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GEOMETRIC STABLE AND SEMISTABLE DISTRIBUTIONS ON $\mathbf{Z}_+^d$

BY

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Abstract. The aim of this article is to study geometric $\mathcal{F}$-semistable and geometric $\mathcal{F}$-stable distributions on the d-dimensional lattice $\mathbf{Z}_+^d$. We obtain several properties for these distributions, including characterizations in terms of their probability generating functions. We describe a relation between geometric $\mathcal{F}$-semistability and geometric $\mathcal{F}$-stability and their counterparts on $\mathbf{R}_+^d$ and, as a consequence, we derive some mixture representations and construct some examples. We establish limit theorems and discuss the related concepts of complete and partial geometric attraction for distributions on $\mathbf{Z}_+^d$. As an application, we derive the marginal distribution of the innovation sequence of a $\mathbf{Z}_+^d$-valued stationary autoregressive process of order p with a geometric $\mathcal{F}$-stable marginal distribution.

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1. INTRODUCTION

Let $d \geqslant 1$ be a natural number. An $\mathbf{R}^d$-valued random vector $\mathbf{X}$ is said to have a *strictly geometric semistable distribution* if there exist $p \in (0,1)$ and $\alpha \in (0,1)$ such that

$$\mathbf{X} \stackrel{d}{=} \alpha \sum_{i=1}^{N_p} \mathbf{X}_i, \tag{1.1}$$

where $(\mathbf{X}_i, i \geqslant 1)$ is a sequence of iid $\mathbf{R}^d$-valued random vectors, $\mathbf{X}_i \stackrel{d}{=} \mathbf{X}$, N_p has the geometric distribution with parameter p, and $(\mathbf{X}_i, i \geqslant 1)$ and N_p are independent.

$\mathbf{X}$ is said to have a *strictly geometric stable distribution* if for every $p \in (0,1)$ there exists $\alpha \in (0,1)$ such that (1.1) holds.

Geometric stable and semistable distributions on $\mathbf{R}^d$ have been studied by several authors. Klebanov et al. [11] introduced strictly geometric stable laws on the real line and Ramachandran [19] studied the larger class of univariate geometric stable laws. Rachev and Samorodnitsky [18] conducted an in-depth study of geometric stable distributions in Banach spaces while Kozubowski and Rachev [13], [14] treated geometric stability in Euclidean spaces. Geometric semistability was discussed in Mohan et al. [17] and Borowiecka [3]. In the area of applications, geometric stable distributions arise as useful models in finance, reliability theory, and queueing theory. We refer to the survey article by Kozubowski [12] for a fairly exhaustive list of references on these topics.

Strictly geometric stable and semistable distributions are geometric infinitely divisible, and thus infinitely divisible. If a random vector $\mathbf{X}$ has a strictly geometric stable distribution, then there exists $\gamma \in (0,2)$ such that for any $\alpha, p \in (0,1)$ for which (1.1) holds we have $p = \alpha^\gamma$. We refer to γ as the exponent of the distribution and to α as its order.

Strictly geometric stability and semistability relate to their classical counterparts through the following key characterization.

A distribution on $\mathbf{R}^d$ with characteristic function $f(\mathbf{t})$ is strictly geometric semistable for some $p, \alpha \in (0,1)$ (resp., strictly geometric stable) if and only if $f(\mathbf{t})$ admits the representation

$$f(\mathbf{t}) = [1 - \ln f_1(\mathbf{t})]^{-1}, \quad \mathbf{t} \in \mathbf{R}^d, \tag{1.2}$$

where $f_1(\mathbf{t})$ is the characteristic function of a strictly semistable distribution on $\mathbf{R}^d$ (resp., strictly stable distribution).

Let D be the closed unit ball on $\mathbf{R^d}$ and S the unit sphere on $\mathbf{R^d}$. We denote by $\mathcal{B}(\mathbf{R}_+)$ and $\mathcal{B}(S)$ the σ-algebra of Borel sets in $\mathbf{R}_+$ and S, respectively. For $E \subset \mathbf{R}_+$ and $B \subset S$, we let $EB = \{u\mathbf{x} : u \in E, \mathbf{x} \in B\}$.

It follows by representation (1.2), Proposition 2.3 in Choi [7], and Theorems 14.3 and 14.7 in Sato [21] that a distribution on $\mathbf{R}^d$ is strictly geometric semistable with exponent $\gamma \in (0,2)$ and order $\alpha \in (0,1)$ if and only if its characteristic function $f(\mathbf{t})$ has the representation

$$f(\mathbf{t}) = \Big[1 - i\mathbf{a}\cdot\mathbf{t} + \int\limits_{\mathbf{R}^d} \big(1 + iI_D(\mathbf{x})\,\mathbf{t}\cdot\mathbf{x} - \exp(i\,\mathbf{t}\cdot\mathbf{x})\big)\,\Lambda(d\mathbf{x})\Big]^{-1}, \tag{1.3}$$

where $\mathbf{a} \in \mathbf{R}^d$ (with $a = 0$ if $\gamma \neq 1$), Λ is the Lévy measure of a strictly semistable distribution on $\mathbf{R}^d$ with exponent $\gamma \in (0,2)$ and order $\alpha \in (0,1)$, and I_D is the indicator function of D. The measure Λ satisfies, for any $B \in \mathcal{B}(S)$ and $E \in \mathcal{B}(\mathbf{R}_+)$,

$$\Lambda(EB) = \int\limits_B \mu(d\mathbf{x}) \int\limits_E d\big(-N(\mathbf{x};u)u^{-\gamma}\big), \tag{1.4}$$

where μ is a finite measure on S, $N(\mathbf{x};u)$ is nonnegative, right-continuous in u, and Borel measurable in $\mathbf{x}$, $N(\mathbf{x};u)u^{-\gamma}$ is nonincreasing in u, $N(\mathbf{x};1) = 1$, and $N(\mathbf{x};\alpha u) = N(\mathbf{x};u)$.

In the case of strict stability, the Lévy measure Λ in (1.4) simplifies to (see Theorem 14.3 in Sato [21])

$$\Lambda(B) = \int_S \mu(d\mathbf{x}) \int_0^\infty I_B(u\mathbf{x}) u^{-1-\gamma}\, du \quad (B \in \mathcal{B}(S)). \tag{1.5}$$

Bouzar [6] proposed discrete analogues of stability and semistability for distributions on the d-dimensional lattice $\mathbf{Z}_+^d := \mathbf{Z}_+ \times \ldots \times \mathbf{Z}_+$, where $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$ (see also Krapavitskaite [15] and Bouzar [5] for the case $d = 1$). These authors' definitions are based on the $\mathbf{Z}_+$-valued multiple $\alpha \odot_{\mathcal{F}} X$, $\alpha \in (0, 1)$, of a $\mathbf{Z}_+$-valued random variable X. The operator $\odot_{\mathcal{F}}$ is due to van Harn et al. [9] (see also Steutel and van Harn [22]) and is defined as follows:

$$\alpha \odot_{\mathcal{F}} X = \sum_{k=1}^{X} Y_k(t) := Z_X(t) \quad (t = -\ln \alpha), \tag{1.6}$$

where $Y_1(\cdot), Y_2(\cdot), \ldots$ are independent copies of a continuous-time Markov branching process, independent of X, such that, for every $k \geqslant 1$, $P\big(Y_k(0) = 1\big) = 1$. The processes $\big(Y_k(\cdot), k \geqslant 1\big)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geqslant 0)$:

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leqslant 1;\ s, t \geqslant 0). \tag{1.7}$$

For every $k \geqslant 1$ and $t \geqslant 0$, $F_t(z)$ is the pgf of $Y_k(t)$, and the transition matrix $\{p_{ij}(t)\}$ of the Markov process $Y_k(\cdot)$ is determined by the equation

$$\sum_{j=0}^{\infty} p_{ij}(t) z^j = \{F_t(z)\}^i \quad (|z| \leqslant 1;\ i \geqslant 0). \tag{1.8}$$

Note that the process $Z_X(\cdot)$ of (1.6) is itself a Markov branching process driven by $\mathcal{F}$ and starting with X individuals ($Z_X(0) = X$).

The multiplication $\odot_{\mathcal{F}}$ of (1.6) is extended to the multivariate setting as follows.

