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A NOTE ON PRESERVATION OF THE GENERALIZED TTT TRANSFORM ORDER

BY

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Dedicated to deceased Professor Jarosław Bartoszewicz

Abstract. The generalized total time on test (GTTT) transform with respect to the nonnegative function was introduced by Li and Shaked (2007). Comparing GTTT transforms of two distributions is useful in reliability theory and actuarial science, see, e.g., Li and Shaked (2007), Shaked et al. (2010). In 2009, Bartoszewicz and Benduch formulated the preserving theorem for some orders under the GTTT transform. We complete it with result for the GTTT transform order with respect to the nonnegative function. We give also some conditions for preservation under mixtures of exponential distributions and extend the theorem proved by Bartoszewicz and Skolimowska (2006).

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1. PRELIMINARIES

Consider the class $\mathcal{F}$ of absolutely continuous probability distributions with differentiable quantile functions. Throughout the paper we identify probability distribution $F \in \mathcal{F}$ with its cumulative distribution function (cdf). By f and F^{-1} we denote the density and the quantile functions, respectively. Their composition is denoted by fF^{-1}. For all other distributions, we use an analogous notation. We put the term *increasing* (*decreasing*) in place of *nondecreasing* (*nonincreasing*).

Let Φ be the class of nonnegative measurable functions defined on $[0,1]$. According to Li and Shaked [6], the function $H_F^{-1}(u;\varphi) = \int_{-\infty}^{F^{-1}(u)} \varphi\big(F(x)\big)dx$, for $u \in (0,1)$, is called the *generalized total time on test* (GTTT) transform of $F \in \mathcal{F}$ with respect to $\varphi \in \Phi$. We regard the GTTT transform as the quantile function of some distribution $H_F(\cdot;\varphi)$. Thus, the quantile-density of $H_F(\cdot;\varphi)$ is equal to $1/h_F H_F^{-1}(u;\varphi) = \varphi(u)/fF^{-1}(u)$.

A distribution F is said to be *smaller than G in the generalized TTT transform order with respect to* $h \in \Phi$ ($F \leqslant_{\mathrm{TTT}}^{(h)} G$) if $H_F^{-1}(u;h) \leqslant H_G^{-1}(u;h)$, $u \in (0,1)$, provided both integrals are well defined. The GTTT transform order was introduced by Li and Shaked [6] and for some h it appears to be a well-known order, e.g., when $h(y) = y$ it is the location independent riskier order (Jewitt [5]). Some authors (e.g., Shaked et al. [8]) studied the case when the function h is a distortion function, i.e., increasing, $h(0) = 0$ and $h(1) = 1$. We denote the class of distortion functions by $\Phi_d \subset \Phi$. It is easy to see that the inequality

$$\int_0^u \frac{h(y)}{gG^{-1}(y)}dy - \int_0^u \frac{h(y)}{fF^{-1}(y)}dy \geqslant 0, \quad 0 < u < 1, \tag{1.1}$$

is an equivalent condition when $F, G \in \mathcal{F}$ and $h \in \Phi_d$. Writing $F \leqslant_{\mathrm{TTT}}^{(h)} G$, we always assume that both GTTT transforms with respect to h exist.

A distribution F is said to be *smaller than G in the excess wealth order* ($F \leqslant_{\mathrm{ew}} G$) if $\int_{F^{-1}(u)}^{\infty} \big(1 - F(x)\big)dx \leqslant \int_{G^{-1}(u)}^{\infty} \big(1 - G(x)\big)dx$, $u \in (0,1)$. The excess wealth order could be rewritten in terms of the GTTT transform order, viz. the location independent riskier order (see Shaked and Shanthikumar [7], p. 165, and Example 4.B.32, p. 226).

A distribution F is said to be *smaller than G in the dispersive order* ($F \leqslant_{\mathrm{disp}} G$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leqslant G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leqslant \beta < 1$. For $F, G \in \mathcal{F}$ it is known (Shaked and Shanthikumar [7], (3.B.11), p. 149) that $F \leqslant_{\mathrm{disp}} G$ if and only if $fF^{-1}(u)/gG^{-1}(u) \geqslant 1$, $u \in (0,1)$. This characterization and (1.1) show the relation between the dispersive and the GTTT transform orders.

LEMMA 1.1. *Let $F, G \in \mathcal{F}$. If $F \leqslant_{\mathrm{disp}} G$, then $F \leqslant_{\mathrm{TTT}}^{(h)} G$ for all $h \in \Phi_d$.*

The GTTT transform preserves the dispersive order.

