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FROBENIUS–PERRON OPERATOR DESCRIPTION OF MARKOV CHAINS

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Abstract. We consider canonical shift space representation of discrete-time Markov chain given by transition kernels. Markov shifts and eigenfunctions of skew products above them are characterized by terms of Frobenius–Perron operator. The results are applied to the exactness property of Markov chains. We introduce also the notion of quasi-Markov chain and apply it to Gauss endomorphisms.

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1. INTRODUCTION

We start with consideration of Markov chains from ergodic theory point of view. A probabilistic measure space (X, μ) and transition probability $P(x, \cdot)$ determine the dynamical system on product space. Here the measure is given by μ and $P(x, \cdot)$, the transformation is a one-sided shift. If μ is stationary, then the shift is measure preserving. For details see Section 2. To study our dynamical system (called, as before, a Markov chain), we use the Frobenius–Perron operator instead of the Markov one. In Section 2 we construct the Frobenius–Perron (F–P) operator for Markov chain and characterize the Markov chain by using the F–P operator. Section 3 is assigned for skew products with Markov chain in the base. As for discrete Markov chains (see [4]) we get characterization of Markov chains by densities of absolutely continuous invariant measures (a.c.i.m.). This is also a generalization of results in [7] where a mutually independent process is considered. In Section 4 we characterize the exactness property of Markov chains. Theorem 4.3 gives sufficient conditions for exactness of some position-dependent random maps. At the end of the section we describe exactness of Gauss endomorphisms in language of F–P operator. In Section 5 we introduce the notion of quasi-Markov chain. Among others we give the example of exact Gauss endomorphism which is not a quasi-Markov chain.

2. PRELIMINARIES

This section consists of basic notions and definitions connected with Markov chain and the dynamical system determined by this process. In particular, we introduce the Frobenius–Perron operator for Markov chain.

DEFINITION 2.1. Let us assume that $(X, \mathcal{B}, \mu)$ is a probability space. A function $P : X \times \mathcal{B} \to [0, 1]$ is called a *stochastic transition function* if it has the following properties:

(i) for any $A \in \mathcal{B}$, $P(\cdot, A) : X \to [0, 1]$ is a $\mathcal{B}$-measurable function,

(ii) for any $x \in X$, $P(x, \cdot) : \mathcal{B} \to [0, 1]$ is a probability measure.

DEFINITION 2.2. We say that the measure μ is *nonsingular* if

$$\mu(A) = 0 \Rightarrow \int\limits_X P(x, A) d\mu(x) = 0.$$

The measure μ is *invariant* if

$$\mu(A) = \int\limits_X P(x, A) d\mu(x)$$

for every $A \in \mathcal{B}$.

Let $\Omega = X^N$, and let $\mathcal{D} = \mathcal{B}^N$ be a product σ-field. Here $N = \{0, 1, \ldots\}$. For any sets $A_0, \ldots, A_n \in \mathcal{B}$, we will put

$$[A_0 \ldots A_n] = \{\omega \in \Omega : \omega_0 \in A_0, \ldots, \omega_n \in A_n\}.$$

We define the measure P on $\mathcal{D}$ by

$$P([A_0 \ldots A_n]) = \int\limits_{A_0} d\mu(\omega_0) \int\limits_{A_1} P(\omega_0, d\omega_1) \ldots \int\limits_{A_n} P(\omega_{n-1}, d\omega_n) \tag{2.1}$$

for every $n \geqslant 1$ and $A_0, \ldots, A_n \in \mathcal{B}$. Let σ be a one-sided shift on Ω, i.e.,

$$\sigma(\omega)_n = \omega_{n+1} \quad \text{for } n \in N.$$

Finally, we have obtained the measure theoretical dynamical system $(\Omega, \mathcal{D}, P, \sigma)$ which is called a *Markov chain*. More details can be found in Chapter I of [8]. For $f \in L_1(P)$ and

$$\omega_1^\infty = (\omega_1, \omega_2, \ldots) \in X^{N-\{0\}}$$

we define the signed measure $\nu_{(f,\omega_1^\infty)}$ on $\mathcal{B}$ such that

$$\nu_{(f,\omega_1^\infty)}(A) = \int\limits_X d\mu(x) \int\limits_A f(x\omega) P(x, d\omega_0),$$

where $x\omega = (x, \omega_0, \omega_1, \ldots)$. Consequently,

$$\nu_{(f,\omega_1^\infty)} = \int\limits_X d\mu(x) \int\limits_{(\cdot)} f(x\omega) P(x, d\omega_0).$$

We will assume throughout the rest of the paper that μ is nonsingular. Therefore, $\nu_{(f,\omega_1^\infty)} \ll \mu$ for every $f \in L_1(P)$ and ω_1^∞. Now, we are in the position to define the F–P operator $\mathcal{L}$ for (σ, P), i.e., such that

$$\int g\mathcal{L}f dP = \int fg(\sigma) dP$$

for any $g \in L^\infty(P)$ and for any $f \in L_1(P)$. Namely,

$$\mathcal{L}f(\omega) = \frac{d\nu_{(f,\omega_1^\infty)}}{d\mu}(\omega_0) = \frac{d}{d\mu} \int\limits_X d\mu(x) \int\limits_{(\cdot)} f(x\omega) P(x, d\omega_0). \tag{2.2}$$

REMARK 2.1. *If $P(x, \cdot) \ll \mu$ for a.e. x and $dP(x, \cdot)/d\mu = \rho(x, \cdot)$, then*

$$\mathcal{L}f(\omega) = \int\limits_X f(x\omega)\rho(x, \omega_0) d\mu(x).$$

LEMMA 2.1. *The F–P operator $\mathcal{L}$ given by* (2.2) *is the F–P operator for* (σ, P).