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a $\mathbf{Z}_+^d$-valued random vector, and $\alpha \in (0, 1)$. Then

$$\alpha \odot_{\mathcal{F}} \mathbf{X} = (\alpha \odot_{\mathcal{F}} X_1, \ldots, \alpha \odot_{\mathcal{F}} X_d) = \big(Z_{X_1}(t), \ldots, Z_{X_d}(t)\big). \tag{1.9}$$

The multiplications $\alpha \odot_{\mathcal{F}} X_j$ in (1.9) are performed independently for each j. More precisely, we suppose the existence of d independent sequences $\big(Y_k^{(j)}(t), t \geqslant 0, k \geqslant 1\big)$, $j = 1, 2, \ldots, d$, of iid continuous-time Markov branching processes driven by the semigroup $\mathcal{F}$ (see (1.6)), independent of $\mathbf{X}$, such that

$$Z_{X_j}(t) = \alpha \odot_{\mathcal{F}} X_j = \sum_{k=1}^{X_j} Y_k^{(j)}(t) \quad (t = -\ln \alpha). \tag{1.10}$$

The operator $\alpha \odot_{\mathcal{F}} \mathbf{X}$ of (1.9) was first introduced by Rao and Shanbhag [20] for the binomial thinning semigroup $\mathcal{F}_1 = (F_t^{(1)}, t \geqslant 0)$,

$$F_t^{(1)}(z) = 1 - e^{-t} + e^{-t}z \quad (t \geqslant 0; |z| \leqslant 1). \tag{1.11}$$

From the assumptions and a conditioning argument it follows that the pgf of $\alpha \odot_{\mathcal{F}} \mathbf{X}$ is shown to be

$$P_{\alpha \odot_{\mathcal{F}} \mathbf{X}}(\mathbf{z}) = P\big(F_t(\mathbf{z})\big), \tag{1.12a}$$

where $P(\mathbf{z})$ is the pgf of $\mathbf{X}$, $t = -\ln \alpha$, and

$$F_t(\mathbf{z}) = \big(F_t(z_1), \ldots, F_t(z_d)\big) \quad (\mathbf{z} = (z_1, \ldots, z_d); |z_j| \leqslant 1). \tag{1.12b}$$

Unless noted otherwise, the notation (1.12b) for $F_t(\mathbf{z})$ and $\mathbf{z}$ will be used throughout the paper without further reference.

We have, by (1.7) and (1.12a),

$$\alpha \odot_{\mathcal{F}} (\beta \odot_{\mathcal{F}} \mathbf{X}) \stackrel{d}{=} (\alpha\beta) \odot_{\mathcal{F}} \mathbf{X} \quad (\alpha, \beta \in (0,1)). \tag{1.13a}$$

Moreover, if $\mathbf{X}$ and $\mathbf{Y}$ are independent $\mathbf{Z}_+^d$-valued random vectors and $\alpha \in (0,1)$, then

$$\alpha \odot_{\mathcal{F}} (\mathbf{X} + \mathbf{Y}) \stackrel{d}{=} \alpha \odot_{\mathcal{F}} \mathbf{X} + \alpha \odot_{\mathcal{F}} \mathbf{Y}. \tag{1.13b}$$

A distribution on $\mathbf{Z}_+^d$ (or its pgf) is said to be *$\mathcal{F}$-semistable* if its pgf $P(\mathbf{z})$ satisfies $0 < P(\mathbf{0}) < 1$ and

$$P(\mathbf{z}) = \big[P\big(F_t(\mathbf{z})\big)\big]^{\lambda} \quad (\mathbf{z} \in [0,1]^d) \tag{1.14}$$

for some $t > 0$ and $\lambda > 0$.

A distribution on $\mathbf{Z}_+^d$ with pgf $P(\mathbf{z})$, such that $0 < P(\mathbf{0}) < 1$, is said to be *$\mathcal{F}$-stable* if for every $t > 0$ there exists $\lambda > 0$ such that (1.14) holds.

Rao and Shanbhag ([20], pp. 160–161) introduced the notion of $\mathcal{F}$-stability on $\mathbf{Z}_+^d$ for the binomial thinning semigroup $\mathcal{F}_1$ of (1.11) (see also Gupta et al. [8]).

The aim of this article is to study geometric $\mathcal{F}$-semistable and geometric $\mathcal{F}$-stable distributions on the d-dimensional lattice $\mathbf{Z}_+^d$. We obtain several properties for these distributions, including characterizations in terms of their probability generating functions. We describe a relation between geometric $\mathcal{F}$-semistability and geometric $\mathcal{F}$-stability and their counterparts on $\mathbf{R}_+^d := \mathbf{R}_+ \times \ldots \times \mathbf{R}_+$ and, as a consequence, we derive some mixture representations and construct some examples. We establish limit theorems and discuss the related concepts of complete and partial geometric attraction for distributions on $\mathbf{Z}_+^d$. The paper is organized as follows. In Section 2 we discuss the property of geometric infinite divisibility on

$\mathbf{Z}_+^d$ and state several characterizations. Geometric $\mathcal{F}$-semistable and geometric $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ are introduced in Section 3. In Section 4, we establish a relation between geometric (semi)stability for distributions on $\mathbf{Z}_+^d$ and geometric $\mathcal{F}$-(semi)stability on $\mathbf{R}_+^d$. In Section 5, we present limit theorems and discuss the notions of complete and partial geometric attraction for distributions on $\mathbf{Z}_+^d$. In Section 6, we derive, as an application, the marginal distribution of the innovation sequence of a $\mathbf{Z}_+^d$-valued stationary autoregressive process of order p with a geometric $\mathcal{F}$-stable marginal distribution.

Many of the results in this paper are extensions of results obtained in the univariate case by Bouzar [5] for a general semigroup of pgf's $\mathcal{F}$, and by Aly and Bouzar [1] and Bouzar [4] for the binomial thinning semigroup $\mathcal{F}_1$ of (1.11). In several instances, the proofs in the multivariate case are mere adaptations of their univariate counterparts, and thus, wherever warranted, these proofs are either sketched or omitted and the relevant references are provided.

In the remainder of this section we introduce some definitions and recall several basic facts about the semigroup $\mathcal{F} = (F_t, t \geqslant 0)$. For proofs and further details we refer to van Harn et al. [9] and Steutel and van Harn [22] and references therein.

Following Steutel and van Harn [22], Chapter V, Section 8, we impose the following limit conditions on the composition semigroup $\mathcal{F}$:

$$\lim_{t\downarrow 0} F_t(z) = F_0(z) = z, \quad \lim_{t\to\infty} F_t(z) = 1. \tag{1.15}$$

The first part of (1.15) implies the continuity of the semigroup $\mathcal{F}$ and the second part is equivalent to assuming that $E\big(Y_1(1)\big) = F_1'(1) \leqslant 1$, which implies the (sub)criticality of the continuous-time Markov branching processes $Y_k^{(j)}(\cdot)$ in (1.10). We will restrict ourselves to the subcritical case ($F_1'(1) < 1$) and we will assume without loss of generality that $F_1'(1) = e^{-1}$ (see Remark 3.1 in van Harn et al. [9]). In this case,

$$F_t'(1) = e^{-t} \quad (t > 0). \tag{1.16}$$

By convention, and in compatibility with (1.15), we set

$$0 \odot_{\mathcal{F}} \mathbf{X} = \mathbf{0} \quad \text{and} \quad 1 \odot_{\mathcal{F}} \mathbf{X} = \mathbf{X}.$$

The infinitesimal generator U of the semigroup $\mathcal{F}$ is defined by

$$U(z) = \lim_{t\downarrow 0} \big(F_t(z) - z\big)/t \quad (|z| \leqslant 1) \tag{1.17}$$

and satisfies $U(z) > 0$ for $0 \leqslant z < 1$.

The related A-function is defined by

$$A(z) = \exp\Big\{-\int_0^z \big(U(x)\big)^{-1}\,dx\Big\} \quad (0 \leqslant z \leqslant 1). \tag{1.18}$$

The functions $U(z)$ and $A(z)$ satisfy, for any $t > 0$,

$$\frac{\partial}{\partial t} F_t(z) = U\big(F_t(z)\big) = U(z) F_t'(z) \quad (|z| \leqslant 1) \tag{1.19a}$$

and

$$A\big(F_t(z)\big) = e^{-t} A(z) \quad (0 \leqslant z < 1). \tag{1.19b}$$

Moreover, by (1.18), the function $A(z)$ decreases from 1 to 0.

2. GEOMETRIC INFINITELY DIVISIBLE DISTRIBUTIONS ON $\mathbf{Z}_+^d$

DEFINITION 2.1. A $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ is said to have a *geometric infinitely divisible distribution* if for every $p \in (0,1)$ there exits a sequence of iid $\mathbf{Z}_+^d$-valued random vectors $(\mathbf{X}_i^{(p)}, i \geqslant 1)$ such that

$$\mathbf{X} \stackrel{d}{=} \sum_{i=1}^{N_p} \mathbf{X}_i^{(p)}, \tag{2.1}$$

where N_p has the geometric distribution

$$P(N_p = k) = p(1-p)^{k-1} \quad (k = 1, 2, \ldots), \tag{2.2}$$

and N_p and $(\mathbf{X}_i^{(p)}, i \geqslant 1)$ are independent.

It is easily seen that a $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ with pgf $P(\mathbf{z})$ has a geometric infinitely divisible distribution if and only if for any $p \in (0,1)$ there exists a pgf $G_p(\mathbf{z})$ such that

$$P(\mathbf{z}) = \frac{p G_p(\mathbf{z})}{1 - (1-p) G_p(\mathbf{z})} \tag{2.3}$$

or, equivalently, if and only if

$$G_p(\mathbf{z}) = \frac{P(\mathbf{z})}{p + (1-p) P(\mathbf{z})} \tag{2.4}$$

is a pgf for every $p \in (0,1)$.

We start out by stating a useful lemma. Its proof is straightforward.

LEMMA 2.1. *Any compound-geometric distribution on $\mathbf{Z}_+^d$, i.e., any distribution with pgf*

$$P(\mathbf{z}) = \big\{1 + c\big(1 - Q(\mathbf{z})\big)\big\}^{-1}, \tag{2.5}$$

for some constant $c > 0$ and some pgf $Q(\mathbf{z})$, is necessarily geometric infinitely divisible.