LEMMA 1.2 (Bartoszewicz and Benduch [3]). *Let $F, G \in \mathcal{F}$ and $\varphi_1, \varphi_2 \in \Phi$. If $F \leqslant_{\mathrm{disp}} G$ and $\varphi_2(u)/\varphi_1(u) \geqslant 1$, $u \in (0,1)$, then $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{disp}} H_F(\cdot;\varphi_2)$ and $H_F(\cdot;\varphi_2) \leqslant_{\mathrm{disp}} H_G(\cdot;\varphi_2)$.*

Now, let us assume that $F, G \in \mathcal{F}$ and $F(0) = G(0) = 0$. Then it follows that $\widetilde{F}(x) = \int_0^{\infty}(1 - e^{-x\lambda})f(\lambda)d\lambda$, $x > 0$, is a cdf of a mixture of exponential distributions with means $1/\lambda$, $\lambda > 0$, and a mixing distribution F. Analogously for $\widetilde{G}$ and G.

LEMMA 1.3 (Bartoszewicz and Skolimowska [4]). *Let $F(0) = G(0) = 0$. If $F \leqslant_{\mathrm{disp}} G$ and either F or G is IFR, then $\widetilde{G} \leqslant_{\mathrm{disp}} \widetilde{F}$.*

A distribution F is said to be *smaller than G in the convex transform order* ($F \leqslant_{\mathrm{c}} G$) if $G^{-1}F$ is a convex function on the support of F or fF^{-1}/gG^{-1} is increasing on $(0,1)$, equivalently.

A distribution F is said to be IFR (DFR) if $f(x)/\big(1 - F(x)\big)$ is increasing (decreasing) in x on the support of F. The equivalent characteristic, obtained by the substitution $x = F^{-1}(u)$, is the monotonicity of $fF^{-1}(u)/(1-u)$ in $u \in (0,1)$. Let H be an exponential distribution with mean one. Since $hH^{-1}(u) = 1-u$, a distribution F is IFR (DFR) if $F \leqslant_{\mathrm{c}} H$ ($H \leqslant_{\mathrm{c}} F$).

2. RESULTS

Some transformations and conditions for the GTTT transform order to be preserved were given by Shaked et al. [8]. Bartoszewicz and Benduch [3] formulated conditions for the GTTT transform to preserve a few stochastic orders, including the dispersive order.

2.1. The GTTT transform. We prove that the GTTT transform order is preserved under the GTTT transform. Note that if $-\infty < F^{-1}(0) = G^{-1}(0)$ and $F \leqslant_{\mathrm{TTT}}^{(h)} G$, then $H_F(\cdot;\varphi) \leqslant_{\mathrm{TTT}}^{(h)} H_G(\cdot;\varphi)$ holds if and only if

$$\int_0^u \varphi(y)\frac{h(y)}{fF^{-1}(y)}dy \leqslant \int_0^u \varphi(y)\frac{h(y)}{gG^{-1}(y)}dy, \quad 0<u<1. \tag{2.1}$$

If $F \leqslant_{\mathrm{TTT}}^{(h)} G$ and $\varphi \in \Phi$ is differentiable and decreasing, then $\varphi'(y)H_F^{-1}(y;h) \geqslant \varphi'(y)H_G^{-1}(y;h)$, $y \in (0,1)$. Integrating by parts, we obtain (2.1) as follows:

$$\begin{aligned}\int_0^u \varphi(y)\frac{h(y)}{fF^{-1}(y)}dy &= \varphi(u)H_F^{-1}(u;h) - \int_0^u \varphi'(y)H_F^{-1}(y;h)dy \\ &\leqslant \varphi(u)H_G^{-1}(u;h) - \int_0^u \varphi'(y)H_G^{-1}(y;h)dy = \int_0^u \varphi(y)\frac{h(y)}{gG^{-1}(y)}dy.\end{aligned}$$

We state the theorem without the assumption of differentiability of φ.

THEOREM 2.1. *Let $F, G \in \mathcal{F}$, $-\infty < F^{-1}(0) = G^{-1}(0)$ and $h \in \Phi_d$. If $\varphi \in \Phi$ is decreasing and $F \leqslant_{\mathrm{TTT}}^{(h)} G$, then $H_F(\cdot;\varphi) \leqslant_{\mathrm{TTT}}^{(h)} H_G(\cdot;\varphi)$.*

P r o o f. If $F \leqslant_{\mathrm{TTT}}^{(h)} G$, then (1.1) holds. Thus,

$$\int_0^u \left(\frac{h(y)}{gG^{-1}(y)} - \frac{h(y)}{fF^{-1}(y)}\right)dy \geqslant 0 \quad \text{for all } u \in (0,1).$$

Since the function φ is nonnegative and decreasing, by Barlow and Proschan [1] (Lemma 7.1 (b), p. 101) we have

$$\int_0^u \varphi(y)\left(\frac{h(y)}{gG^{-1}(y)} - \frac{h(y)}{fF^{-1}(y)}\right)dy \geqslant 0 \quad \text{for all } u \in (0,1),$$

so (2.1) holds true. ■

The following lemma precedes stating the preserving rule for distributions ordered in the convex transform order.

