2\pi]$. Thus the corresponding sequence converges with respect to ρ also, and so (3.16) holds. The proof of the first condition is concluded.

It remains to demonstrate (3.10). Here we use the union bound and Chebyshev's inequality. This is because the various $Z_{N,\epsilon}(\varphi_r)$ in the sequences are dependent, as they are determined via the same θ_q. Nevertheless, they are of course themselves the sum of i.i.d. random variables. In determining the variance, we may work with $\varphi_r = 0$ without loss of generality. The variance of one of the i.i.d. summands in (3.13) is determined as

$$\sigma^2_\epsilon := \frac{2}{2\pi}\int_0^\epsilon \log^2\Big(2\big(1-\cos(\psi)\big)\Big)d\psi - \mu^2_\epsilon < \infty. \tag{3.17}$$

Since for small $\epsilon > 0$ we have $\log\big(2\big(1-\cos(\psi)\big)\big) = O\big(2\log(\psi)\big) + o(\psi)$, the integral is $\sigma^2_\epsilon = O(\epsilon\log^2\epsilon)$ as the integral of $\log^2 x$ is $x\log^2 x - 2x\log x + 2x$.

It follows that $\sigma_\epsilon^2 \to 0$ as $\epsilon \to 0$, which is the variance of the entire sum by independence and as it has been scaled.

Now, fix $\eta > 0$. By definition of ρ and from (3.14) we obtain

$$\rho(\mathbf{T}_{N,\epsilon}, \mathbf{T}_N) = \sum_{r=0}^{\infty} \frac{|Z_{N,\epsilon}(\varphi_r)|}{1 + |Z_{N,\epsilon}(\varphi_r)|} 2^{-r}.$$

Let R_η be such that $\sum_{r=R_\eta+1}^{\infty} 2^{-r} < \eta/2$. Now we apply the union bound to the remaining $R_\eta + 1$ summands to obtain

$$\begin{aligned}\mathbb{P}\Big(\sum_{r=0}^{R_\eta} \frac{|Z_{N,\epsilon}(\varphi_r)|}{1 + |Z_{N,\epsilon}(\varphi_r)|} 2^{-r} \geqslant \eta/2\Big) &\leqslant \sum_{r=0}^{R_\eta} \mathbb{P}\Big(|Z_{N,\epsilon}(\varphi_r)| 2^{-r} \geqslant \frac{\eta}{2\,(R_\eta+1)}\Big) \\ &\leqslant \sum_{r=0}^{R_\eta} \sigma_\epsilon^2 \frac{4\,(R_\eta+1)^2}{\eta^2 2^{2r}} \leqslant \sigma_\epsilon^2 \frac{16\,(R_\eta+1)^2}{3\eta^2}.\end{aligned}$$

Hence,

$$\limsup_{N\to\infty} \mathbb{P}\big(\rho(\mathbf{T}_{N,\epsilon}, \mathbf{T}_N) > \eta\big) \leqslant \frac{16(R_\eta+1)^2\sigma_\epsilon^2}{3\eta^2}$$

and the right-hand side goes to zero as ϵ goes to zero for each $\eta > 0$. Hence we obtain (3.10) as required. Therefore, we have verified all the conditions and Theorem 3.1 is proved. ∎

It is rather easy to see that $I^* > 0$ almost surely. For instance, as in the proof of Lemma 3.1 it can be shown that

$$\int_0^{2\pi}\int_0^{2\pi} \left| W_\varphi^o(\psi) \frac{\sin(\psi)}{1 - \cos(\psi)} \right| d\psi\, d\varphi < \infty.$$

Then from Fubini's theorem we get $\int_0^{2\pi} I_\varphi\, d\varphi = 0$ since $\int_0^{2\pi} W_\varphi^o(\psi) d\varphi = 0$. But I_φ is almost surely continuous, and hence $I^* = 0$ if and only if $I_\varphi = 0$ for every φ. It thus follows that $I^* > 0$ almost surely, as required.

The proof of the following is straightforward.

THEOREM 3.2. *Given* $\tau \geqslant 0$,

$$\liminf_{N\to\infty} \mathbb{P}(T_N^* > \tau) \geqslant \mathbb{P}(I^* > \tau)$$

with equality if τ *is a continuity point for the random variable* I^*. *Moreover*,

$$\frac{1}{s_N} \log \max\{|P_N(z)|^2 \colon |z| = 1\} \Rightarrow I^*. \tag{3.18}$$

It therefore follows that

$$\max\{|P_N(z)|^2 \colon |z| = 1\} \approx \exp\Big(\sqrt{n_1^2 + \ldots + n_N^2} I^*\Big)$$

for N sufficiently large. A sample histogram for the probability distribution function of the random variable I^* is shown in Figure 3.

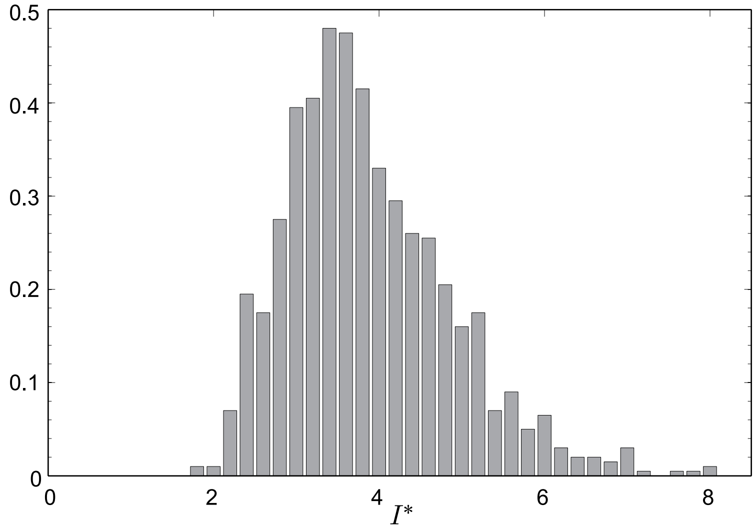

FIGURE 3. Sample histogram for the probability distribution function of the random variable I^*