Next, we give several characterizations of geometric infinitely divisible distributions.

THEOREM 2.1. *A* $\mathbf{Z}_+^d$*-valued random vector* $\mathbf{X}$ *with pgf* $P(\mathbf{z})$ *has a geometric infinitely divisible distribution if and only if there exist a sequence of pgf's* $\big(Q_n(\mathbf{z}), n \geqslant 1\big)$ *and a sequence of positive numbers* $(c_n, n \geqslant 1)$ *such that*

$$P(\mathbf{z}) = \lim_{n\to\infty} \Big(1 + c_n\big(1 - Q_n(\mathbf{z})\big)\Big)^{-1}. \tag{2.6}$$

Proof. We start out with the "if" part. Let $P_n(\mathbf{z}) = \big(1 + c_n\big(1 - Q_n(\mathbf{z})\big)\big)^{-1}$. By Lemma 2.1 and (2.4), the function

$$G_p^{(n)}(\mathbf{z}) = \frac{P_n(\mathbf{z})}{p + (1-p)P_n(\mathbf{z})}$$

is a pgf for every $n \geqslant 1$ and $0 < p < 1$. Thus $G_p(\mathbf{z}) = \lim_{n\to\infty} G_p^{(n)}(\mathbf{z})$ exists. By the continuity theorem, $G_p(\mathbf{z})$ is a pgf that satisfies (2.4) for every $0 < p < 1$. For the "only if" part, if $P(\mathbf{z})$ is the pgf of a geometric infinitely divisible distribution, then $G_p(\mathbf{z})$ of (2.4) is a pgf for any $0 < p < 1$. It is easily shown that $P(\mathbf{z}) = \lim_{p\to 0} \big(1 + p^{-1}\big(1 - G_p(\mathbf{z})\big)\big)^{-1}$. ∎

THEOREM 2.2. *Let* $\mathbf{X}$ *be a* $\mathbf{Z}_+^d$*-valued random vector with pgf* $P(\mathbf{z})$, $0 < P(\mathbf{0}) < 1$. *The following assertions are equivalent*:

(i) $\mathbf{X}$ *has a geometric infinitely divisible distribution.*

(ii) $P(\mathbf{z})$ *admits the representation*

$$P(\mathbf{z}) = [1 - \ln H(\mathbf{z})]^{-1}, \tag{2.7}$$

where $H(\mathbf{z})$ *is the pgf of an infinitely divisible distribution on* $\mathbf{Z}_+^d$.

(iii) $\mathbf{X}$ *has a compound geometric distribution* (*with pgf given by* (2.5)).

(iv) $\mathbf{X}$ *satisfies the stability equation*

$$\mathbf{X} \overset{d}{=} B\,(\mathbf{X} + \mathbf{S}) \tag{2.8}$$

for some $\mathbf{Z}_+^d$*-valued random vector* $\mathbf{S}$ *and a binary random variable* B, *with values in* $\{0,1\}$, *whose distribution is a mixture of a Bernoulli distribution and a continuous distribution on* $(0,1)$. *The rv's* $\mathbf{X}$, B, *and* $\mathbf{S}$ *are assumed independent.*

(v) *For every* $p \in (0,1)$, *there exists a sequence of iid* $\mathbf{Z}_+^d$*-valued random vectors* $(\mathbf{X}_i^{(p)}, i \geqslant 1)$ *independent of a geometric random variable* N_p *with parameter* p (*as in* (2.2)), *and with common pgf* $Q_p(\mathbf{z})$, *such that* $\sum_{i=1}^{N_p} \mathbf{X}_i^{(p)}$ *converges in distribution to* $\mathbf{X}$ *as* $p \to 0$.

(vi) *There exists a sequence* $(b_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d)$ *such that* $b_{\mathbf{n}} \geqslant 0$ *for all nonzero* $\mathbf{n}$ *and*

$$P(\mathbf{z}) = \big[1 - \sum_{\mathbf{n} \in \mathbf{Z}_+^d} b_{\mathbf{n}} z_1^{n_1} \dots z_d^{n_d}\big]^{-1} \tag{2.9}$$

with $\mathbf{z} = (z_1, \dots, z_d)$ *and* $\mathbf{n} = (n_1, \dots, n_d)$. *In addition,* $\sum_{\mathbf{n} \in \mathbf{Z}_+^d} b_{\mathbf{n}} < \infty$.

Proof. A mere adaptation of the proof of the univariate result (Proposition 2.1 in Aly and Bouzar [1]), with an appeal to Theorem 2.1 and Lemma 2.1 above, establishes the chain (i)⇔(ii)⇔(iii)⇔(iv).

(i)⇔(v). Assuming (i), it is easily seen that (v) holds by letting $\mathbf{X}_i^{(p)} \stackrel{d}{=} \mathbf{X}$ for every $p \in (0,1)$. For the converse we use an argument due to Rachev and Samorodnitsky [18]. For $c \in (0,1)$, let $\big(N_c(j), j \geqslant 1\big)$ be iid random variables such that $N_c(j) \stackrel{d}{=} N_c$. A straightforward pgf argument shows that, for any $0 < p' < p < 1$,

$$N_{p'} \stackrel{d}{=} 1 + \sum_{j=1}^{N_p} \big(N_c(j) - 1\big), \qquad c = \frac{p'}{p + (1-p)p'}.$$

Note that $0 < c < p'/p$. It follows that

$$\sum_{i=1}^{N_{p'}} \mathbf{X}_i^{(p')} \stackrel{d}{=} \widetilde{\mathbf{X}}_1^{(p')} + \sum_{j=1}^{N_p} \sum_{i=1}^{N_c(j)-1} \mathbf{X}_{ij}^{(p')}, \tag{2.10}$$

where $\widetilde{\mathbf{X}}_1^{(p')}$ and $\{\mathbf{X}_{ij}^{(p')}\}$ are iid random vectors with common pgf $Q_{p'}(\mathbf{z})$. Let $K_{p,p'}(\mathbf{z})$ denote the pgf of $\sum_{i=1}^{N_c(j)-1} \mathbf{X}_{ij}^{(p')}$. Since, by (v), $\lim_{p' \to 0} Q_{p'}(\mathbf{z}) = 1$, we see by (2.10) (and again by (v)) that

$$\lim_{p' \to 0} \frac{pK_{p,p'}(\mathbf{z})}{1 - (1-p)K_{p,p'}(\mathbf{z})} = P(\mathbf{z}),$$

which in turn implies

$$G_p(\mathbf{z}) = \lim_{p' \to 0} K_{p,p'}(\mathbf{z}) = \frac{P(\mathbf{z})}{p + (1-p)P(\mathbf{z})}.$$

By the continuity theorem, $G_p(\mathbf{z})$ is a pgf for any $p \in (0,1)$, and thus (i) follows.

(ii)⇔(vi). Apply Theorem 2.1 in Horn and Steutel [10] to the pgf $H(\mathbf{z})$. ∎

COROLLARY 2.1. *The property of geometric infinite divisibility is closed under weak convergence within the class of distributions on* $\mathbf{Z}_+^d$.

COROLLARY 2.2. *Any geometric infinitely divisible distribution on* $\mathbf{Z}_+^d$ *is necessarily infinitely divisible.*

P r o o f. Let $P(\mathbf{z})$ be the pgf of a geometric infinitely divisible distribution. By Theorem 2.2 ((i)⇔(iii)) and (2.5), it follows that $P(\mathbf{z}) = \big(P_n(\mathbf{z})\big)^n$, where $P_n(\mathbf{z}) = \big(1 + c\big(1 - Q(\mathbf{z})\big)\big)^{1/n}$ for some pgf $Q(\mathbf{z})$. It is clearly seen that $P_n(\mathbf{z})$ is the pgf of a compound negative binomial distribution. ∎

3. MULTIVARIATE GEOMETRIC $\mathcal{F}$-SEMISTABILITY

DEFINITION 3.1. A $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ is said to have a *geometric $\mathcal{F}$-semistable distribution* if there exist $p \in (0,1)$ and $\alpha \in (0,1)$ such that

$$\mathbf{X} \stackrel{d}{=} \alpha \odot_{\mathcal{F}} \sum_{i=1}^{N_p} \mathbf{X}_i, \tag{3.1}$$

where $(\mathbf{X}_n, n \geqslant 1)$ is a sequence of iid $\mathbf{Z}_+^d$-valued random vectors, $\mathbf{X}_n \stackrel{d}{=} \mathbf{X}$, N_p has the geometric distribution with parameter p (as in (2.2)), and $(\mathbf{X}_n, n \geqslant 1)$ and N_p are independent.

$\mathbf{X}$ is said to have a *geometric $\mathcal{F}$-stable distribution* if for every $p \in (0,1)$ there exists $\alpha \in (0,1)$ such that (3.1) holds.

By definition, a distribution on $\mathbf{Z}_+^d$ is geometric $\mathcal{F}$-stable if and only if it is geometric $\mathcal{F}$-semistable for every $p \in (0,1)$.

We state a useful characterization of geometric $\mathcal{F}$-semistability that is easily derived from Definition 3.1.