P r o o f. It is enough to show that

$$\int\limits_\Omega \mathcal{L}1_{[A_0 \ldots A_n]}(\omega) 1_{[B_0 \ldots B_n]}(\omega) dP = \int\limits_\Omega 1_{[A_0 \ldots A_n]}(\omega) 1_{[B_0 \ldots B_n]}\big(\sigma(\omega)\big) dP$$

for cylinder sets $[A_0 \ldots A_n], [B_0 \ldots B_n]$ and $n \geqslant 0$. Let us observe that

$$\begin{aligned} \mathcal{L}1_{[A_0 \ldots A_n]}(\omega) &= \frac{d}{d\mu} \int\limits_X d\mu(x) \int\limits_{(\cdot)} 1_{[A_0 \ldots A_n]}(x\omega) P(x, d\omega_0) \\ &= \frac{d}{d\mu} \int\limits_X d\mu(x) \int\limits_{(\cdot)} 1_{[A_0]}(x) 1_{[A_1 \ldots A_n]}(\omega) P(x, d\omega_0) \\ &= 1_{[A_1 \ldots A_n]}(\omega) \frac{d}{d\mu} \int\limits_X d\mu(x) 1_{[A_0]}(x) P(x, d\omega_0) \\ &= 1_{[A_1 \ldots A_n]}(\omega) \mathcal{L}1_{[A_0]}(\omega_0). \end{aligned} \tag{2.3}$$

Therefore,

$$\begin{aligned}
&\int_\Omega \mathcal{L}1_{[A_0\ldots A_n]}(\omega)1_{[B_0\ldots B_n]}(\omega)dP\\
&= \int_{B_0\cap A_1} \mathcal{L}1_{[A_0]}(\omega_0)d\mu(\omega_0) \int_{B_1\cap A_2} P(\omega_0,d\omega_1)\ldots \int_{B_{n-1}\cap A_n} P(\omega_{n-2},d\omega_{n-1})\\
&\quad\times \int_{B_n} P(\omega_{n-1},d\omega_n)\\
&= \int_{A_0} P(x,B_0\cap A_1)d\mu(x) \int_{B_1\cap A_2} P(\omega_0,d\omega_1)\ldots \int_{B_{n-1}\cap A_n} P(\omega_{n-2},d\omega_{n-1})\\
&\quad\times \int_{B_n} P(\omega_{n-1},d\omega_n)\\
&= \int_{A_0} d\mu(x) \int_{B_0\cap A_1} P(x,d\omega_0)\ldots \int_{B_{n-1}\cap A_n} P(\omega_{n-2},d\omega_{n-1}) \int_{B_n} P(\omega_{n-1},d\omega_n)\\
&= \int_\Omega 1_{[A_0\ldots A_n]}(\omega)1_{[B_0\ldots B_n]}\big(\sigma(\omega)\big)dP. \quad \blacksquare
\end{aligned}$$

The existence of $\mathcal{L}$ implies nonsingularity of (σ, P), i.e., $P(A) = 0 \Rightarrow P(\sigma^{-1}A) = 0$. If (σ, P) is nonsingular, then it is easy to see that μ is nonsingular. Therefore, the following holds:

PROPOSITION 2.1. *(σ, P) is nonsingular if and only if μ is nonsingular.*

As the F–P operator, $\mathcal{L}$ has the following properties for any $f \in L_1(P)$:
(a) $f \geqslant 0$ implies $\mathcal{L}f \geqslant 0$,
(b) $\int \mathcal{L}f dP = \int f dP$,
(c) $\int |\mathcal{L}f| dP \leqslant \int |f| dP$,
(d) $\mathcal{L}f = f$ for $f \geqslant 0$ if and only if f is the density of absolutely continuous invariant measure (a.c.i.m.).

Here and throughout all the paper we will assume that relations between sets and functions hold modulo a set of measure zero. A Markov chain can be characterized by the F–P operator as follows. Let $(\Omega, \mathcal{D}, \sigma)$ be a measurable space defined as previously. Let ν be a probability measure such that $\nu|\mathcal{B} = \mu$ and (σ, ν) is nonsingular. Denote by $\mathcal{L}_\nu$ the F–P operator with respect to (σ, ν).

DEFINITION 2.3. We say that (σ, ν) is a *Markov chain* if ν is determined by (2.1) for some stochastic transition function.

THEOREM 2.1. *Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space. (σ, ν) is a Markov chain if and only if $\mathcal{L}_\nu : L_1(\mu) \to L_1(\mu)$.*

Proof. We need only to show that if $\mathcal{L}_\nu : L_1(\mu) \to L_1(\mu)$, then (σ, ν) is a Markov chain. By reasoning similar to that in the proof of Proposition 1.8 in [8], p. 120, we can define $P(x, \cdot)$ in such a way that

$$P(x, A) = \nu([XA]|\mathcal{B})(x)$$

for every $A \in \mathcal{B}$ and every x from a set of μ-measure one. Using mathematical induction, we show that ν is determined by μ and $P(x, \cdot)$. By assumption, $\nu([A]) = \mu(A)$. Let us note that

$$\begin{aligned}\nu([A_0A_1]) &= \int_{[A_0]} 1_{[A_1]}\big(\sigma(\omega)\big)d\nu = \int_{[A_0]} \nu([XA_1]|\mathcal{B})(\omega_0)d\mu(\omega_0) \\ &= \int_{A_0} P(\omega_0, A_1)d\mu(\omega_0).\end{aligned}$$

On the other hand,

$$\int_{[A_0]} 1_{[A_1]}\big(\sigma(\omega)\big)d\nu = \int \mathcal{L}_\nu 1_{[A_0]}(\omega)1_{[A_1]}(\omega)d\nu = \int_{A_1} \mathcal{L}_\nu 1_{[A_0]}(\omega_0)d\mu(\omega_0).$$

