PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 34, Fasc. 2 (2014), pp. 279–291

BARGMANN MEASURES FOR t-DEFORMED PROBABILITY*

BY

ANNA DOROTA KRYSTEK AND ŁUKASZ JAN WOJAKOWSKI

Abstract. It is shown that the Bargmann representation of a t-deformed probability measure can be obtained by taking away some t-dependent amount of mass at zero of the Bargmann representation of the original measure and scaling of the remaining part. This allows us to formulate conditions on existence of the Bargmann representation of a t-deformed probability measure and to study some prominent examples.

2000 AMS Mathematics Subject Classification: Primary: 30E10; Secondary: 60E10.

Key words and phrases: Bargmann representation, deformation, complex moment problem.

1. MOTIVATION

Bargmann [3], [4] showed that there is a unitary isomorphism from the Hilbert space $\mathbf{L}^2\big(\mathbb{R}^n, (2\pi)^{-n/2}\exp(-\|x\|^2/2)\mathrm{d}x\big)$ with the usual scalar product onto the Hilbert space of all holomorphic functions in n complex variables, equipped with the scalar product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\gamma_n(\mathrm{d}z),$$

where $\gamma_n(\mathrm{d}z) = \pi^{-n}\exp(-\|z\|^2)\mathrm{d}z$ for $z \in \mathbb{C}^n$, which maps orthogonal polynomials of the first space onto monomials of the same degree in the second. That mapping is usually called the *Segal–Bargmann transform*. A similar result was shown by Asai et al. [2] for the Gaussian and Poisson measures. There are also other contributions to the subject, for instance Królak [11], who studied q-Gaussian measures, and Penson and Solomon [12], with Gaussian and q-Gaussian in a more physical setting. It is therefore natural to ask if the Segal–Bargmann transform has the desired properties for Hilbert spaces constructed with other measures on $\mathbb{R}$. A necessary condition for that is the existence of an analogue of the measure γ_n

* This paper was sponsored by the Polish National Science Center grant No. 2012/05/B/ST1/00626.

as a solution to a complex moment problem depending on the initial measure. In the present paper we are studying some general facts about such moment problems, and calculating a few examples for central limit measures of convolutions appearing around the free probability theory.

2. INTRODUCTION

Let μ be a probability measure on $\mathbb{R}$ having finite moments $m_\mu(n)$ of all orders. Then the Cauchy transform of the measure μ can be conveniently written in the continued fraction form

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(\mathrm{d}x)}{z-x} = \cfrac{1}{z-\alpha_\mu(1) - \cfrac{\lambda_\mu(1)}{z-\alpha_\mu(2) - \cfrac{\lambda_\mu(2)}{z-\alpha_\mu(3) - \cfrac{\lambda_\mu(3)}{\ddots}}}}, \tag{2.1}$$

which has to be understood either formally in the general case or as a completely convergent continued fraction in the case when the measure μ is determined by its moments, due to a theorem that can be found, for instance, in [13].

From [1] and [9] it is known that for such a μ there exists a complete orthogonal system $\{P_n^\mu(x)\}_{n=0}^\infty$ of polynomials for $\mathbf{L}^2\big(\mathbb{R};\mu(\mathrm{d}x)\big)$, a sequence $\{\lambda_\mu(n)\}_{n=0}^\infty$, $\lambda_\mu(n) \geqslant 0$, and a sequence $\{\alpha_\mu(n)\}_{n=0}^\infty$, $\alpha_\mu(n) \in \mathbb{R}$, the same as in the Cauchy transform, such that

$$\begin{aligned} P_0^\mu(x) &= 1, \quad P_1^\mu(x) = x - \alpha_\mu(1), \\ \big(x-\alpha_\mu(n+1)\big)P_n^\mu(x) &= P_{n+1}^\mu(x) + \lambda_\mu(n) P_{n-1}^\mu(x), \\ \langle P_n^\mu(x), P_m^\mu(x)\rangle_{\mathbf{L}^2(\mu)} &= \delta_{n,m}\lambda_\mu(1)\ldots\lambda_\mu(n) =: \delta_{n,m}\Lambda_\mu(n). \end{aligned}$$

The sequence of the orthonormal polynomials associated with μ is given by

$$\begin{aligned} p_0^\mu(x) &= 1, \quad p_1^\mu(x) = \frac{x-\alpha_\mu(1)}{\sqrt{\lambda_\mu(1)}}, \\ \big(x-\alpha_\mu(n+1)\big)p_n^\mu(x) &= \sqrt{\lambda_\mu(n+1)}\, p_{n+1}^\mu(x) + \sqrt{\lambda_\mu(n)}\, p_{n-1}^\mu(x). \end{aligned}$$