PROPOSITION 3.1. *A $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ has a geometric $\mathcal{F}$-semistable distribution for some $p, \alpha \in (0,1)$ if and only if its pgf $P(\mathbf{z})$ satisfies the functional equation*

$$P\big(F_t(\mathbf{z})\big) = \frac{P(\mathbf{z})}{p + (1-p)P(\mathbf{z})} \quad (t = -\ln \alpha). \tag{3.2}$$

In the case of geometric $\mathcal{F}$-stability, (3.2) *holds for every $p \in (0,1)$ and some $\alpha = \alpha(p) > 0$.*

Since the i-th marginal distribution of a distribution on $\mathbf{Z}_+^d$ with pgf $P(\mathbf{z})$ has pgf $P_i(z) = P(\mathbf{z}_i)$ with $\mathbf{z}_i = (1, \ldots, 1, z, 1, \ldots, 1)$, z being the i-th coordinate, it follows by Proposition 3.1 that the marginal distributions of a geometric $\mathcal{F}$-semistable (resp., geometric $\mathcal{F}$-stable) distribution on $\mathbf{Z}_+^d$ are univariate geometric $\mathcal{F}$-semistable (resp., geometric $\mathcal{F}$-stable).

LEMMA 3.1. *The parameters $p, \alpha \in (0,1)$ of a geometric $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ satisfy the condition $0 < \alpha \leqslant p < 1$.*

P r o o f. It suffices to prove the claim for $d = 1$. Let $P(z)$ be the pgf of a geometric $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+$ for some $p, \alpha \in (0,1)$. By differentiating (3.2), we have

$$F_t'(z)P'\big(F_t(z)\big) = \frac{pP'(z)}{\big(p+(1-p)P(z)\big)^2} \qquad (t = -\ln\alpha).$$

Now, by (1.19b) (note $A(z)$ of (1.18) is decreasing), $F_t(z) > z$ for all $0 \leqslant z < 1$. Hence,

$$F_t'(z) = p\frac{P'(z)}{P'\big(F_t(z)\big)}\frac{1}{\big(p+(1-p)P(z)\big)^2} \leqslant \frac{p}{\big(p+(1-p)P(z)\big)^2} \qquad (0 \leqslant z < 1).$$

Letting $z \uparrow 1$, we infer, via (1.16), that $F_t'(1) = e^{-t} = \alpha \leqslant p$. ∎

The property of infinite divisibility is stated next. The proof in the univariate case (see Proposition 4.1 in Bouzar [5]) extends straightforwardly.

PROPOSITION 3.2. *Geometric $\mathcal{F}$-semistable and geometric $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ are geometric infinitely divisible, and hence infinitely divisible.*

The following representation theorem links in fundamental fashion the concepts of $\mathcal{F}$-semistability and geometric $\mathcal{F}$-semistability. Its proof is essentially the same as the one given for the univariate version (see Theorem 4.1 in Bouzar [5]).

THEOREM 3.1. *A distribution on $\mathbf{Z}_+^d$ with pgf $P(\mathbf{z})$ is geometric $\mathcal{F}$-semistable for some $p, \alpha \in (0,1)$ (resp., geometric $\mathcal{F}$-stable) if and only if $P(\mathbf{z})$ admits the representation* (2.7), *where the function $H(\mathbf{z})$ is the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ that satisfies* (1.14) *with $\lambda = 1/p$ and $t = -\ln\alpha$ (resp., $\mathcal{F}$-stable distribution).*

Additional representation theorems for geometric $\mathcal{F}$-semistable and geometric $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$ are given next.

THEOREM 3.2. *A distribution on $\mathbf{Z}_+^d$ with pgf $P(\mathbf{z})$ is geometric $\mathcal{F}$-semistable for some $p, \alpha \in (0,1)$ if and only if, for any $\mathbf{z} = (z_1, \ldots, z_d) \in [0,1)^d$,*

$$P(\mathbf{z}) = \Big[1 + \Big(\prod_{i=1}^{d} A(z_i)\Big)^{\gamma/d} g_{\gamma,t}\big(|\ln A(z_1)|, \ldots, |\ln A(z_d)|\big)\Big]^{-1}, \tag{3.3}$$

where $\gamma = \ln p/\ln\alpha \in (0,1]$, $A(z)$ is the A-function of $\mathcal{F}$ (see (1.18)), *and $g_{\gamma,t}(\tau)$ is a continuous function from $\mathbf{R}_+^d$ to $\mathbf{R}_+$ such that*

$$g_{\gamma,t}(\underline{\tau} + \underline{t}) = g_{\gamma,t}(\underline{\tau}) \qquad (\underline{\tau} \in \mathbf{R}_+^d;\ t = -\ln\alpha;\ \underline{t} = (t, \ldots, t)). \tag{3.4}$$

P r o o f. Assume that $P(\mathbf{z})$ is the pgf of a geometric $\mathcal{F}$-semistable distribution for some $p, \alpha \in (0,1)$. By Lemma 3.1, $\gamma = \ln p/\ln\alpha \in (0,1]$. By Theorem 3.1, the function $H(\mathbf{z})$ in (2.7) is the pgf of an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$

with exponent γ and order $t = -\ln\alpha$. The representation (3.3) follows then from Theorem 3.1 in Bouzar [6]. This proves the "only if" part. The "if" part is a direct consequence of Theorem 3.1. ■

THEOREM 3.3. *A distribution on $\mathbf{Z}_+^d$ with pgf $P(\mathbf{z})$ is geometric $\mathcal{F}$-stable if and only if, for any $\mathbf{z} = (z_1, \ldots, z_d) \in [0,1)^d$,*

$$(3.5)\qquad P(\mathbf{z}) = \left[1 + \Big(\prod_{i=1}^{d} A(z_i)\Big)^{\gamma/d} Q_\gamma\left(\ln\frac{A(z_1)}{A(z_2)}, \ldots, \ln\frac{A(z_1)}{A(z_d)}\right)\right]^{-1}$$

for some $\gamma \in (0,1]$ and some nonnegative function $Q_\gamma(\mathbf{x})$ that is defined on $\mathbf{R}^{d-1}$ if $d \geqslant 2$, and that reduces to a constant if $d = 1$.

Proof. It is essentially the same as that of Theorem 3.2 above, except that we rely on the representation for $\mathcal{F}$-stable distributions given in Theorem 3.2 in Bouzar [6]. ■

We will refer to $\gamma \in (0,1)$ in the representation (3.3) (resp., (3.5)) above as the exponent of a geometric $\mathcal{F}$-semistable (resp., geometric $\mathcal{F}$-stable) distribution. In the semistable case, we will also refer to $\alpha \in (0,1)$ or, interchangeably, $t = -\ln\alpha > 0$, as the order of the distribution. We note that in this case $p = \alpha^\gamma$.

It is important to note that the exponent of a geometric $\mathcal{F}$-semistable distribution is unique in the following sense.

PROPOSITION 3.3. *Let $P(\mathbf{z})$ be the pgf of a geometric $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$. Then there exists a unique $\gamma \in (0,1]$ such that $\gamma = \ln p/\ln\alpha$ for any $p, \alpha \in (0,1)$ for which $P(\mathbf{z})$ satisfies* (3.2).

Proof. By Lemma 3.1 in Bouzar [6], the exponent γ of an $\mathcal{F}$-semistable distribution satisfies the equation $\lambda = e^{\gamma t}$ for all pairs $\lambda, t > 0$ for which (1.14) holds. By Theorem 3.1, this result is applicable to the pgf $H(\mathbf{z})$ of (2.7). Since in this case $\lambda = 1/p$ and $t = -\ln\alpha$, the conclusion ensues. ■

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a $\mathbf{Z}_+^d$-valued random vector. A $\mathbf{Z}_+$-valued random variable Y is said to be a *linear combination* of the X_i's if

$$(3.6)\qquad Y = \sum_{j=1}^{d} \alpha_j \odot_{\mathcal{F}} X_j \quad (\text{with } 1 \odot_{\mathcal{F}} X = X)$$

for some $\alpha_1, \ldots, \alpha_d \in (0,1]$. The multiplications $\alpha_j \odot_{\mathcal{F}} X_j$ in (3.6) are performed independently for each j (see equation (1.10) and the discussion preceding it).

By the assumptions and a conditioning argument, the pgf $P_Y(z)$ of the linear combination (3.6) is given by

$$(3.7)\qquad P_Y(z) = P\big(F_{t_1}(z), \ldots, F_{t_d}(z)\big) \quad (t_j = -\ln\alpha_j;\ j = 1, \ldots, d),$$

where $P(\mathbf{z})$ is the pgf of $\mathbf{X}$.

THEOREM 3.4. *A* $\mathbf{Z}_+^d$*-valued random vector* $\mathbf{X}$ *has a geometric* $\mathcal{F}$*-semistable* (*resp.*, *geometric* $\mathcal{F}$*-stable*) *distribution with exponent* $\gamma \in (0, 1]$ *and order* $\alpha \in (0, 1)$ (*resp.*, *exponent* γ) *if and only if the linear combination* (3.6) *is univariate geometric* $\mathcal{F}$*-semistable* (*resp.*, *geometric* $\mathcal{F}$*-stable*) *with exponent* γ *and order* α (*resp.*, *exponent* γ) *for every* $\alpha_1, \ldots, \alpha_d \in (0, 1]$.