Therefore,

$$\int_{A_0} d\mu(\omega_0) \int \varphi(\omega_1)P(\omega_0, d\omega_1) = \int \varphi(\omega_0)\mathcal{L}_\nu 1_{[A_0]}(\omega_0)d\mu(\omega_0)$$

for any $\varphi \in L_1(\mu)$ and $A_0 \in \mathcal{B}$. Let us assume that $\nu([A_0 \dots A_{n-1}])$ satisfies (2.1) for any $A_0, \dots, A_{n-1} \in \mathcal{B}$. Then

$$\begin{aligned}\nu([A_0 \dots A_n]) &= \int_{[A_0]} 1_{[A_1 \dots A_n]}\big(\sigma(\omega)\big)d\nu \\ &= \int \mathcal{L}_\nu 1_{[A_0]}(\omega)1_{[A_1 \dots A_n]}(\omega)d\nu = \int_{[A_1 \dots A_n]} \mathcal{L}_\nu 1_{[A_0]}(\omega)d\nu \\ &= \int_{A_1} \mathcal{L}_\nu 1_{[A_0]}(\omega_0)d\mu(\omega_0) \int_{A_2} P(\omega_0, d\omega_1) \dots \int_{A_n} P(\omega_{n-2}, d\omega_{n-1}) \\ &= \int_{A_0} d\mu(\omega_0) \int_{A_1} P(\omega_0, d\omega_1) \int_{A_2} P(\omega_1, d\omega_2) \dots \int_{A_n} P(\omega_{n-1}, d\omega_n). \quad \blacksquare\end{aligned}$$

Let us consider Gauss endomorphisms. By Gauss endomorphism we understand the one-sided version of Gauss automorphism. For the definition and basic properties see Chapter 8 in [3]. For reader's convenience we note that $X = R$ and that the joint distribution of any family of random variables $X_{s_1}, \dots, X_{s_r}$ is an r-dimensional Gaussian distribution. Here $X_n(\omega) = \omega_n$, $n \in N$, and $(\Omega, \mathcal{D}, \nu, \sigma)$ denotes the Gauss dynamical system. The measure ν is σ-invariant. It is well known (see Proposition 2.1 in [7]) that

$$\mathcal{L}_\nu(f)(\sigma) = E(f|\sigma^{-1}\mathcal{D}).$$

Hence for the Gauss endomorphism we have

$$\mathcal{L}_\nu(X_0)(\sigma) = P_{H_1}(X_0),$$

where the operator P_{H_1} projects perpendicularly onto H_1, which is the real complete subspace of $L_2(\nu)$ generated by finite combinations

$$\sum_{k\in I} a_k X_k \quad \text{for } I \subset \{1,2,\ldots\}.$$

Therefore, we can use the methods of prediction theory.

PROPOSITION 2.2. *The Gauss endomorphism* (σ,ν) *is a Markov chain if and only if*

$$\mathcal{L}_\nu X_0 = aX_0$$

for some a *such that* $|a| < 1$.

The proposition follows from Example 2 in [11], p. 40, where it is shown that

$$P_{H_1}(X_0) = aX_1 = aX_0(\sigma)$$

for Markov chain Gauss endomorphism. The converse implication can be obtained by covariance argumentation for real Gaussian process.

3. SKEW PRODUCTS WITH MARKOV CHAIN IN THE BASE

We present description of Markov chain $(\Omega,\mathcal{D},P,\sigma)$ by skew products. Let $(Y,\mathcal{A},m)$ be a probability measure space, and $\{T_x\}_{x\in X}$ a measurable family of nonsingular transformations of Y into Y. Measurability means that the map $X\times Y \ni (x,y) \to T_x(y) \in Y$ is measurable, and nonsingularity means that $m(A)=0$ implies $m\big(T_x^{-1}(A)\big)=0$ for every $x\in X$ and $A\in\mathcal{A}$. Now, we define a skew product

$$T(\omega,y) = \big(\sigma(\omega), T_{\omega(0)}(y)\big), \tag{3.1}$$

where T is a nonsingular transformation of $(\Omega\times Y)$ into itself with respect to $P\times m$. The F–P operator is described as follows:

$$\mathcal{L}_T f(\omega,y) = \mathcal{L}(\mathcal{L}_{\omega(0)} f)(\omega,y),$$

where $\mathcal{L}_x$ denotes the F–P operator for T_x, $x\in X$, and $\mathcal{L}$ is the F–P operator for the Markov chain (σ,P).

THEOREM 3.1. *If* $\mathcal{L}_T f = \lambda f$, *where* $|\lambda|=1$ *and* $f\in L_1(P\times m)$, *then* $f = h(\omega_0,y)$, *where* $h\in L_1(\mu\times m)$.

The proof will be preceded by an auxiliary lemma.

LEMMA 3.1. *If* $f(\omega) = f(\omega_0,\ldots,\omega_n)$ *and* $f\in L_1(P)$, *then*

$$\mathcal{L}^{n+1} f(\omega) = h(\omega_0),$$

where $h\in L_1(\mu)$.

P r o o f. By (2.3) we see that

$$\mathcal{L}^{n+1}1_{[A_0\dots A_n]}(\omega) = \varphi_{n+1}(\omega_0)$$

for any cylinder set $[A_0 \dots A_n]$. By property (c) of $\mathcal{L}$ and by using the approximation argumentation we obtain the desired assertion. ■

P r o o f o f T h e o r e m 3.1. Let $\epsilon > 0$. There exist $n, l \in N$ such that

$$\Big\| f - \sum_{k=1}^{l} f_k g_k \Big\| < \epsilon,$$

where $\|\cdot\|$ means the L_1-norm, $f_k(\omega) = f_k(\omega_0, \dots, \omega_n) \in L_1(P)$ and $g_k \in L_1(m)$. Since $\mathcal{L}_T^n = \mathcal{L}_{T^n}$, we have

$$\mathcal{L}_T^{n+1}(f_k g_k)(\omega, y) = \mathcal{L}^{n+1}\big(f_k(\omega)\mathcal{L}_{\varpi_{n+1}} g_k(y)\big),$$

where

$$\mathcal{L}_{\varpi_{n+1}} g(y) = \mathcal{L}_{\omega_0} \circ \dots \circ \mathcal{L}_{\omega_n} g(y).$$