LEMMA 2.1. *Let* $F, G \in \mathcal{F}$ *and* $h \in \Phi_d$. *If* $F \leqslant_{\mathrm{TTT}}^{(h)} G$ *and* fF^{-1}/gG^{-1} *is increasing*, *then* $F \leqslant_{\mathrm{disp}} G$.

P r o o f. Remind that $F \leqslant_{\mathrm{disp}} G$ if and only if $fF^{-1}(u)/gG^{-1}(u) \geqslant 1$, $u \in (0,1)$. Suppose that there exists $u_0 \in (0,1)$ such that $fF^{-1}(u_0)/gG^{-1}(u_0) < 1$. Since h and fF^{-1} are nonnegative and fF^{-1}/gG^{-1} is increasing, $h(u)/gG^{-1}(u) < h(u)/fF^{-1}(u)$ for all $u \in (0, u_0)$. Thus,

$$\int_0^{u_0} \frac{h(y)}{gG^{-1}(y)} dy < \int_0^{u_0} \frac{h(y)}{fF^{-1}(y)} dy.$$

But $F \leqslant_{\mathrm{TTT}}^{(h)} G$, so (1.1) holds true and a supposition leads to a contradiction. ∎

THEOREM 2.2. *Let* $F, G \in \mathcal{F}$ *and* $\varphi_1, \varphi_2 \in \Phi_d$. *If* $F \leqslant_{\mathrm{TTT}}^{(h_0)} G$ *for some* $h_0 \in \Phi_d$, fF^{-1}/gG^{-1} *is increasing and* $\varphi_2(u)/\varphi_1(u) \geqslant 1$, $u \in (0,1)$, *then we infer for all* $h \in \Phi_d$ *that* $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{TTT}}^{(h)} H_G(\cdot;\varphi_2)$.

P r o o f. Using Lemmas 2.1 and 1.2, we have $H_F(\cdot;\varphi_1) \leqslant_{\mathrm{disp}} H_G(\cdot;\varphi_2)$. Then Lemma 1.1 completes the proof. ∎

2.2. The mixture of exponential distributions. We examine if mixtures $\widetilde{F}$ and $\widetilde{G}$ preserve the GTTT transform order. For some other stochastic orders it is known (Bartoszewicz and Skolimowska [4]) that mixtures of exponential distributions are ordered in the reversed way to that of mixing distributions, like the dispersive order in Lemma 1.3. We prove that the GTTT transform order has the same property under some assumptions.

THEOREM 2.3. *Let* $F, G \in \mathcal{F}$ *and* $F(0) = G(0) = 0$. *If* $F \leqslant_{\mathrm{TTT}}^{(h_0)} G$ *for some* $h_0 \in \Phi_d$, *either* F *or* G *is IFR*, *and* $F \leqslant_{\mathrm{c}} G$, *then* $\widetilde{G} \leqslant_{\mathrm{TTT}}^{(h)} \widetilde{F}$ *for all* $h \in \Phi_d$.

P r o o f. If $F \leqslant_{\mathrm{c}} G$, then fF^{-1}/gG^{-1} is increasing and from Lemma 2.1 we infer that $F \leqslant_{\mathrm{disp}} G$. It follows from Lemma 1.3 that $\widetilde{G} \leqslant_{\mathrm{disp}} \widetilde{F}$. Thus, by Lemma 1.1, $\widetilde{G} \leqslant_{\mathrm{TTT}}^{(h)} \widetilde{F}$. ∎

Bartoszewicz and Skolimowska [4] formulated a preserving rule for the excess wealth order. We prove it using previous lemmas, since the excess wealth order is equivalent to the GTTT transform order (with respect to the identity function) between mirror-images of input distributions.

COROLLARY 2.1. *Let* $F, G \in \mathcal{F}$, $F(0) = G(0) = 0$. *If* $G \leqslant_{\mathrm{ew}} F$, F *is IFR and* G *is DFR*, *then both of the following relations are true*:

(i) $\widetilde{F} \leqslant_{\mathrm{TTT}}^{(h)} \widetilde{G}$ *for all* $h \in \Phi$;

(ii) $\widetilde{F} \leqslant_{\mathrm{ew}} \widetilde{G}$ (Bartoszewicz and Skolimowska [4]).

P r o o f. Let $F_{\leftrightarrow}$ be the mirror-image distribution of the distribution F, i.e., $F_{\leftrightarrow}(t) = 1 - F(-t)$. Mirror-images of $G, \widetilde{F}$, and $\widetilde{G}$ are denoted by $G_{\leftrightarrow}, \widetilde{F}_{\leftrightarrow}$, and $\widetilde{G}_{\leftrightarrow}$, respectively. Then $G \leqslant_{\mathrm{ew}} F$ if and only if $G_{\leftrightarrow} \leqslant_{\mathrm{TTT}}^{(h_0)} F_{\leftrightarrow}$ for $h_0(y) = y$ (Shaked and Shanthikumar [7], p. 165, and Example 4.B.32, p. 226).