P r o o f. It suffices to establish the result for semistability. The "only if" part follows easily from (3.2), (3.7) and the semigroup property (1.7). Assume that for every $\alpha_1, \ldots, \alpha_d \in (0, 1]$ the linear combination (3.6) is univariate $\mathcal{F}$-semistable with exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$. Let $P(\mathbf{z})$ be the pgf of $\mathbf{X}$. By (3.2) (in its univariate version) and (3.7), we have for any $s_1, \ldots, s_d > 0$ and $z \in [0, 1]$

$$(3.8) \qquad P\big(F_{t_1+t}(z), \ldots, F_{t_d+t}(z)\big) = \frac{P\big(F_{t_1}(z), \ldots, F_{t_d}(z)\big)}{p + (1-p)P\big(F_{t_1}(z), \ldots, F_{t_d}(z)\big)},$$

where $t = -\ln \alpha$ and $p = \alpha^\gamma$. Choose $\mathbf{z}$ arbitrarily in $[0, 1)^d$. By (1.18) and (1.19b), the function $\varphi(t) = F_t(0)$ is one-to-one from $[0, \infty)$ onto $[0, 1)$. Its inverse is $\varphi^{-1}(z) = \int_0^z \big(1/U(x)\big)\, dx$, $z \in [0, 1)$. Therefore, there exists $\mathbf{t} \in [0, \infty)$ such that $z_j = F_{t_j}(0)$, $j = 1, \ldots, d$. By setting $z = 0$ in (3.8), we have shown that (3.2) holds for any $\mathbf{z} \in [0, 1)^d$, and thus, by the principle of analytic continuation, for any $\mathbf{z}$ such that $|z_i| \leqslant 1$. ∎

We conclude the section with a characterization of geometric $\mathcal{F}$-stability in terms of geometric $\mathcal{F}$-semistability.

THEOREM 3.5. *A* $\mathbf{Z}_+^d$*-valued random vector* $\mathbf{X}$ *has a geometric* $\mathcal{F}$*-stable distribution with exponent* $\gamma \in (0, 1]$ *if and only if* $\mathbf{X}$ *has a geometric* $\mathcal{F}$*-semistable distribution with exponent* γ *and two distinct orders* $\alpha_1, \alpha_2 \in (0, 1)$ *such that* $\ln \alpha_1 / \ln \alpha_2$ *is irrational.*

P r o o f. The "only if" part is easily proved. Assume now that $\mathbf{X}$ has a geometric $\mathcal{F}$-semistable distribution with two different orders $\alpha_1, \alpha_2 \in (0, 1)$ such that $\ln \alpha_1 / \ln \alpha_2$ is irrational. By Theorem 3.1, the function $H(\mathbf{z})$ of (2.7) is the pgf of an $\mathcal{F}$-semistable distribution with orders α_1 and α_2. It follows by Corollary 3.2 in Bouzar [6] that $H(\mathbf{z})$ is $\mathcal{F}$-stable, which in turn implies, again by Theorem 3.1, that $\mathbf{X}$ is geometric $\mathcal{F}$-stable. This proves the "if" part. ∎

4. A RELATION BETWEEN GEOMETRIC SEMISTABLE DISTRIBUTIONS ON $\mathbf{R}_+^d$ AND THOSE ON $\mathbf{Z}_+^d$

We start out by recalling a useful result (see Lemma 3.3 in Bouzar [6]).

LEMMA 4.1. *Let* $\phi(\mathbf{u})$ *be the Laplace–Stieltjes transform* (*LST*) *of a distribution on* $\mathbf{R}_+^d$, *and* $A(z)$ *the* A*-function of* $\mathcal{F}$ (see (1.18)). *Then, for any* $\theta > 0$,

$$(4.1) \qquad P_\theta(\mathbf{z}) = \phi\big(\theta A(z_1), \ldots, \theta A(z_d)\big) \qquad (\mathbf{z} = (z_1, \ldots, z_d) \in [0, 1]^d)$$

is the pgf of a distribution on $\mathbf{Z}_+^d$.

The following result establishes a relation between geometric $\mathcal{F}$-semistable (resp., geometric $\mathcal{F}$-stable) distributions on $\mathbf{Z}_+^d$ and their continuous counterparts with support on $\mathbf{R}_+^d$.

THEOREM 4.1. *A function $\phi(\mathbf{u})$ defined on $\mathbf{R}_+^d$ is the LST of a geometric semistable (resp., geometric stable) distribution on $\mathbf{R}_+^d$ with exponent γ and order $\alpha \in (0,1)$ (resp., exponent $\gamma \in (0,1]$) if and only if, for every $\theta > 0$, $P_\theta(\mathbf{z})$ of (4.1) is the pgf of a geometric $\mathcal{F}$-semistable (resp., geometric $\mathcal{F}$-stable) distribution on $\mathbf{Z}_+^d$ with exponent γ and order α (resp., exponent γ).*

P r o o f. The "only if" part readily follows from Proposition 3.1 and its counterpart for the LST of a semistable distribution on $\mathbf{R}_+^d$. For the "if" part, assume that $\phi(\mathbf{u})$ is an LST with the property that there exist $\alpha \in (0,1)$ and $\gamma \in (0,1]$ such that, for any $\theta > 0$, $P_\theta(\mathbf{z})$ of (4.1) satisfies (3.2) with $t = -\ln\alpha$ and $p = \alpha^\gamma$. Select $u_1, \ldots, u_d \geqslant 0$ and choose $\theta > \max_{1 \leqslant i \leqslant d} u_i$. Let us define $z_i = A^{-1}(u_i/\theta)$, $i = 1, \ldots, d$. Letting $\mathbf{u} = (u_1, \ldots, u_d)$ and $\mathbf{z} = (z_1, \ldots, z_d)$, we have, by (4.1) and (1.19b),

$$\phi(\alpha\mathbf{u}) = P_\theta\big(F_t(\mathbf{z})\big),$$

which, combined with (3.2), yields

$$\phi(\alpha\mathbf{u}) = \frac{P_\theta(\mathbf{z})}{p + (1-p)P_\theta(\mathbf{z})} = \frac{\phi(\mathbf{u})}{p + (1-p)\phi(\mathbf{u})}.$$

This implies that $\phi(\mathbf{u})$ is the LST of a geometric semistable distribution. ∎

A geometric $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ can only arise from a continuous counterpart on $\mathbf{R}_+$ via equation (4.1).

THEOREM 4.2. *A $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ has a geometric $\mathcal{F}$-semistable distribution with exponent $0 < \gamma \leqslant 1$ and order $\alpha \in (0,1)$ if and only if its pgf $P(\mathbf{z})$ admits the representation (4.1) (for some $\theta > 0$), where $\phi(\mathbf{u})$ is the LST of a geometric semistable distribution on $\mathbf{R}_+^d$ with exponent γ and order α.*

P r o o f. The result is a direct consequence of Theorem 4.1 above and Corollary 4.1 in Bouzar [6]. ∎

Let S be the unit sphere on $\mathbf{R}^d$. We define $S_+ = S \cap \mathbf{R}_+^d$ and we denote by $\mathcal{B}(\mathbf{R}_+)$ and $\mathcal{B}(S_+)$ the σ-algebra of Borel sets in $\mathbf{R}_+$ and S_+, respectively. For $E \subset \mathbf{R}_+$ and $B \subset S_+$, we let $EB = \{u\mathbf{x} : u \in E, \mathbf{x} \in B\}$.

Adapting the canonical representation (1.3) to LST's, we see that a distribution on $\mathbf{R}_+^d$ is geometric semistable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$ if and only if its LST admits the representation

$$\phi(\underline{\tau}) = \Big[1 + \int_{\mathbf{R}_+^d} \big(1 - \exp(\underline{\tau}\cdot\mathbf{x})\big)\,\Lambda(d\mathbf{x})\Big]^{-1} \quad (\underline{\tau} \in \mathbf{R}_+^d), \tag{4.2}$$

with a Lévy measure Λ that satisfies (1.4) (where $N(\mathbf{x};u)$ is defined over $S_+ \times \mathbf{R}_+$ and B is restricted to $\mathcal{B}(S_+)$).

COROLLARY 4.1. *A distribution on* $\mathbf{Z}_+^d$ *is geometric* $\mathcal{F}$*-semistable with exponent* $\gamma \in (0,1]$ *and order* $\alpha \in (0,1)$ *if and only if its pgf* $P(\mathbf{z})$ *admits the representation* (3.3) *with the function* $g_{\gamma,t}(\underline{\tau})$ *given by*

$$g_{\gamma,t}(\underline{\tau}) = \exp\left(\frac{\gamma}{d}\sum_{i=1}^{d}\tau_i\right)\int_{\mathbf{R}_+^d}\Big(1-\exp\big(-\theta\sum_{i=1}^{d}e^{-\tau_i}x_i\big)\Big)\Lambda(d\mathbf{x}), \tag{4.3}$$

where Λ *is the Lévy measure* (*as in* (4.2)) *of a semistable distribution on* $\mathbf{R}_+^d$ *with exponent* γ *and order* $\alpha = e^{-t}$, $\underline{\tau} = (\tau_1,,\ldots,\tau_d)$, *and* $\theta > 0$.