Therefore, by Lemma 3.1,

$$\mathcal{L}_T^{n+1}(f_k g_k) = \varphi_k(\omega_0, y).$$

Finally,

$$\begin{aligned} \epsilon > \Big\| f - \sum_{k=1}^{l} f_k g_k \Big\| &\geqslant \Big\| \mathcal{L}_T^{n+1} f - \sum_{k=1}^{l} \mathcal{L}_T^{n+1}(f_k g_k) \Big\| \\ = \Big\| \lambda^{n+1} f - \sum_{k=1}^{l} \varphi_k(\omega_0, y) \Big\| &= \Big\| f - \sum_{k=1}^{l} \bar{\lambda}^{n+1} \varphi_k(\omega_0, y) \Big\|, \end{aligned}$$

which completes the proof. ■

COROLLARY 3.1. *If ν is a T-a.c.i.m., then*

$$\frac{d\nu}{dP \times m} = h(\omega_0, y),$$

where $h \in L_1(\mu \times m)$.

COROLLARY 3.2. *If $\mathcal{L}f = \lambda f$, where $|\lambda| = 1$ and $f \in L_1(P)$, then $f(\omega) = f(\omega_0)$.*

By Corollary 3.2 we see that a σ-a.c.i.m. is given by some $\nu \ll \mu$ and the same transition probability as P. Let us return to the skew product T. Let $h(\omega_0, y)$ be a density of a.c.i.m., i.e., $\mathcal{L}_T h = h$. Therefore,

$$h(\omega_0, y) = \frac{d}{d\mu} \int \mathcal{L}_x\big(h(x, y)\big) P(x, d\omega_0) d\mu(x).$$

The above equality implies

COROLLARY 3.3. *If* $P(x,\cdot)=\mu$ *for a.e.* x, *then* $h(\omega_0,y)=h(y)$.

COROLLARY 3.4. *If* $P(\cdot,A)$ *is* $\mathcal{B}_0$*-measurable for every* $A\in\mathcal{B}$, *where* $\mathcal{B}_0\subset\mathcal{B}$, *then* $h(\omega_0,y)$ *is* $(\mathcal{B}_0\times\mathcal{A})$*-measurable.*

It is interesting if the converse of Theorem 3.1 holds. Let $(\Omega,\mathcal{D},\sigma)$ be the measurable shift space defined in Section 2. Let ν be a probability invariant measure such that $\nu|\mathcal{B}=\mu$, and let $\mathcal{L}_\nu$ be the F–P operator with respect to (σ,ν).

THEOREM 3.2. *Let* $(X,\mathcal{B},\mu)$ *be a Lebesgue space.* (σ,ν) *is a Markov chain if and only if for any skew product* T *given by formula* (3.1) *densities of* T*-a.c.i.m. are* $(\mathcal{B}\times\mathcal{A})$*-measurable.*

The proof will be preceded by an auxiliary lemma. We start with construction of special skew product. Namely, let $\alpha=\{A_i\}_{i=1}^s$ be a measurable partition of Ω, $Z_s=\{1,\dots,s\}$, and m be the measure on Z_s such that $m(j)=\frac{1}{s}$ for $j\in Z_s$. Let $\{T_i\}_{i=1}^s$ be the family of maps such that $T_i(j)=i$ for $j\in Z_s$. Define the skew product

$$T(\omega,j)=\big(\sigma(\omega),T_{\alpha(\omega)}(j)\big),$$

where $\alpha(\omega)=i$ if and only if $\omega\in A_i$.

LEMMA 3.2. T *has a.c.i.m. with the density*

$$g(\omega,i)=s\mathcal{L}_\nu(1_{A_i})(\omega).$$

Proof. We need only to check that $\mathcal{L}_T g(\omega,i)=g(\omega,i)$. Therefore, we firstly determine $\mathcal{L}_T$. Let $G\in L_1(\nu\times m)$ and $H\in L^\infty(\nu\times m)$. Then

$$\begin{aligned}\int\limits_{\Omega\times Z_s} G(\omega,j)H\big(T(\omega,j)\big)d\nu\times m &= \int\limits_{Z_s} dm\sum_{i=1}^s\int\limits_\Omega 1_{A_i}(\omega)G(\omega,j)H\big(\sigma(\omega),i\big)d\nu\\ &=\int\limits_{Z_s} dm\sum_{i=1}^s\int\limits_\Omega \mathcal{L}_\nu\big(1_{A_i}(\omega)G(\omega,j)\big)H(\omega,i)d\nu\\ &=\int\limits_\Omega\sum_{i=1}^s H(\omega,i)\bigg(\frac{1}{s}\sum_{k=1}^s\mathcal{L}_\nu\big(1_{A_i}(\omega)G(\omega,k)\big)\bigg)d\nu\\ &=\int\limits_{\Omega\times Z_s}\Big(\sum_{k=1}^s\mathcal{L}_\nu\big(1_{A_i}(\omega)G(\omega,k)\big)\Big)H(\omega,i)d\nu\times m.\end{aligned}$$

Hence

$$\mathcal{L}_T G(\omega,i)=\sum_{k=1}^s\mathcal{L}_\nu\big(1_{A_i}(\omega)G(\omega,k)\big).$$

Finally,

$$\mathcal{L}_T g(\omega, i) = s \sum_{k=1}^{s} \mathcal{L}_\nu\big(1_{A_i}(\omega)\big)\mathcal{L}_\nu\big(1_{A_k}(\omega)\big)$$
$$= s\mathcal{L}_\nu\big(1_{A_i}(\omega)\big)\mathcal{L}_\nu\big(\sum_{k=1}^{s} 1_{A_k}(\omega)\big) = s\mathcal{L}_\nu\big(1_{A_i}(\omega)\big) = g(\omega, i).$$