4. NUMERICAL RESULTS

In this section, we present some simulations of our results. In Figure 4 we show the logarithm squared magnitude for a random polynomial with $N = 500$ and constant sequence $n_k = 1$. Here the maximum value is $\approx 10^{60}$ and occurs near $\theta = 0.3$. Our final plots (Figure 5) show the logarithm of the maximum magnitude as a function of the degree for the sequences $n_k = k$ and the sequence $n_k = 1$ with 100 realizations per degree. The black curves are $(n_1^2 + \ldots + n_N^2)^{1/2}$ and $5(n_1^2 + \ldots + n_N^2)^{1/2}$ respectively for both cases.

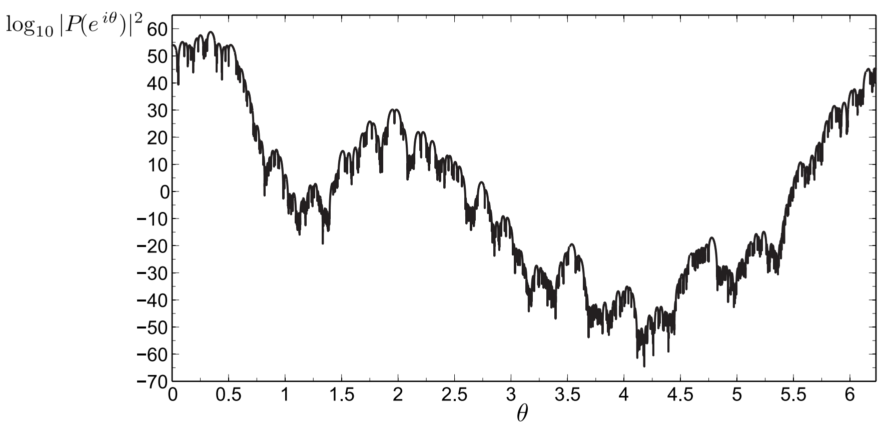

FIGURE 4. Log magnitude squared as a function of the phase ($N = 500$ and $n_k = 1$)

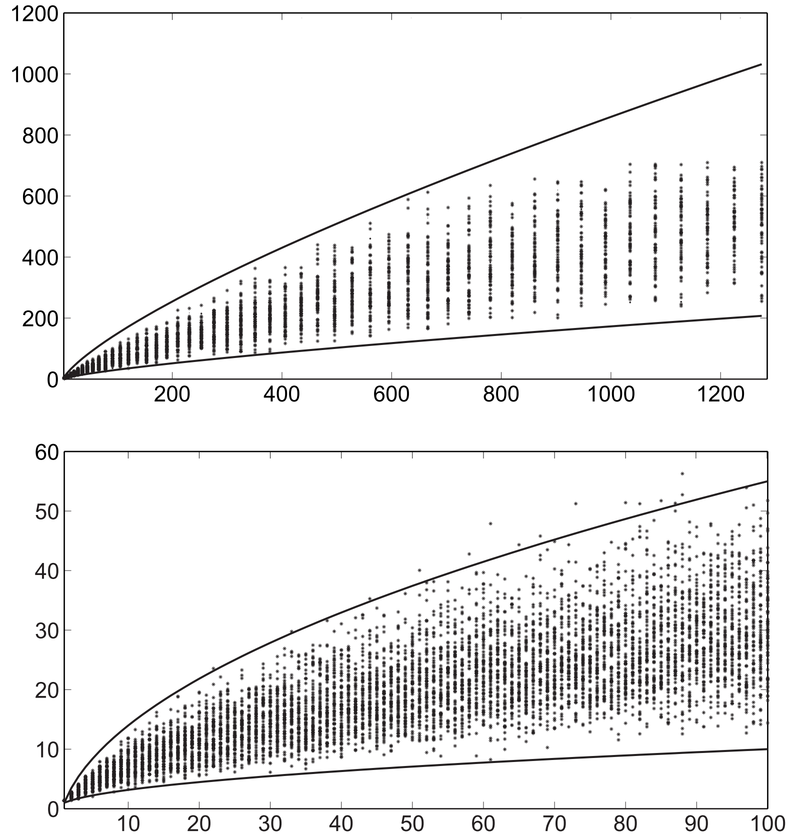

FIGURE 5. Logarithm of the maximum magnitude as a function of the degree for the sequences $n_k = k$ (top) and the sequence $n_k = 1$ (bottom) and 100 realizations per degree. The black curves are $(n_1^2 + \ldots + n_N^2)^{1/2}$ and $5(n_1^2 + \ldots + n_N^2)^{1/2}$ respectively for both cases

5. CONCLUSIONS AND FURTHER WORK

Here we make a few brief comments and suggestions. In most prior studies of random polynomials the approach has been to choose the coefficients at random (e.g., i.i.d.) and then examine the statistical properties of the roots. This paper goes the other way, it supposes the statistical properties of the roots and then looks at a property of the polynomial, namely its maximum magnitude on the unit circle. It is clear that there is some scope for continuation of this program. For example, one may attempt to derive statistical properties of the coefficients themselves.

With respect to applications, there is the possibility that the results of this paper may find connections to signal processing. This is because of links to complex Vandermonde matrices, generalized versions of which have been applied to sensor networks, see [11]; for other applications see [15].

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