Proof. The “only if” part follows from Theorems 3.1 and 4.2 above, and Corollary 4.3 in Bouzar [6]. The “if” part is easy to verify. ■

In the univariate case ($d = 1$), the function $g_{\gamma,t}$ of (4.3) reduces to

$$g_{\gamma,t}(\tau) = e^{\gamma\tau}\int_0^\infty\big(1-\exp(-\theta e^{-\tau}x)\big)\Lambda(dx) \quad (\tau \geqslant 0) \tag{4.4}$$

for some $\theta > 0$. The measure Λ is given by

$$\Lambda(E) = \int_E d\big(-N(u)u^{-\gamma}\big) \quad (E \in \mathcal{B}(\mathbf{R}_+)), \tag{4.5}$$

where $N(u)$ is nonnegative and right-continuous in u, $N(u)u^{-\gamma}$ is nonincreasing, $N(1) = 1$, and $N(\alpha u) = N(u)$ with $\alpha = e^{-t}$.

The results are similar in the case of geometric $\mathcal{F}$-stability.

THEOREM 4.3. *A* $\mathbf{Z}_+^d$*-valued random vector* $\mathbf{X}$ *has a geometric* $\mathcal{F}$*-stable distribution with exponent* $0 < \gamma \leqslant 1$ *if and only if its pgf admits the representation* (4.1) (*for some* $\theta > 0$), *where* $\phi(\mathbf{u})$ *is the LST of a geometric stable distribution on* $\mathbf{R}_+^d$ *with exponent* γ.

Proof. The proof is the same as that of Theorem 4.2, except that we call on Corollary 4.2 in Bouzar [6], instead. ■

By using Theorems 3.1 and 4.3 above, and Corollary 4.5 in Bouzar [6], we obtain the following characterization of geometric $\mathcal{F}$-stable distributions on $\mathbf{Z}_+^d$.

COROLLARY 4.2. *Let* $0 < \gamma \leqslant 1$ *and* $d > 1$. *A distribution on* $\mathbf{Z}_+^d$ *is geometric* $\mathcal{F}$*-stable with exponent* γ *if and only if its pgf* $P(\mathbf{z})$ *admits the representation*

(3.5) *and the function* $Q_\gamma(\mathbf{x})$ *satisfies* (*for any* $\tau_1, \ldots, \tau_d \geqslant 0$)

$$(4.6)\quad Q_\gamma(\tau_2 - \tau_1, \ldots, \tau_d - \tau_1) = \exp\left(\frac{\gamma}{d}\sum_{i=1}^{d}\tau_i\right)\int_{\mathbf{R}_+^d}\Big(1 - \exp\big(-\theta\sum_{i=1}^{d} e^{-\tau_i}x_i\big)\Big)\,\Lambda(d\mathbf{x}),$$

where Λ *is the Lévy measure given by* (1.5) (*with* μ *being defined over* $\mathcal{B}(\mathcal{S}_+)$) *of a stable distribution on* $\mathbf{R}_+^d$ *with exponent* γ *and* $\theta > 0$.

We conclude the section with some examples.

EXAMPLE 4.1. Let $\alpha \in (0,1)$ and $\gamma \in (0,1)$. We denote by $|\mathbf{a}|$ the Euclidean norm of $\mathbf{a} \in \mathbf{R}^d$. We define

$$(4.7)\qquad S_n(\alpha) = \{\mathbf{x} \in \mathbf{R}_+^d :\ \alpha^{-n} \leqslant |\mathbf{x}| < \alpha^{-n-1}\} \quad (n \in \mathbf{Z}).$$

From the well-known formula of the volume of a d-ball we have $\int_{S_n(\alpha)} d\mathbf{x} = C_d\alpha^{-nd}$ ($n \in \mathbf{Z}$) for some constant $C_d > 0$. We also have $\mathbf{R}_+^d = \bigcup_{n\in\mathbf{Z}} S_n(\alpha)$.

Let f_0 be a nonnegative bounded (Lebesgue) measurable function defined on the interval $[0, \alpha^{-1})$. We define $h(\mathbf{x})$ on $\mathbf{R}_+^d$ by

$$(4.8)\qquad h(\mathbf{x}) = \alpha^{n(\gamma+d)} f_0(\alpha^n|\mathbf{x}|) \quad (\mathbf{x} \in S_n(\alpha);\ n \in \mathbf{Z}),$$

and the absolutely continuous measure $\Lambda(\cdot)$ on $\mathcal{B}(\mathbf{R}_+^d)$ by

$$(4.9)\qquad \Lambda(B) = \int_B h(\mathbf{x})\,d\mathbf{x} \quad (B \in \mathcal{B}(\mathbf{R}_+^d)).$$

By using the partition $\big(S_n(\alpha),\ n \in \mathbf{Z}\big)$ of $\mathbf{R}_+^d$ and the formula for the volume of $S_n(\alpha)$, it is a straightforward argument to show that $\Lambda(\alpha B) = \alpha^{-\gamma}\Lambda(B)$ and that $\int \min(\mathbf{x}, 1)\Lambda(d\mathbf{x}) < \infty$. Therefore, $\Lambda(\cdot)$ is the Lévy measure of a semistable distribution on $\mathbf{R}_+^d$ with exponent γ and order α (see Theorem 14.3 in Sato [21]), which in turn gives rise to an $\mathcal{F}$-semistable distribution on $\mathbf{Z}_+^d$ with exponent γ and order α (via Theorem 4.2 and representation (4.1)).

EXAMPLE 4.2 (due to Sato [21]). Let $\alpha \in (0,1)$, $\gamma \in (0,1]$, and $\mathbf{x}_0 \in \mathbf{R}_+^d$. We define the Lévy measure

$$(4.10)\qquad \Lambda(d\mathbf{x}) = \sum_{n=-\infty}^{\infty} \alpha^{-n\gamma}\delta_{\alpha^n\mathbf{x}_0}(d\mathbf{x}),$$

where $\delta_{(\cdot)}$ denotes the Dirac measure. As in Example 4.1, one can easily verify that $\Lambda(d\mathbf{x})$ gives rise to a semistable distribution on $\mathbf{Z}_+^d$ with exponent γ and order α.

5. LIMIT THEOREMS

We start out with a limit theorem that characterizes geometric $\mathcal{F}$-semistability on $\mathbf{Z}_+^d$.

THEOREM 5.1. *Let $\mathbf{X}$ be a $\mathbf{Z}_+^d$-valued random vector with pgf $P(\mathbf{z})$ such that $0 < P(\mathbf{0}) < 1$. The following assertions are equivalent*:

(i) *$\mathbf{X}$ has a geometric $\mathcal{F}$-semistable distribution with some exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$.*

(ii) *There exist a sequence of $\mathbf{Z}_+^d$-valued iid random vectors $(\mathbf{X}_n, n \geqslant 1)$, independent of geometric random variables $(N_{p_n}, n \geqslant 1)$, and real numbers $(\alpha_n, n \geqslant 1)$ in $(0, 1)$ such that $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{N_{p_n}} \mathbf{X}_i$ converges in distribution to $\mathbf{X}$ and*

$$\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = \lim_{n\to\infty} (p_n/p_{n+1})^{1/\gamma} = \alpha \tag{5.1}$$

for some $\alpha \in (0, 1)$ and $\gamma \in (0, 1]$.

(iii) *There exist a pgf $Q(\mathbf{z})$, a sequence of real numbers $(\alpha_n, n \geqslant 1)$ in $(0, 1)$, and a sequence $(k_n, n \geqslant 1)$ in $\mathbf{Z}_+$, $k_n \uparrow \infty$, such that*

$$P(\mathbf{z}) = \lim_{n\to\infty} \Big[1 + k_n\Big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\Big)\Big]^{-1} \qquad (t_n = -\ln \alpha_n), \tag{5.2}$$

and

$$\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = \lim_{n\to\infty} (k_n/k_{n+1})^{1/\gamma} = \alpha \tag{5.3}$$

for some $\alpha \in (0, 1)$ and $\gamma \in (0, 1]$.

Proof. (i) $\Rightarrow$(ii). For $n \geqslant 1$, let $\alpha_n = \alpha^n$, $p_n = \alpha^{n\gamma}$, and $\mathbf{X}_n \stackrel{d}{=} \mathbf{X}$. Clearly, (5.1) holds, and since for every $n \geqslant 1$, $\mathbf{X}$ has a geometric $\mathcal{F}$-semistable distribution with exponent γ and order α_n (see equation (3.3) and Proposition 3.3), (ii) ensues.

(ii) $\Rightarrow$(iii). Letting $t_n = -\ln \alpha_n$, we have by (ii)

$$P(\mathbf{z}) = \lim_{n\to\infty} \frac{Q\big(F_{t_n}(\mathbf{z})\big)}{Q\big(F_{t_n}(\mathbf{z})\big) + \frac{1}{p_n}\big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\big)},$$

from which we easily deduce that $\lim_{n\to\infty} Q\big(F_{t_n}(\mathbf{z})\big) = 1$ and (noting that $P(\mathbf{z}) \neq 0$ on $[0, 1]^d$ since $0 < P(\mathbf{0}) < 1$)

$$\lim_{n\to\infty} \frac{1}{p_n}\Big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\Big) = \frac{1 - P(\mathbf{z})}{P(\mathbf{z})}.$$

Therefore,

$$P(\mathbf{z}) = \lim_{n\to\infty} \Big[1 + \frac{1}{p_n}\Big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\Big)\Big]^{-1}.$$

Letting $k_n = [1/p_n]$, where $[x]$ is the greatest integer function, by a straightforward argument we see that (5.2) and (5.3) hold.