Here, we use the equality $\mathcal{L}_\nu 1 = 1$. ■

P r o o f o f T h e o r e m 3.2. If (σ, ν) satisfies the assumptions of Theorem 3.2, then $\mathcal{L}_\nu : L_1(\mu) \to L_1(\mu)$ by Lemma 3.2. Therefore, (σ, ν) is a Markov chain by Theorem 2.1. ■

4. EXACTNESS OF MARKOV CHAINS

Let $\mathcal{L}$ be the F–P operator for the Markov chain (σ, P). Let us denote by $\hat{\mathcal{L}}$ the restriction of $\mathcal{L}$ to $L_1(\mu)$,

$$\hat{\mathcal{L}} f(x) = \frac{d}{d\mu} \int f(y) P(y, \cdot) d\mu(y).$$

LEMMA 4.1. *$\hat{\mathcal{L}}$ is dual to the Markov operator*

$$\mathcal{P} h(x) = \int h(y) P(x, dy).$$

P r o o f. Let $h \in L^\infty(\mu)$. Then

$$\int \hat{\mathcal{L}}(f) h d\mu = \int h d\nu_f = \int d\mu(y) \int h(x) f(y) P(y, dx)$$
$$= \int f(y) d\mu(y) \int h(x) P(y, dx) = \int f \mathcal{P}(h) d\mu. \quad ■$$

DEFINITION 4.1. (σ, P) is *exact* if

$$\bigcap_{n=0}^{\infty} \sigma^{-n}\mathcal{D} = \{\emptyset, \Omega\},$$

i.e., the zero-one law holds.

THEOREM 4.1. *Let μ be invariant. For the system (σ, P) the following holds*:

(i) (σ, P) *is ergodic* $\Leftrightarrow$ $\hat{\mathcal{L}}$ *is ergodic* $\Leftrightarrow$ $\mathcal{P}$ *is ergodic.*

(ii) (σ, P) *is weakly mixing* $\Leftrightarrow$ $\hat{\mathcal{L}}$ *is weakly mixing* $\Leftrightarrow$ $\mathcal{P}$ *is weakly mixing.*

(iii) (σ, P) *is mixing* $\Leftrightarrow$ $\hat{\mathcal{L}}$ *is mixing* $\Leftrightarrow$ $\mathcal{P}$ *is mixing.*

(iv) (σ, P) *is exact* $\Leftrightarrow$ $\lim_{n\to\infty} \hat{\mathcal{L}}^n f = \int f d\mu$ *in* L_1 *for every* $f \in L_1(\mu)$.

P r o o f. It is well known that ergodicity, weak mixing, mixing and exactness property of (σ, P) is equivalent to suitable convergence of $\mathcal{L}^n$. For example, let us consider (iii). We have

$$(\sigma, P) \text{ is mixing } \Leftrightarrow \lim_{n\to\infty} \int (\mathcal{L}^n f) g dP = \int f dP \int g dP$$

for $f \in L_1(P)$ and $g \in L^\infty(P)$. As $\mathcal{L}$ is contractive, it is enough to consider functions $f(\omega) = f(\omega_0, \ldots, \omega_k)$, $k \in N$. Thus, by Lemma 3.1,

$$\begin{aligned}\lim_{n\to\infty} \int (\mathcal{L}^n f) g dP &= \lim_{n\to\infty} \int \big(\hat{\mathcal{L}}^n(\mathcal{L}^{k+1} f)\big) g dP \\ &= \lim_{n\to\infty} \int \big(\hat{\mathcal{L}}^n(\mathcal{L}^{k+1} f)\big) E(g|\mathcal{B}) d\mu = \lim_{n\to\infty} \int (\mathcal{L}^{k+1} f) \mathcal{P}^n \big(E(g|\mathcal{B})\big) d\mu,\end{aligned}$$

where $E(g|\mathcal{B})$ denotes the conditional expectation. ∎

Let us fix our attention on the exactness property. Let (σ, P) satisfy the assumptions of Theorem 4.1. The assertion (iv) allows us to apply the theory of asymptotic stability of operators considered in [6]. Let us assume that μ is nonsingular instead of being invariant.

DEFINITION 4.2. $\hat{\mathcal{L}}$ is said to be *asymptotically stable* if there exists a unique density f^*, i.e., $f^* \geqslant 0$, $\int f^* d\mu = 1$, such that

$$\lim_{n\to\infty} \big\| \hat{\mathcal{L}}^n(f) - \big(\int f d\mu\big) f^* \big\| = 0$$

for every $f \in L_1(\mu)$.

The following theorem can be easily seen.

THEOREM 4.2. *If $\hat{\mathcal{L}}$ is asymptotically stable and f^* is the unique $\hat{\mathcal{L}}$-invariant density, then the Markov chain (σ, P), where P is determined by*

$$\mu^*(A) = \int_A f^*(x) d\mu(x) \quad \text{for } A \in \mathcal{B},$$

is exact.

P r o o f. Let $\hat{\mathcal{L}}_{\mu^*}$ be the restriction of the F–P operator for (σ, P) to $L_1(\mu^*)$. Then

$$\hat{\mathcal{L}}^n_{\mu^*}(f) = \frac{1}{f^*} \hat{\mathcal{L}}^n(f f^*) \quad \text{for } n = 1, 2, \ldots$$

and $f \in L_1(\mu^*)$. Hence, by Theorem 4.1, we obtain exactness of (σ, P). ∎

For application let us consider position-dependent random maps. Let $X = [0, 1]$, Λ be the Lebesgue measure, and $\mathcal{B}$ the σ-algebra of Borel sets. Let us consider nonsingular transformations $\tau_i : X \to X, i = 0, 1$, and a probabilistic vector