DEFINITION 2.1. The *Bargmann representation* of a probability measure μ on $\mathbb{R}$ with symmetric Jacobi coefficients $\lambda_\mu(n)$ is a probability measure β_μ on the complex plane $\mathbb{C}$ such that for all $m, n \in \mathbb{N}$

$$\int_{\mathbb{C}} z^n \overline{z}^m \beta_\mu(\mathrm{d}z) = \delta_{n,m}\, \lambda_\mu(1)\ldots\lambda_\mu(n) = \delta_{n,m}\Lambda_\mu(n). \tag{2.2}$$

REMARK 2.1. *Since the Bargmann representation β_μ depends only on the symmetric Jacobi coefficients $\lambda_\mu(n)$ of the original measure μ, it is also the Bargmann representation of all other probability measures ν that differ from μ by the coefficients $\alpha_\nu(n)$. In the sequel, we shall therefore restrict our attention to the symmetric probability measures, that is, measures with $\alpha_\nu(n) = 0$ for all n.*

A discussion of when a complex bisequence is a complex moment sequence can be found in [15]. The authors show in Theorem 1 that the sequence $\{c_{m,n}\}_{m,n=0}^{\infty}$ is a complex moment sequence, that is, there exists a positive Borel measure γ on $\mathbb{C}$ fulfilling

$$c_{n,m} = \int_{\mathbb{C}} z^n \overline{z}^m \gamma(\mathrm{d}z)$$

if and only if there exists a sequence $\{\tilde{c}_{n,m}\}_{m+n\geqslant 0}^{\infty} \subset \mathbb{C}$ such that

$$\tilde{c}_{n,m} = c_{n,m} \quad \text{for } m, n = 0, 1, \ldots$$

and for any finite $\{z_{m,n}\}_{m+n\geqslant 0} \subset \mathbb{C}$

$$\sum_{\substack{m+n\geqslant 0 \\ p+q\geqslant 0}} \tilde{c}_{m+q,n+p}\, z_{m,n}\, \overline{z}_{p,q} \geqslant 0. \tag{2.3}$$

Moreover, Stochel and Szafraniec [15] show in Corollary 4 that if the bisequence $\{c_{m,n}\}$ admits a decomposition $c_{m,n} = a_{m+n} b_{m-n}$ with $\{a_n\}_{n\in\mathbb{N}}$ positive definite on the semigroup $\mathbb{N}$ and $\{b_m\}_{n\in\mathbb{Z}}$ positive definite on the group $\mathbb{Z}$, then the extension $\{\tilde{c}_{m,n}\}$ satisfying (2.3) can be found. Consequently, we have

PROPOSITION 2.1. *If $\{\Lambda_\mu(n)\}_{n=0}^{\infty}$ and $\{\Lambda_\mu(n)\}_{n=1}^{\infty}$ are positive definite, μ admits a Bargmann representation β_μ.*

Proof. Take

$$b_m = \begin{cases} 1, & m = 0, \\ 0, & m \in \mathbb{Z} \setminus 0, \end{cases} \qquad a_n = \begin{cases} \Lambda_\nu(k), & n = 2k, \\ 0, & n = 2k+1, \end{cases}$$

since $\{\Lambda_\mu(n)\}_{n=0}^{\infty}$ are moments of a Stieltjes measure, a_n are moments of the symmetrization of its composition with square root, and are positive definite. Now, use the above-mentioned Corollary 4 of [15]. ■

Assume that for a given μ the Bargmann measure β_μ exists, that is, there exists a solution of the following moment problem:

$$\int_{\mathbb{C}} z^n \overline{z}^m \beta_\mu(\mathrm{d}z) = \delta_{n,m} \Lambda_\mu(n).$$

We would like to make a polar decomposition of the measure on the complex plane. Let us take a random variable Z whose distribution is the Bargmann measure β_μ and some integrable function f. We have

$$\begin{aligned}\int f(z)\beta_\mu(\mathrm{d}z) &= \mathbb{E}\, f(Z) = \mathbb{E}\,[\mathbb{E}\, f(Z)\,|\,|Z| = r] \\ &= \int\limits_{[0,\infty)} \big[\mathbb{1}_{\{0\}}(r) f(0) + \mathbb{1}_{(0,\infty)}(r) \int\limits_{[0,2\pi)} f(r\, e^{i\,\theta})\Theta_\mu^r(\mathrm{d}\theta)\big]\,\varrho_\mu(\mathrm{d}r).\end{aligned}$$

DEFINITION 2.2. The Bargmann measure will be called *rotation-invariant* (or *radial*) if $\Theta_\mu^r = \frac{1}{2\pi}\lambda_{[0,2\pi)}$ for ϱ_μ-almost all r.