The quantile function of $F_{\leftrightarrow}$ is equal to $-F^{-1}(1-u)$, $u \in (0,1)$. Thus, its quantile-density is equal to $1/fF^{-1}(1-u)$. Analogously for the distribution $G_{\leftrightarrow}$. We assumed that F is IFR and G is DFR, so fF^{-1}/gG^{-1} is increasing. Consequently, the function $gG^{-1}(1-u)/fF^{-1}(1-u)$ is also increasing and we can use Lemma 2.1. We obtain $G_{\leftrightarrow} \leqslant_{\mathrm{disp}} F_{\leftrightarrow}$, which is equivalent to $G \leqslant_{\mathrm{disp}} F$ (Shaked and Shanthikumar [7], Theorem 3.B.6, p. 151). Now we get $\widetilde{F} \leqslant_{\mathrm{disp}} \widetilde{G}$ by Lemma 1.3.

Recall that $\widetilde{F} \leqslant_{\mathrm{disp}} \widetilde{G}$ is equivalent to $\widetilde{F}_{\leftrightarrow} \leqslant_{\mathrm{disp}} \widetilde{G}_{\leftrightarrow}$. By Lemma 1.1, we obtain $\widetilde{F} \leqslant_{\mathrm{TTT}}^{(h)} \widetilde{G}$ and $\widetilde{F}_{\leftrightarrow} \leqslant_{\mathrm{TTT}}^{(h)} \widetilde{G}_{\leftrightarrow}$ for all $h \in \Phi_d$. The first provides (i). The second corresponds to (ii) in the special case $h(y) = y$. ■

2.3. The representation in terms of $\tilde{\Lambda}$-order. Jewitt [5] proved that when $h(y) = y$, then $F \leqslant_{\mathrm{TTT}}^{(h)} G$ if and only if $u^{-1} \int_0^u \big(F^{-1}(t) - G^{-1}(t)\big)\, dt$ is decreasing in $u \in (0,1)$. We recall the condition for the GTTT transform order (e.g., Shaked et al. [8]) and give our proof.

PROPOSITION 2.1. *Let* $h \in \Phi_d$ *be differentiable. Then* $F \leqslant_{\mathrm{TTT}}^{(h)} G$ *if and only if*

$$\frac{1}{h(u)} \int_0^u h'(t)\big(G^{-1}(t) - F^{-1}(t)\big)\, dt \tag{2.2}$$

is increasing in $u \in (0,1)$.

P r o o f. Integrating the left-hand side of (1.1) by parts, we can infer that the inequality

$$h(u)\big(G^{-1}(u) - F^{-1}(u)\big) - \int_0^u h'(t)\big(G^{-1}(t) - F^{-1}(t)\big)\, dt \geqslant 0, \qquad u \in (0,1),$$

is equivalent to $F \leqslant_{\mathrm{TTT}}^{(h)} G$. The derivative of the function (2.2) is equal to

$$\frac{h'(u)}{h^2(u)} \Big[h(u)\big(G^{-1}(u) - F^{-1}(u)\big) - \int_0^u h'(t)\big(G^{-1}(t) - F^{-1}(t)\big)\, dt\Big], \quad u \in (0,1).$$

Thus, the monotonicity of (2.2) indicates the GTTT transform order and refers to the monotonicity of h. ■

Let $\mathcal{F}_S \subset \mathcal{F}$ be a subclass of distributions with a common support S, which is an interval. Assume that $\tilde{\mathcal{H}}$ is a class of real functions defined on $(0,1)$. Let $F, G \in \mathcal{F}_S$. Bartoszewicz [2] introduced the $\tilde{\Lambda}$-hazard function of the distribution F as $\tilde{\Lambda}(F)(u)$, $u \in (0,1)$, where $\tilde{\Lambda} : \mathcal{F}_S \to \tilde{\mathcal{H}}$ is a one-to-one operator. It was used to define the $\tilde{\Lambda}$-order as follows. A distribution F is said to be *smaller than G in the $\tilde{\Lambda}$-order* if $\tilde{\Lambda}(G)(u) - \tilde{\Lambda}(F)(u)$ is increasing on $u \in (0,1)$. Using Proposition 2.1, it is possible to regard the GTTT transform order with respect to an increasing and differentiable function $h \in \Phi_d$ as the $\tilde{\Lambda}$-order with the operator

$$\tilde{\Lambda}(F)(u) = \frac{1}{h(u)} \int_0^u |h'(t)| F^{-1}(t)dt, \qquad 0 < u < 1.$$

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