$\big(p(x), q(x)\big)$. Here $p(x)+q(x)=1$ and $p(x)\geqslant 0, q(x)\geqslant 0$ for $x\in X$. The transition probability is defined as follows:

$$P(x,A)=p(x)1_A\big(\tau_0(x)\big)+q(x)1_A\big(\tau_1(x)\big).$$

Λ and P determine Markov chain $(\Omega,\mathcal{D},P_\Lambda,\sigma)$. The F–P operator for (σ,P_Λ) is given by

$$\mathcal{L}f(\omega)=\mathcal{L}_0\big(p(\cdot)f(\cdot\omega)\big)(\omega_0)+\mathcal{L}_1\big(q(\cdot)f(\cdot\omega)\big)(\omega_0),$$

where $\mathcal{L}_i$ is the F–P operator for $\tau_i, i=0,1$. Therefore,

$$\hat{\mathcal{L}}f(x)=\mathcal{L}_0\big(p(x)f(x)\big)+\mathcal{L}_1\big(q(x)f(x)\big)\quad \text{for } x\in X.$$

The operator as above was considered in [2] and used for the model of asset prices. Now, for the reader's convenience let us assume that $\tau_j, j=0,1$, are piecewise monotonic mappings and satisfy the following conditions:

(1) There is a partition $0=a_0^j<a_1^j<\ldots<a_{r_j}^j=1$ of $[0,1]$ such that the restriction of τ_j to the interval (a_{i-1}^j,a_i^j), $i=1,\ldots,r_j$, is a C^2-function.

(2) $\tau_j\big((a_{i-1}^j,a_i^j)\big)=(0,1)$.

(3) There is a $\lambda_j>1$ such that $|\tau_j'(x)|\geqslant\lambda_j$ for $x\neq a_i^j, i=0,\ldots,r_j$.

(4) There exists a finite constant c_j such that

$$\frac{|\tau_j''(x)|}{[\tau_j'(x)]^2}\leqslant c_j$$

for $x\neq a_i^j, i=0,\ldots,r_j$.

THEOREM 4.3. *Let us assume that* $\tau_j, j=0,1$, *satisfy the conditions* (1)–(4). *If* $p\in C^1$ *and there exists a constant* A *such that* $|p'(x)|\leqslant Ap(x)$ *and* $|q'(x)|\leqslant Aq(x)$ *for every* $x\in(0,1)$, *then* $\hat{\mathcal{L}}$ *is asymptotically stable.*

P r o o f. We apply the reasoning from the proof of Theorem 6.2.2 in [6]. Let D_0 be the set of all bounded continuously differentiable densities such that

$$|f'(x)|\leqslant k_f f(x)\quad \text{for } 0<x<1,$$

where the constant k_f depends on f. For $f\in D_0$ and for $x\in(0,1)$ we have

$$\big|\big(p(x)f(x)\big)'\big|\leqslant(A+k_f)p(x)f(x)\quad\text{and}\quad\big|\big(q(x)f(x)\big)'\big|\leqslant(A+k_f)q(x)f(x).$$

Therefore, by the proof of Theorem 6.2.2 in [6] we have

$$\big|\big(\mathcal{L}_j(pf)\big)'\big|\leqslant\left[c_j+\frac{A+k_f}{\lambda}\right]\mathcal{L}_j(pf)\quad \text{for } j=0,1.$$

Hence

$$\left|\left(\hat{\mathcal{L}}(f)\right)'\right| \leqslant \left[c + \frac{A + k_f}{\lambda}\right]\hat{\mathcal{L}}(f),$$

where $c = \max\{c_j : j = 0, 1\}$ and $\lambda = \min\{\lambda_j : j = 0, 1\}$. By using an induction argument we have

$$\left|\left(\hat{\mathcal{L}}^n(f)\right)'\right| \leqslant \left[\frac{c\lambda + A}{\lambda - 1} + \frac{k_f}{\lambda^n}\right]\hat{\mathcal{L}}^n(f),$$

and thus Proposition 5.8.2 in [6] completes our proof. ■

The problem of exactness for Gauss endomorphism $(\Omega, \mathcal{D}, \nu, \sigma)$ has been completely solved in the language of the spectral measure of ν. Let us put

$$b_n = \int X_0 X_n d\nu \quad \text{for } n = 0, 1, 2, \ldots$$

We assume that $\int X_0 d\nu = 0$. It is well known that the sequence (b_n) is determined by the symmetric measure γ on the unit circle. Namely,

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itn} d\gamma(t) \quad \text{for } n = 0, 1, \ldots$$

If $\gamma \ll \Lambda$, where Λ is the Lebesgue measure, then we put

$$p(t) = \frac{d\gamma(t)}{dt}.$$

THEOREM 4.4. *The Gauss system* $(\Omega, \mathcal{D}, \nu, \sigma)$ *with the spectral measure* γ *is exact if and only if*

$$\gamma \ll \Lambda \quad \textit{and} \quad \int_{-\pi}^{\pi} \log\left(p(t)\right) dt > -\infty.$$

The result as above belongs to folklore theorems in ergodic theory.

THEOREM 4.5. *The Gauss system* $(\Omega, \mathcal{D}, \nu, \sigma)$ *is exact if and only if*

$$\lim_{n\to\infty} \mathcal{L}_\nu^n(X_0) = 0 \quad \textit{in } L_2(\nu) \textit{ convergence.}$$

Proof. Assume that $(X_n)_{n\in N}$ is a sequence given by Gauss endomorphism $(\Omega, \mathcal{D}, \nu, \sigma)$. By Theorem 7.4 in [11] we see that the sequence $(X_n)_{n\in N}$ is completely undetermined, i.e.,

$$\lim_{n\to\infty} P_{H_1}^n(X_0) = 0 \quad \text{in } L_2(\nu)$$

if and only if it satisfies the assumptions of Theorem 4.4. Since

$$\mathcal{L}_\nu^n(X_0)(\sigma^n) = P_{H_1}^n(X_0),$$

we get the assertion. ■

In particular, every Gaussian Markov chain is exact (see Proposition 2.2).