(iii) ⇒(i). Assume that (5.2) and (5.3) hold and define

$$P_n(\mathbf{z}) = \Big[1 + k_n\Big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\Big)\Big]^{-1}.$$

Note that $P_n(\mathbf{z})$ is geometric infinitely divisible, and thus $P_n(\mathbf{z}) \neq 0$ (see Lemma 2.1 and Corollary 2.2). Let $H_n(\mathbf{z}) = \exp\{1 - 1/P_n(\mathbf{z})\}$, $n \geqslant 1$. By Theorem 2.2, $H_n(\mathbf{z})$ is an infinitely divisible pgf and

$$\lim_{n\to\infty} H_n(\mathbf{z}) = \lim_{n\to\infty} \exp\Big\{k_n\Big(Q\big(F_{t_n}(\mathbf{z})\big) - 1\Big)\Big\} = \exp\{1 - 1/P(\mathbf{z})\}.$$

Since we again have $\lim_{n\to\infty} Q\big(F_{t_n}(\mathbf{z})\big) = 1$, it follows that

$$\lim_{n\to\infty} \exp\Big\{k_n\Big(Q\big(F_{t_n}(\mathbf{z})\big) - 1\Big)\Big\} = \lim_{n\to\infty}\big[Q\big(F_{t_n}(\mathbf{z})\big)\big]^{k_n}.$$

Therefore, by Theorem 4.2 in Bouzar [6], $H(\mathbf{z}) = \exp\{1 - 1/P(\mathbf{z})\}$ is $\mathcal{F}$-semistable with exponent $\gamma \in (0, 1]$ and order $\alpha \in (0, 1)$, and thus, by Theorem 3.1, $P(\mathbf{z})$ is geometric $\mathcal{F}$-semistable with the same exponent and order. ∎

The next theorem gathers some characterization results for geometric $\mathcal{F}$-stable distribution on $\mathbf{Z}_+^d$.

THEOREM 5.2. *Let* $\mathbf{X}$ *be a* $\mathbf{Z}_+^d$*-valued random vector with pgf* $P(\mathbf{z})$ *such that* $0 < P(\mathbf{0}) < 1$. *The following assertions are equivalent*:

(i) $\mathbf{X}$ *has a geometric* $\mathcal{F}$*-stable distribution with some exponent* $\gamma \in (0, 1]$.

(ii) *There exist a sequence of* $\mathbf{Z}_+^d$*-valued iid random vectors* $(\mathbf{X}_n, n \geqslant 1)$, *independent of geometric random variables* $\big(N_p, p \in (0, 1)\big)$ (see (2.2)), *and real numbers* $\big(\alpha(p), p \in (0, 1)\big)$ *in* $(0, 1)$ *such that* $\alpha(p) \odot_{\mathcal{F}} \sum_{i=1}^{N_p} \mathbf{X}_i$ *converges in distribution to* $\mathbf{X}$.

(iii) *There exist a sequence of* $\mathbf{Z}_+^d$*-valued iid random vectors* $(\mathbf{X}_n, n \geqslant 1)$, *independent of geometric random variables* $(N_{p_n}, n \geqslant 1)$ *with* $p_n = 1/n$, *and real numbers* $(\alpha_n, n \geqslant 1)$ *in* $(0, 1)$ *such that* $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{N_{p_n}} \mathbf{X}_i$ *converges in distribution to* $\mathbf{X}$.

(iv) *There exist a pgf* $Q(\mathbf{z})$ *and a sequence of real numbers* $(\alpha_n, n \geqslant 1)$ *in* $(0, 1)$ *such that*

$$P(\mathbf{z}) = \lim_{n\to\infty}\Big[1 + n\Big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\Big)\Big]^{-1} \qquad (t_n = -\ln\alpha_n). \tag{5.4}$$

In this case, *we have necessarily* $\lim_{n\to\infty} \alpha_n = 0$.

P r o o f. The implication (i) ⇒(ii) is easily seen to follow from the definition of geometric $\mathcal{F}$-stability by letting $\mathbf{X}_n \overset{d}{=} \mathbf{X}$. Assuming (ii) and letting p approach zero along the sequence $p_n = 1/n$ yields (iii).

(iii) ⇒(iv). Let $Q(\mathbf{z})$ be the common pgf of the $\mathbf{X}_n$'s. We have, by (iii),

$$P(\mathbf{z}) = \lim_{n\to\infty} \frac{Q\big(F_{t_n}(\mathbf{z})\big)}{Q\big(F_{t_n}(\mathbf{z})\big) + n\big(1 - Q\big(F_{t_n}(\mathbf{z})\big)\big)},$$

where $t_n = -\ln \alpha_n$. By using the same argument as in the proof of (ii) ⇒(iii) in Theorem 5.1 (with $1/p_n = n$), it is shown that (5.4) holds. Now, if $\lim_{n\to\infty} \alpha_n \neq 0$, then by a compactness argument, there would exist a subsequence $\alpha_{n'}$ of α_n such that $\lim_{n'\to\infty} \alpha_{n'} = \alpha$ for some $\alpha \in (0,1)$. It would follow by (5.4), applied along the subsequence $\{n'\}$, and by the continuity of the semigroup $\mathcal{F}$, that $Q\big(F_t(\mathbf{z})\big) = 1$, $t = -\ln\alpha$. This would imply that $F_t(z) = 1$ for any $z \in [0,1]$. That is a contradiction, and thus $\lim_{n\to\infty} \alpha_n = 0$.

Finally, the proof of (iv) ⇒(i) is identical to the proof of (iii) ⇒(i) of Theorem 5.1. In this case, we have $k_n = n$ and we would appeal to Theorem 4.3 instead of Theorem 4.2 in Bouzar [6]. ■

As in classical stability, a theory of attraction can be developed in connection with the notions of $\mathcal{F}$-stability and geometric $\mathcal{F}$-stability.

A distribution on $\mathbf{Z}_+^d$ with pgf $Q(\mathbf{z})$ is said to belong to the *domain of* (*complete*) *attraction of a distribution on* $\mathbf{Z}_+^d$ *with pgf* $P(\mathbf{z})$ if there exist a sequence of iid $\mathbf{Z}_+^d$-valued random vectors $(\mathbf{X}_n, n \geqslant 1)$ with common pgf $Q(\mathbf{z})$, a $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ with pgf $P(\mathbf{z})$, and real numbers $\alpha_n \in (0,1)$, $n \geqslant 1$, such that $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{n} \mathbf{X}_i$ converges in distribution to $\mathbf{X}$. In this case, the sequence $(\alpha_n, n \geqslant 1)$ necessarily satisfies $\lim_{n\to\infty} \alpha_n = 0$. This can be seen by using the same argument as in the proof of (iii) ⇒(iv) of Theorem 5.2.

The *domain of partial attraction of a distribution on* $\mathbf{Z}_+^d$ is defined similarly. In this case, the convergence in distribution is required to occur along a subsequence $(k_n, n \geqslant 1)$, i.e., $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{k_n} \mathbf{X}_i$ converges in distribution to $\mathbf{X}$.

The following result was essentially proved in Bouzar [6] (see Theorem 4.3 in that paper).

THEOREM 5.3. *A distribution on* $\mathbf{Z}_+^d$ *has a nonempty domain of attraction if and only if it is* $\mathcal{F}$*-stable.*

PROPOSITION 5.1. *A geometric* $\mathcal{F}$*-stable distribution belongs to the domain of attraction of an* $\mathcal{F}$*-stable distribution.*

P r o o f. Let $(\mathbf{X}_n, n \geqslant 1)$ be a sequence of iid $\mathbf{Z}_+^d$-valued random vectors with a common distribution that is geometric $\mathcal{F}$-stable with exponent $\gamma \in (0,1]$ and order $\alpha \in (0,1)$. Let $\mathbf{Y}$ be a $\mathbf{Z}_+^d$-valued random vector with pgf $H(\mathbf{z})$ defined by (2.7). By Theorem 3.1, $\mathbf{Y}$ has an $\mathcal{F}$-stable distribution. For $n \geqslant 1$, let $P_n(\mathbf{z})$

denote the pgf of $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{n} \mathbf{X}_i$ with $\alpha_n = n^{-1/\gamma}$. By (2.7) and Corollary 3.2 in Bouzar [6], $P_n(\mathbf{z})$ satisfies

$$\ln P_n(\mathbf{z}) = -n \ln \Big(1 - \ln H\big(F_{t_n}(\mathbf{z})\big)\Big) = -n \ln \Big(1 - \big(\ln H(\mathbf{z})\big)/n\Big)$$

with $t_n = (\ln n)/\gamma$. Therefore, $\lim_{n\to\infty} P_n(\mathbf{z}) = H(\mathbf{z})$. ■

The analogous result for the property of partial attraction follows from Theorem 4.2 in Bouzar [6].