The following theorem has been noted in [16].

THEOREM 2.1. *If a measure μ admits a Bargmann representation $\beta_\mu(\mathrm{d}z) = \Theta_\mu^r(\mathrm{d}\theta)\varrho_\mu(\mathrm{d}r)$, then it also admits a rotation-invariant Bargmann representation.*

P r o o f. Take $f(z) = z^k \overline{z}^l$. Observe that for $k = l$

$$\begin{aligned}&\int\limits_{\mathbb{C}} z^k \overline{z}^l \beta_\mu(\mathrm{d}z) \\ &\quad= \int\limits_{[0,\infty)} \big[\mathbb{1}_{\{0\}}(r) 0^{k+l} + \mathbb{1}_{(0,\infty)}(r) \int\limits_{[0,2\pi)} r^{k+l}\, e^{i\,(k-l)\,\theta}\Theta_\mu^r(\mathrm{d}\theta)\big]\,\varrho_\mu(\mathrm{d}r) \\ &\quad= \int\limits_{[0,\infty)} r^{k+l}\varrho_\mu(\mathrm{d}r) = \Lambda_\mu(k).\end{aligned}$$

This means that $m_{\varrho_\mu}(2k) = \Lambda_\mu(k)$, and integration with respect to the product measure $\frac{1}{2\pi}\lambda_{[0,2\pi)}(\mathrm{d}\theta)\varrho_\mu(\mathrm{d}r)$ yields for $k = l$

$$\begin{aligned}&\int\limits_{[0,\infty)} \left[\mathbb{1}_{\{0\}}(r) 0^{k+l} + \mathbb{1}_{(0,\infty)}(r) \int\limits_{[0,2\pi)} r^{k+l}\, e^{i\,(k-l)\,\theta} \frac{1}{2\pi}\lambda_{[0,2\pi)}(\mathrm{d}\theta)\right] \varrho_\mu(\mathrm{d}r) \\ &\qquad\qquad = \int\limits_{[0,\infty)} r^{k+l}\varrho_\mu(\mathrm{d}r) = \Lambda_\mu(k),\end{aligned}$$

and zero for $k \neq l$, so it is also a Bargmann representation for μ. ∎

The radial measure ϱ_μ is convenient for taking into product measures, but since its moments are only partially specified, it is inconvenient to handle it directly. Instead, we may use, along the lines of [6], the bijection $j\colon \mathbb{R}^+ \to \mathbb{R}^+$, $j(x) = x^2$, that produces a bijection between Stieltjes measures via the equation

$$\int f(x)\nu^j(\mathrm{d}x) = \int f\big(j(x)\big)\nu(\mathrm{d}x).$$

Then let us define the following:

DEFINITION 2.3. Define the *square-radial measure* ξ_μ to be

$$\xi_\mu = \varrho_\mu^j.$$

Now we have

$$m_{\xi_\mu}(n) = \Lambda_\mu(n) \tag{2.4}$$

and the existence and uniqueness of ξ_μ can be studied by using standard Stieltjes moment problem techniques, while the measure ϱ_μ can be easily recovered from ξ_μ by inverting j. Hence, as noticed in Proposition 2.1 of [6], we have the following:

COROLLARY 2.1. *There is a one-to-one correspondence between radial solutions of* (2.2) *and solutions of the Stieltjes moment problem*:

$$\begin{aligned}&\Lambda_\mu(0) = 1,\\ &\Lambda_\mu(n) = \lambda_\mu(1)\dots\lambda_\mu(n) = \int_0^\infty x^n \xi_\mu(\mathrm{d}x).\end{aligned}$$

From the above discussion we have the following remark:

REMARK 2.2. *The Bargmann measure β_μ exists if and only if the sequences $\{\Lambda_\mu(n)\}_{n=0}^\infty$ and $\{\Lambda_\mu(n)\}_{n=1}^\infty$ are positive definite.*

We remind a criterion for the determinate moment problem (see [14], Proposition 1.5):

PROPOSITION 2.2. *If $\{d_n\}_{n=0}^\infty$ is a Stieltjes moment sequence* (*that is*, *if there is a measure ρ_d on $[0,\infty)$ such that $d_n = \int_0^\infty x^n \rho_d(\mathrm{d}x)$*) *and for some $C, R > 0$*

$$|d_n| \leqslant CR^n\,(2n)!,$$

then the Stieltjes moment problem is determinate.