5. QUASI-MARKOV CHAIN

Let ν be an invariant probability measure on $(\Omega, \mathcal{D}, \sigma)$ such that $\nu|\mathcal{B} = \mu$, and $\mathcal{L}_\nu$ the F–P operator with respect to (σ, ν). Similarly to quasi-Markovian processes (see [4]) we define quasi-Markov chain.

DEFINITION 5.1. We say that (σ, ν) is a *quasi-Markov chain* if for every skew product T given by (3.1) the following condition holds: if $\mathcal{L}_T f = f$, where $f \geqslant 0$ and $\int f d\nu \times m = 1$, then

$$\{(\omega, y) : f(\omega, y) > 0\} \in \mathcal{B} \times \mathcal{A},$$

where $\mathcal{L}_T$ is the F–P operator for T.

The difference between quasi-Markovian process and quasi-Markov chain lies in that in the latter case we do not assume positive nonsingularity of $\{T_x\}_{x \in X}$, i.e., $m(A) = 0 \Rightarrow m\big(T_x(A)\big) = 0$ for every $x \in X$ and $A \in \mathcal{A}$. Moreover, the cardinality of X may be greater than $\aleph_0$. By a similar argument to that in the proof of Theorem 2 in [4] we get:

THEOREM 5.1. *If (σ, ν) is a quasi-Markov chain, then for every skew product T and for every T-eigenfunction H from $L_1(\eta)$*

$$1_{D_\eta} H = h(\omega_0, y),$$

where η is T-a.c.i.m. and D_η denotes the support of η.

DEFINITION 5.2. A function $f \in L_1(\nu)$ will be called a *σ-quasi-eigenfunction* with quasi-eigenvalue h if h is measurable, $|h| = 1$ and $f \circ \sigma = hf$.

COROLLARY 5.1. *If (σ, ν) is a quasi-Markov chain then every σ-quasi-eigenfunction $f \in L_1(\nu)$ with quasi-eigenvalue $h(\omega) = h(\omega_0)$ has the form $f(\omega) = f(\omega_0)$.*

Proof. We use similar argumentation to that in the proof Lemma 3 in [4]. ■

PROPOSITION 5.1. *If (σ, ν) is a quasi-Markov chain, then the support of $\mathcal{L}_\nu f$ belongs to $\mathcal{B}$ for every $f \in L_1(\mu)$ and $f \geqslant 0$.*

Proof. Let us denote by $\{f > 0\}$ the set $\{\omega \in \Omega : f(\omega) > 0\}$. By Lemma 3.2, $\{\mathcal{L}_\nu 1_A > 0\} \in \mathcal{B}$ for every $A \in \mathcal{B}$. Hence for every simple function $f \geqslant 0$

$$\{\mathcal{L}_\nu f > 0\} \in \mathcal{B}.$$

For $f \in L_1(\mu)$, $f \geqslant 0$, there exists a nondecreasing sequence $0 \leqslant f_n \leqslant f$ of simple functions such that

$$\lim_{n \to \infty} \mathcal{L}_\nu f_n = \mathcal{L}_\nu f \text{ a.e.}$$

Since $\mathcal{L}_\nu f_n \leqslant \mathcal{L}_\nu f$, we have

$$\{\mathcal{L}_\nu f > 0\} \supset \{\mathcal{L}_\nu f_n > 0\} = A_n \in \mathcal{B}.$$

Due to the nondecreasing of $\mathcal{L}_\nu f_n$ and a.e. convergence we obtain

$$\{\mathcal{L}_\nu f > 0\} = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}. \ \blacksquare$$

REMARK 5.1. *The converse implication is not true in general.*

We can see the above by using symbolic representation of Examples 2 and 3 in [5]. Similarly we obtain examples of quasi-Markov chains from quasi-Markovian processes (see [4] and [1]). Here card$X \leqslant \aleph_0$. For the case card$X > \aleph_0$ we may take stationary Markov chain of order r (for the definition see [8], p. 21).

PROPOSITION 5.2. *If the support of $\mathcal{L}_\nu^r f$ belongs to $\mathcal{B}$ for any function $f \in L_1\big(\prod_{n=0}^{r-1} X, \nu\big)$ $(0 \leqslant f)$, and (σ, ν) is a Markov chain of order $r \geqslant 1$, then (σ, ν) is a quasi-Markov chain.*

Proof. If (σ, ν) is a Markov chain of order r, then (σ^r, ν) is a Markov chain on $\big(\prod_{n=0}^{r-1} X\big)^N$. Let T be a skew product given by (3.1). Then T^r is the skew product of type (3.1) over (σ^r, ν). Let η be a T-invariant measure absolutely continuous with respect to $\nu \times m$. The measure η is also T^r-invariant. Therefore, by Theorem 3.1,

$$h = \frac{d\eta}{d\nu \times m} \text{ is } \Big(\prod_{n=0}^{r-1} \mathcal{B} \times \mathcal{A}\Big)\text{-measurable.}$$

Thus $\mathcal{L}_T^r h = h$, i.e., $\mathcal{L}_\nu^r(\mathcal{L}_{\overline{\omega}_r} h) = h$, which under the assumptions implies that the support of h belongs to $\mathcal{B} \times \mathcal{A}$. $\blacksquare$

Let us consider Gauss endomorphisms.