3. THE t-DEFORMATION OF MEASURES AND OF BARGMANN REPRESENTATIONS

DEFINITION 3.1. Let $t \geqslant 0$. The *t-deformation of a probability measure $\mu \in \mathcal{P}(\mathbb{R})$* is the measure $U_t\mu$ corresponding to the reciprocal of the Cauchy transform given by

$$\frac{1}{G_{U_t\mu}(z)} = \frac{t}{G_\mu(z)} + (1-t)z. \tag{3.1}$$

The properties of the transformation U_t were studied in [7], [8], and [17].

Given a measure μ with all moments finite, the t-deformed measure $U_t\mu$ has the Cauchy transform given by

$$G_{U_t\mu}(z) = \cfrac{1}{z - t\,\alpha_\mu(1) - \cfrac{t\,\lambda_\mu(1)}{z - \alpha_\mu(2) - \cfrac{\lambda_\mu(2)}{z - \alpha_\mu(3) - \cfrac{\lambda_\mu(3)}{\ddots}}}}.$$

Using the t-transformation Bożejko and Wysoczański [7] defined the associated deformation of convolutions in the following way:

DEFINITION 3.2. Given a convolution $\oplus$ and $t > 0$, the *t-deformed convolution* $\oplus_t$ is defined as

$$\mu \oplus_t \nu = U_{1/t}\big((U_t\mu) \oplus (U_t\nu)\big)$$

for any two probability measures μ and ν.

Let us now recall the fundamental observation of [7], the central limit theorem.

THEOREM 3.1. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be such that $m_\mu(1) = 0,\ m_\mu(2) = 1$, and let $t > 0$. Then*

$$\mathbb{D}_{1/\sqrt{n}}\mu \oplus_t \ldots \oplus_t \mathbb{D}_{1/\sqrt{n}}\mu \stackrel{n\to\infty}{\longrightarrow} \nu_t$$

in the weak-* topology, where the limiting measure ν_t is a related measure appearing in the central limit theorem for the convolution $\oplus$ by $\nu_t = U_{1/t}\mathbb{D}_{\sqrt{t}}\nu$, where the dilation $\mathbb{D}$ is defined as usual by $\mathbb{D}_s\nu(A) = \nu(s^{-1}A)$.

The above theorem suggests that instead of considering the original deformation U_t it is preferable to use $U_{1/t}\mathbb{D}_{\sqrt{t}}$. In the remaining part of the present paper we shall be concerned with the existence of Bargmann representations of $(U_{1/t}\mathbb{D}_{\sqrt{t}})$-deformations of probability measures. To this end we have the following

PROPOSITION 3.1. *Assume that a measure μ is symmetric with finite moments of all orders. If the symmetric Jacobi coefficients of μ are equal to $\lambda_\mu(n)$, then to the measure $\mu_t = U_{1/t}\mathbb{D}_{\sqrt{t}}\,\mu$ there corresponds the sequence of symmetric Jacobi coefficients*

$$\lambda_{\mu_t}(1) = \lambda_\mu(1), \quad \lambda_{\mu_t}(n) = t\,\lambda_\mu(n),\ n \geqslant 2.$$

Hence, the moments associated with the Bargmann measures are related by

$$\Lambda_{\mu_t}(0) = \Lambda_\mu(0) = 1, \quad \Lambda_{\mu_t}(n) = t^{n-1}\Lambda_\mu(n).$$

THEOREM 3.2. *The measures ξ_{μ_t}, ϱ_{μ_t}, when they exist, are given by*

$$\xi_{\mu_t} = \left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\mathbb{D}_t \xi_\mu,$$
$$\varrho_{\mu_t} = \left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\mathbb{D}_{\sqrt{t}} \varrho_\mu.$$

P r o o f. It is easily checked by computing the moments of the measures. ■

4. POSITIVE DEFINITENESS OF THE t-TRANSFORMED MOMENTS

Assume that the Stieltjes moment problem $\{m_\nu(n)\}_{n=0}^\infty$ with $m_\nu(0) = 1$ has indeed a solution ν, that is, all the determinants

$$D_n = \begin{vmatrix} m_\nu(0) & m_\nu(1) & \dots & m_\nu(n) \\ m_\nu(1) & m_\nu(2) & \dots & m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ m_\nu(n) & m_\nu(n+1) & \dots & m_\nu(2n) \end{vmatrix},$$

$$D'_n = \begin{vmatrix} m_\nu(1) & m_\nu(2) & \dots & m_\nu(n+1) \\ m_\nu(2) & m_\nu(3) & \dots & m_\nu(n+2) \\ \vdots & \vdots & & \vdots \\ m_\nu(n+1) & m_\nu(n+2) & \dots & m_\nu(2n+1) \end{vmatrix}$$

are nonnegative. Let us now consider the sequence $m_\nu^{(t)}(0) = m_\nu(0)$, $m_\nu^{(t)}(n) = t^{n-1}\, m_\nu(n)$, $n \geqslant 1$. We would like to determine whether the moment problem $m_\nu^{(t)}(n)$ also has a Stieltjes solution, that is, whether the determinants

$$(4.1)\qquad \begin{aligned} T_n &= \begin{vmatrix} m_\nu(0) & m_\nu(1) & \dots & t^{n-1}\, m_\nu(n) \\ m_\nu(1) & t\, m_\nu(2) & \dots & t^n\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ t^{n-1}\, m_\nu(n) & t^n\, m_\nu(n+1) & \dots & t^{2n-1}\, m_\nu(2n) \end{vmatrix}, \\ T'_n &= \begin{vmatrix} m_\nu(1) & t\, m_\nu(2) & \dots & t^n\, m_\nu(n+1) \\ t\, m_\nu(2) & t^2\, m_\nu(3) & \dots & t^{n+1}\, m_\nu(n+2) \\ \vdots & \vdots & & \vdots \\ t^n\, m_\nu(n+1) & t^{n+1}\, m_\nu(n+2) & \dots & t^{2n}\, m_\nu(2n+1) \end{vmatrix} \end{aligned}$$

are also nonnegative. Let us start with a few facts on determinants, moments, and orthogonal polynomials. Since

$$
\begin{aligned}
(4.2) \qquad T_n' &= t\cdot t^2\dots t^n \begin{vmatrix} m_\nu(1) & t\,m_\nu(2) & \dots & t^n\,m_\nu(n+1)\\ m_\nu(2) & t\,m_\nu(3) & \dots & t^n\,m_\nu(n+2)\\ \vdots & \vdots & & \vdots\\ m_\nu(n) & t\,m_\nu(n+1) & \dots & t^n\,m_\nu(2n+1)\end{vmatrix}\\
&= t^{[n(n+1)]/2} t\cdot t^2\dots t^n\, D_n' = t^{n(n+1)}\, D_n',
\end{aligned}
$$

we see that we need to care only about the nonnegativity of T_n.

Let $p_n^\nu(x)$ be a sequence of orthonormal polynomials associated with the probability measure ν. Let

$$
\begin{aligned}
h_\nu^n(x,y) &= \sum_{k=0}^n p_k^\nu(x)p_k^\nu(y),\\
h_\nu^\infty(x,x) &= \sum_{n=0}^\infty p_n^\nu(x)p_n^\nu(x) \in (0,\infty]
\end{aligned}
$$

be the Christoffel–Darboux kernel of polynomials $p_n^\nu(x)$. The following proposition links the determinants D_n with the kernel $h_\nu^n(x,y)$:

PROPOSITION 4.1. *Let us denote by D_n^* the following determinant*:

$$
D_n^* = \begin{vmatrix} m_\nu(2) & m_\nu(3) & \dots & m_\nu(n+1)\\ m_\nu(3) & m_\nu(4) & \dots & m_\nu(n+2)\\ \vdots & \vdots & & \vdots\\ m_\nu(n+1) & m_\nu(n+1) & \dots & m_\nu(2n)\end{vmatrix}.
$$

Then we have

$$
D_n^* = h_\nu^n(0,0)\, D_n.
$$

For the proof we refer to [1], Chapter 2, Exercise 3.