PROPOSITION 5.3. *Gauss endomorphism which is Markov chain of order $r \geqslant 1$ is quasi-Markov chain.*

Proof. If (σ, ν) is a Markov chain of order r, then $\mathcal{L}_\nu^r(f) \in L_1\big(\prod_{n=0}^{r-1} R, \nu\big)$ for $f \in L_1\big(\prod_{n=0}^{r-1} R, \nu\big)$. By the proof of Proposition 5.1 and by Proposition 5.2 it is sufficient to show that the support of $\mathcal{L}_\nu^r 1_A$ belongs to $\mathcal{B}$ for any $A \in \mathcal{B}^r$ such that $\nu(A) > 0$. Let us observe that

$$\int (\mathcal{L}_\nu^r 1_A) 1_B \, d\nu = \int 1_A 1_B(\sigma^r) \, d\nu = \int 1_{A \times B} \, d\nu > 0$$

as $\nu | \prod_{n=0}^{2r-1} \mathcal{B}$ is a $2r$-dimensional Gaussian measure. Here $B \in \mathcal{B}^r$ is such that $\nu(B) > 0$. Therefore,

$$\mathcal{L}_\nu^r 1_A > 0 \ \nu \text{ a.e.,}$$

and the support of $\mathcal{L}_\nu^r 1_A$ is $\Omega \in \mathcal{B}$. $\blacksquare$

A non-quasi-Markov chain can be obtained as follows.

PROPOSITION 5.4. *If the Gauss endomorphism* (σ,ν) *is nonergodic*, *then it is not a quasi-Markov chain.*

P r o o f. Suppose, on the contrary, that (σ,ν) is a quasi-Markov chain. If (σ,ν) is not ergodic, then the Gauss automorphism is not ergodic either. By the Corollary in Chapter 8 of [3], p. 191, there exists a non-constant eigenfunction y such that

$$y=\lim_{n\to\infty}\sum_{k=-n}^{n} a_k^n X_k \quad \text{in } L_2.$$

As Gauss automorphism is the natural extension of (σ,ν), it follows from [9] that y is also an eigenvalue for (σ,ν). Therefore, $y(\omega)=y(\omega_0)$ by Corollary 5.1. Every sequence convergent in L_2-norm has a subsequence convergent a.e. Therefore, we can assume without loss of generality that

$$y=\lim_{n\to\infty}\sum_{k=-n}^{n} a_k^n X_k \text{ a.e.}$$

Hence there exists $(\ldots,\omega_{-1}^*,\omega_1^*,\ldots)\in R^{Z-\{0\}}$ such that

$$y(\omega_0)=\lim_{n\to\infty} a_0^n\omega_0+\sum_{k\in Z-\{0\}} a_k^n\omega_k^*$$

for μ a.e. ω_0. Thus

$$y(\omega)=a\omega_0+c$$

for some constants a,c. Here $a\neq 0$ because $y\neq c$. But this contradicts the assertion that y is an eigenfunction. ∎

There are ergodic Gauss endomorphisms which are not quasi-Markov chains. Let us put $S=\{z\in C:|z|=1\}$.

THEOREM 5.2. *The Gauss endomorphism* (σ,ν) *with the spectral measure* γ *of* ν *such that*

$$\frac{1}{e^{i\alpha}-e^{it}}\in L^2(S,\gamma)$$

for some $\alpha\in(-\pi,\pi)$ *is not a quasi-Markov chain.*

P r o o f. Suppose, to the contrary, that (σ,ν) is a quasi-Markov chain. Without loss of generality we assume that $\alpha=0$. Let H denote the real complete subspace of $L_2(\nu)$ generated by finite combinations

$$\sum_{k\in I} a_k X_k \quad \text{for } I\subset\{0,1,2,\ldots\}.$$

We put

$$\xi(t) = \frac{1}{1 - e^{it}}.$$

Then $1 = \xi - z\xi$, and by the canonical isometric mapping Θ of H into $L_2(S, \gamma)$ (see Chapter 8 in [3]) we get $\omega_0 = X_0(\omega) = g(\omega) - g\big(\sigma(\omega)\big)$, where $g = \Theta^{-1}(\xi)$. In fact, $g(\omega) = \sum_{n=0}^{\infty} X_n(\omega)$. Therefore, $f\big(\sigma(\omega)\big) = e^{-i\omega_0} f(\omega)$, where $f = e^{ig}$. Hence $f(\omega) = f(\omega_0)$ by Corollary 5.1, and $f(\omega_1) = e^{-i\omega_0} f(w_0)$ for ν a.e. ω. As $\nu|\mathcal{B} \times \mathcal{B}$ is a 2-dimensional Gaussian measure, we get $f = \mathrm{const}$, $e^{i\omega_0} = 1$, but it is impossible because X_0 has Gaussian distribution. ■

COROLLARY 5.2. *There are exact Gauss endomorphisms which are not quasi-Markov chains.*

To see this let us take (σ, ν) such that the spectral measure γ of ν is absolutely continuous with respect to the Lebesgue measure Λ and put $p(t) = \frac{1}{\pi} \sin^2 t$ for $t \in (-\pi, \pi)$.

6. AN APPLICATION

Stochastic perturbations are used in the modeling of processes connected with real life. For example, Schenk-Hope [10] presents processes in economy. The model has usually the form of skew product

$$T(x, y) = \big(\theta(x), h_x(y)\big),$$

where $\theta : X \to X$ is an ergodic transformation preserving a measure μ, and $h_x : Y \to Y$ is a measurable family of maps and nonsingular with respect to a measure m. By the results of Section 3 we can replace the deterministic base by the random one. Namely, we consider a nonsingular measure μ with respect to transition function $P(x, \cdot)$ on $X \times \mathcal{B}$. Hence we have the Markov chain (σ, P), where σ is a one-sided shift on $\Omega = X^N$, and P is the measure given by μ and $P(x, \cdot)$. We get the skew product

$$T(\omega, y) = \big(\sigma(\omega), h_{\omega(0)}(y)\big).$$

Let us assume that T has an invariant absolutely continuous measure ν. Then the density of ν belongs to $L_1(\mu \times m)$ by Corollary 3.1. Let $g \in L_1(\mu \times m)$ be an observation. Then by the Birkhoff ergodic theorem we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g\big(T^k(\omega, y)\big) = g^*,$$

where $g^* \in L_1(\mu \times m)$. Therefore, the limits of time averages of observations along random trajectory are still in $L_1(\mu \times m)$ as in the deterministic case.

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