Now we have tools to prove the following

THEOREM 4.1. *Let ν be a probability measure with all moments $m_\nu(n)$ and let $p_n^\nu(x)$ be the associated orthonormal polynomials. Then all the determinants T_n of* (4.1) *are all positive if and only if*

$$
t \geqslant 1 - \frac{1}{h_\nu^\infty(0,0)} = 1 - \frac{1}{\sum_{n=0}^\infty p_n^\nu(x)p_n^\nu(x)}.
$$

P r o o f. We have

$$T_n = \begin{vmatrix} t^{-1}+1-t^{-1} & m_\nu(1) & \dots & t^{n-1}\, m_\nu(n) \\ m_\nu(1) & t\, m_\nu(2) & \dots & t^n\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ t^{n-1}\, m_\nu(n) & t^n\, m_\nu(n+1) & \dots & t^{2n-1}\, m_\nu(2n) \end{vmatrix} = R_n + S_n,$$

where

$$R_n = \begin{vmatrix} t^{-1} & m_\nu(1) & \dots & t^{n-1}\, m_\nu(n) \\ m_\nu(1) & t\, m_\nu(2) & \dots & t^n\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ t^{n-1}\, m_\nu(n) & t^n\, m_\nu(n+1) & \dots & t^{2n-1}\, m_\nu(2n) \end{vmatrix},$$

$$S_n = \begin{vmatrix} 1-t^{-1} & m_\nu(1) & \dots & t^{n-1}\, m_\nu(n) \\ 0 & t\, m_\nu(2) & \dots & t^n\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ 0 & t^n\, m_\nu(n+1) & \dots & t^{2n-1}\, m_\nu(2n) \end{vmatrix}.$$

Because by calculations similar to (4.2) we have $R_n = t^{n^2-1}\, D_n$, and

$$\begin{aligned} S_n &= \left(1-\frac{1}{t}\right) \begin{vmatrix} t\, m_\nu(2) & t^2\, m_\nu(3) & \dots & t^n\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ t^n\, m_\nu(n+1) & t^{n+1}\, m_\nu(n+2) & \dots & t^{2n-1}\, m_\nu(2n) \end{vmatrix} \\ &= \left(1-\frac{1}{t}\right) t\cdot t^2 \dots t^n \begin{vmatrix} m_\nu(2) & t\, m_\nu(3) & \dots & t^{n-1}\, m_\nu(n+1) \\ \vdots & \vdots & & \vdots \\ m_\nu(n+1) & t\, m_\nu(n+2) & \dots & t^{n-1}\, m_\nu(2n) \end{vmatrix} \\ &= \left(1-\frac{1}{t}\right) t^{[n(n+1)]/2}\cdot t \dots t^{n-1}\, D_n^* = \left(1-\frac{1}{t}\right) t^{[n(n+1)]/2}\, t^{[n(n-1)]/2}\, D_n^* \\ &= (t-1)\, t^{n^2-1}\, D_n\, h_\nu^n(0,0), \end{aligned}$$

we obtain

$$T_n = t^{n^2-1}\, D_n + (t-1)\, t^{n^2-1}\, D_n\, h_n(0,0) = t^{n^2-1}\, D_n\big(1+(t-1)\, h_\nu^n(0,0)\big)$$

and

$$T_n \geqslant 0 \Leftrightarrow 1+(t-1)\, h_\nu^n(0,0) \geqslant 0,$$

that is, when

$$t \geqslant 1 - \frac{1}{h_\nu^n(0,0)}.$$

Because the sequence $h_\nu^n(0,0)$ is non-decreasing, $1/h_\nu^n(0,0)$ is non-increasing, $1/h_\nu^n(0,0) \to 1/h_\nu^\infty(0,0)$ as $n \to \infty$, and it follows that all T_n are positive if and only if $t \geqslant 1 - 1/h_\nu^\infty(0,0)$. ∎

The above theorem and Theorem 3.2 are linked together by the following propositions:

PROPOSITION 4.2. *For every $x \in \mathbb{R}$, among the solutions of the moment problem given by the sequence $\{m_\nu(n)\}_{n=0}^\infty$ there is at most one N-extremal measure ν with an atom at x, and its weight, dependent only on the moment sequence, is equal to*

$$\nu(\{x\}) = \frac{1}{h_\nu^\infty(x,x)}.$$

Proof. See [14], the note above Theorem 5.21. ∎

PROPOSITION 4.3. *The quantity $1/h_\nu^\infty(0,0)$ is the maximal weight of the atom at zero among all the Stieltjes measures that are solutions of the moment problem given by the sequence $\{m_\nu(n)\}_{n=0}^\infty$.*

Proof. By Proposition 4.2 this is the weight at zero for measures N-extremal in the sense of Hamburger. If the measure ν is determinate in the sense of Hamburger, it is also N-extremal and Stieltjes. If it is not the case, the weight at zero is not larger than $1/h_\nu^\infty(0,0)$ for all the solutions of the moment problem by Theorem 5 of [14], and that weight is reached for the N-extremal Hamburger solution ν_0, and the support of ν_0 is nonnegative. ∎

Applying the above conclusions to the measure ξ_μ of a Bargmann representation β_μ we get the following

THEOREM 4.2. *Assume μ admits a Bargmann representation. Then μ_t admits a Bargmann representation if and only if $t \geqslant 1 - 1/h_{\xi_\mu}^\infty(0,0)$.*

5. CENTRAL MEASURE FOR t-TRANSFORMED CLASSICAL CONVOLUTION

We know (see [7] and [8]) that in the case of t-transformed classical convolution the central measure γ_t has the following Cauchy transform:

$$G_{\gamma_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2t}{z - \cfrac{3t}{z - \ddots}}}}.$$

Consequently,

$$\lambda_{\gamma_t}(1) = 1, \quad \lambda_{\gamma_t}(n) = n\,t,\ n \geqslant 2,$$

and

$$\Lambda_{\gamma_t}(0) = 1, \quad \Lambda_{\gamma_t}(n) = t^{n-1}\,n!,\ n \geqslant 1.$$

The solution of the undeformed even moment problem

$$\int_0^\infty r^{2n} \varrho_\gamma(\mathrm{d}r) = n!$$

is well known (see, e.g., [2]) and the measure ϱ_γ is given by

$$\varrho_\gamma(\mathrm{d}r) = 2\,r\,e^{-r^2}\lambda(\mathrm{d}r).$$

From Proposition 2.2 we infer that the Stieltjes moment problem associated with the moment sequence $\{\Lambda_\gamma(n)\}_{n=0}^\infty$ is determinate. Moreover, since ϱ_γ does not have an atom at zero, ξ_γ also does not. This in turn means that the Bargmann measure β_{γ_t} exists if and only if $t \geqslant 1$.

By Theorem 3.2 we can now see that ϱ_{γ_t} is given by

$$\varrho_{\gamma_t}(\mathrm{d}r) = \left(1 - \frac{1}{t}\right)\delta_0(\mathrm{d}r) + \frac{2}{t^2}\,r\,\exp\left(-\frac{r^2}{t}\lambda(\mathrm{d}r)\right),$$

which can be checked directly for $n \geqslant 1$:

$$\begin{aligned}\int_0^\infty r^{2n}\varrho_{\gamma_t}(\mathrm{d}r) &= \int_0^\infty r^{2n}\frac{2}{t^2}\,r\,\exp\left(-\frac{r^2}{t}\right)\lambda(\mathrm{d}r)\\ &= t^{n-1}\int_0^\infty 2\,s^{2n}\,s\,\exp(-s^2)\lambda(\mathrm{d}s) = t^{n-1}\,n!.\end{aligned}$$

Thus we have

COROLLARY 5.1. *The Bargmann measure $\beta_{\gamma_t}(\mathrm{d}z)$ exists for $t \geqslant 1$ and is of the form*

$$\beta_{\gamma_t}(\mathrm{d}z) = \frac{\lambda(\mathrm{d}\theta)}{2\pi}\left(\left(1 - \frac{1}{t}\right)\delta_0(\mathrm{d}r) + \frac{2}{t^2}r\,\exp\left(-\frac{r^2}{t}\right)\lambda(\mathrm{d}r)\right) \quad \textit{for } z = r\,e^{i\theta}.$$

6. KESTEN MEASURE

From [7] and [8] we know that in the case of t-transformed free convolution the central measure κ_t has the following Cauchy transform:

$$G_{\kappa_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{z - \ddots}}}}.$$

Consequently,

$$\lambda_{\kappa_t}(1) = 1, \quad \lambda_{\kappa_t}(n) = t, \ n \geqslant 2,$$

and

$$\Lambda_{\kappa_t}(0) = 1, \quad \Lambda_{\kappa_t}(n) = t^{n-1}, \ n \geqslant 1.$$

So we look for the measure ξ_{κ_t} that solves the determinate Stieltjes moment problem:

$$\Lambda_{\kappa_t}(n) = \int\limits_0^\infty r^n \xi_{\kappa_t}(\mathrm{d}r).$$

Using the moment function

$$M_{\xi_{\kappa_t}}(z) = \sum_{n=0}^\infty \Lambda_{\kappa_t}(n) z^n = 1 + \frac{z}{1 - t\, z}$$

and the connection

$$G_{\xi_{\kappa_t}}(z) = \frac{1}{z} M_{\xi_{\kappa_t}}\left(\frac{1}{z}\right) = \frac{1}{z} \cdot \frac{z - t + 1}{z - t} = \left(1 - \frac{1}{t}\right) \frac{1}{z} + \frac{1}{t} \frac{1}{z - t}$$

we may see that

$$\xi_{\kappa_t} = \left(1 - \frac{1}{t}\right) \delta_0(\mathrm{d}r) + \frac{1}{t}\, \delta_t(\mathrm{d}r),$$

which is a probability measure if and only if $t \geqslant 1$. For $t < 1$ the corresponding sequence $\{\Lambda_{\xi_{\kappa_t}}(0)\}_{n=0}^\infty$ is not a sequence of moments because it is not positive definite:

$$\begin{vmatrix} \Lambda_{\xi_{\kappa_t}}(0) & \Lambda_{\xi_{\kappa_t}}(1) \\ \Lambda_{\xi_{\kappa_t}}(1) & \Lambda_{\xi_{\kappa_t}}(2) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & t \end{vmatrix} = t - 1 < 0.$$

COROLLARY 6.1. *In the t-transformed free case, that is, for the Kesten measure κ_t, the Bargmann representation exists if and only if $t \geqslant 1$. In this case the* *measure is of the form*

$$\beta_{\kappa_t}(\mathrm{d}z) = \frac{\lambda(\mathrm{d}\theta)}{2\pi}\left(\left(1-\frac{1}{t}\right)\delta_0(\mathrm{d}r) + \frac{1}{t}\,\delta_{\sqrt{t}}(\mathrm{d}r)\right) \quad \textit{for } z = r\,e^{i\theta}.$$

REFERENCES

[1] N. I Akhiezer, *The Classical Moment Problem and some Related Questions in Analysis*, Oliver and Boyd, Edinburgh and London 1965.

[2] N. Asai, I. Kubo, and H. H Kuo, *Segal–Bargmann transforms of one-mode interacting Fock spaces associated with Gaussian and Poisson measures*, Proc. Amer. Math. Soc. 131 (2) (2002), pp. 815–823.

[3] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform I*, Comm. Pure Appl. Math. 14 (1961), pp. 187–214.

[4] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform II*, Comm. Pure Appl. Math. 20 (1967), pp. 1–101.

[5] C. Berg, *Recent results about moment problems*, in: *Probability Measures on Groups and Related Structures, XI Proceedings Oberwolfach 1994*, H. Heyer (Ed.), World Scientific, Singapore 1995, pp. 1–13.

[6] C. Berg and M. Thill, *Rotation invariant moment problem*, Acta Math. 167 (3-4) (1991), pp. 207–227.

[7] M. Bożejko and J. Wysoczański, *New examples of convolutions and non-commutative central limit theorems*, Banach Center Publ. 43 (1998), pp. 95–103.

[8] M. Bożejko and J. Wysoczański, *Remarks on t-transformations of measures and convolutions*, Ann. Inst. H. Poincaré Probab. Statist. 37 (6) (2001), pp. 737–761.

[9] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Math. Appl., Vol. 13, Gordon and Breach, New York 1978.

[10] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York 1966.

[11] I. Królak, *Measures connected with Bargmann's representation of the q-commutation relation for* $q > 1$, Banach Center Publ. 43 (1998), pp. 253–257.

[12] K. A. Penson and A. I. Solomon, *New generalized coherent states*, J. Math. Phys. 40 (5) (1999), pp. 2354–2363.

[13] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Math. Surveys No.1, American Mathematical Society, Providence 1943.

[14] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. 137 (1998), pp. 82–203.

[15] J. Stochel and F. H. Szafraniec, *The complex moment problem and subnormality: A polar decomposition approach*, J. Funct. Anal. 159 (1998), pp. 432–491.

[16] F. H. Szafraniec, *Operators of the q-oscillator*, Banach Center Publ. 78 (2007), pp. 293–307.

[17] Ł. J. Wojakowski, *Probability Interpolating between Free and Boolean*, Dissertationes Math. 446 (2007).

Anna Dorota Krystek
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: Anna.Krystek@math.uni.wroc.pl

Łukasz Jan Wojakowski
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: Lukasz.Wojakowski@math.uni.wroc.pl

Received on 6.1.2014;
revised version on 13.2.2014