THEOREM 5.4. *If the domain of partial attraction of a distribution* $(q_{\mathbf{n}}, \mathbf{n} \in \mathbf{Z}_+^d)$ *on* $\mathbf{Z}_+^d$ *contains a distribution for which the sequences* $(\alpha_n, n \geqslant 1)$ *in* $(0,1)$ *and* $(k_n, n \geqslant 1)$ *in* $\mathbf{Z}_+^d$ *satisfy* (5.3) *for some* $\alpha \in (0,1)$ *and* $\gamma \in (0,1]$, *then* $\{q_{\mathbf{n}}\}$ *must be* $\mathcal{F}$*-semistable with exponent* γ *and order* α.

A distribution on $\mathbf{Z}_+^d$ with pgf $Q(\mathbf{z})$ is said to belong to the *domain of geometric attraction of a distribution on* $\mathbf{Z}_+^d$ *with pgf* $P(\mathbf{z})$ if there exist a sequence of iid $\mathbf{Z}_+^d$-valued random vectors $(\mathbf{X}_n, n \geqslant 1)$ with common pgf $Q(\mathbf{z})$, independent of geometric random variables $\big(N_p, p \in (0,1)\big)$, a $\mathbf{Z}_+^d$-valued random vector $\mathbf{X}$ with pgf $P(\mathbf{z})$, and real numbers $\alpha(p) \in (0,1)$, $p \in (0,1)$, such that $\alpha(p) \odot_{\mathcal{F}} \sum_{i=1}^{N_p} \mathbf{X}_i$ converges in distribution to $\mathbf{X}$ as $p \to 0$.

The *domain of partial geometric attraction of a distribution on* $\mathbf{Z}_+^d$ is defined similarly. In this case, the convergence in distribution is required to occur along a null sequence $(p_n, n \geqslant 1)$ in $(0,1)$, i.e., $\alpha_n \odot_{\mathcal{F}} \sum_{i=1}^{N_{p_n}} \mathbf{X}_i$ converges in distribution to $\mathbf{X}$ as $p_n \to 0$.

The following result is a direct consequence of Theorem 5.2.

THEOREM 5.5. *A distribution on* $\mathbf{Z}_+^d$ *has a nonempty domain of geometric attraction if and only if it is geometric* $\mathcal{F}$*-stable.*

Likewise, we infer from Theorem 5.1 that

THEOREM 5.6. *If the domain of partial geometric attraction of a distribution* $\{q_{\mathbf{n}}\}$ *on* $\mathbf{Z}_+^d$ *contains a distribution for which the sequences* $(\alpha_n, n \geqslant 1)$ *and* $(p_n, n \geqslant 1)$ *satisfy* (5.3) *for some* $\alpha \in (0,1)$ *and* $\gamma \in (0,1]$, *then* $\{q_{\mathbf{n}}\}$ *must be geometric* $\mathcal{F}$*-semistable with exponent* γ *and order* α.

THEOREM 5.7. *A distribution on* $\mathbf{Z}_+^d$ *with pgf* $Q(\mathbf{z})$ *belongs to the domain of geometric attraction of a geometric* $\mathcal{F}$*-stable distribution with pgf* $P(\mathbf{z})$ *if and only if it belongs to the domain of attraction of an* $\mathcal{F}$*-stable distribution with pgf* $H(\mathbf{z}) = \exp\{1 - 1/P(\mathbf{z})\}$.

Proof. First we prove the "only if" part. We have, by assumption, Theorem 5.2, and (5.4),

$$H(\mathbf{z}) = \lim_{n\to\infty} \exp\left\{ n\left(1 - \frac{1}{Q\big(F_{t_n}(\mathbf{z})\big)}\right)\right\} \quad (t_n = -\ln \alpha(1/n)),$$

which implies that $H(\mathbf{z})=\lim_{n\to\infty}\big[Q\big(F_{t_n}(\mathbf{z})\big)\big]^n$, since $\lim_{n\to\infty} Q\big(F_{t_n}(\mathbf{z})\big)=1$. The converse is proved along the same lines. The details are omitted. ∎

One can show similarly that

THEOREM 5.8. *A distribution on $\mathbf{Z}_+^d$ with pgf $Q(\mathbf{z})$ belongs to the domain of partial geometric attraction of a geometric $\mathcal{F}$-semistable distribution with pgf $P(\mathbf{z})$ if and only if it belongs to the domain of partial attraction of an $\mathcal{F}$-semistable distribution with pgf $H(\mathbf{z}) = \exp\{1 - 1/P(\mathbf{z})\}$.*

6. A $\mathbf{Z}_+^d$-VALUED AUTOREGRESSIVE PROCESS OF ORDER p

The binomial thinning operator $\odot_{\mathcal{F}(1)}$ corresponding to the semigroup (1.11) was used by several authors to construct $\mathbf{Z}_+$-valued autoregressive processes. Zhu and Joe [24] and Aly and Bouzar [2] extended some of these models by using the more general $\odot_{\mathcal{F}}$ operator. We refer to the survey articles by McKenzie [16] and Weiß [23] for more on the topic. In this section we present briefly a generalized $\mathbf{Z}_+^d$-valued autoregressive process of order p ($\mathcal{F}$-INAR (p)) based on the $\odot_F$ operator. In particular, we will derive the marginal distribution of the stationary $\mathcal{F}$-INAR (p) process with a geometric $\mathcal{F}$-semistable (stable) marginal.

DEFINITION 6.1. A sequence $\{\mathbf{X}_n\}$ of $\mathbf{Z}_+^d$-valued random vectors is said to be an $\mathcal{F}$-INAR (p) *process* if, for any $n \in \mathbf{Z}$,

$$\mathbf{X}_n = \sum_{i=1}^{p} I(\xi_n = i)\alpha_i \odot_{\mathcal{F}} X_{n-i} + \mathbf{e}_n, \tag{6.1}$$

where $\alpha_i \in (0,1)$, $\{\xi_n\}$ is a sequence of iid random variables such that $P(\xi_n = i) = c_i$, $i = 1, 2, \ldots, p$, $\sum_{i=1}^{p} c_i = 1$, and $\{\mathbf{e}_n\}$ is the innovation sequence, a sequence of iid $\mathbf{Z}_+^d$-valued random vectors independent of $\{\xi_n\}$.

The generalized multiplication $\alpha_i \odot_{\mathcal{F}} \mathbf{X}_{n-i}$ in (6.1), as defined in (1.9), is performed independently for each i. More precisely, we assume the existence of independent arrays $(Y_{j,n}^{(i,h)},\ j \geqslant 0, n \in Z)$, $i = 1, 2, \ldots, p$, $h = 1, \ldots, d$, of iid $\mathbf{Z}_+$-valued rv's, independent of $\{\xi_n\}$ and $\{\mathbf{e}_n\}$, such that for each i, the array's common pgf is $F_{t_i}(z)$, $t_i = -\ln \alpha_i$,

$$\alpha_i \odot_{\mathcal{F}} pr_h(\mathbf{X}_{n-i}) = \sum_{j=1}^{pr_h(\mathbf{X}_{n-i})} Y_{j,n-i}^{(i,h)}, \tag{6.2}$$

where $pr_h(\mathbf{X}_{n-i})$ denotes the h-th coordinate of $\mathbf{X}_{n-i}$.

In terms of pgf's, it follows from (6.1) that

$$P_{\mathbf{X}_n}(\mathbf{z}) = \Big(\sum_{i=1}^{p} c_i P_{\mathbf{X}_{n-i}}\big(F_{t_i}(\mathbf{z})\big)\Big) P_{\mathbf{e}}(\mathbf{z}), \qquad t_i = -\ln \alpha_i. \tag{6.3}$$

The autocorrelation structure of a stationary $\mathbf{Z}_+^d$-valued $\mathcal{F}$-INAR (p) process is given in the following proposition. The proof is based on a conditioning argument and is a mere adaptation of the one given in the univariate case (see Aly and Bouzar [2], Proposition 5.2).

PROPOSITION 6.1. *Let* $\{\mathbf{X}_n\}$ *be a* $\mathbf{Z}_+^d$*-valued stationary* $\mathcal{F}$*-INAR* (p) *process with mean* $\mu_{\mathbf{X}}$ *and covariance matrix* $\Sigma_{\mathbf{X}} = \{\sigma_{\mathbf{X}}(h,l)\}$. *Let* $\mu_{\mathbf{e}}$ *and* $\Sigma_{\mathbf{e}} = \{\sigma_{\mathbf{e}}(h,l)\}$ *be*, *respectively*, *the mean and covariance matrix of the innovation sequence* $\{\mathbf{e}_n\}$. *Let* c_i *and* α_i $(i = 1, \ldots, p)$ *be as in* (6.1). *Then*:

(i) *For any* $n \in \mathbf{Z}$,

$$\mu_{\mathbf{X}} = \big(1 - \sum_{i=1}^{p} c_i \alpha_i\big)^{-1} \mu_{\mathbf{e}}. \tag{6.4}$$

(ii) *The autocovariance matrix function*

$$\Gamma(k) = \{\gamma_k(h,l)\} = E\big((\mathbf{X}_{n-k} - \mu_{\mathbf{X}})'(\mathbf{X}_n - \mu_{\mathbf{X}})\big)$$

of $\{\mathbf{X}_n\}$ *at lag* k, $k \geqslant 1$, *is given by*

$$\gamma_k(h,l) = \begin{cases} \sum_{i=1}^{p} c_i \alpha_i \gamma_i(h,l) + \sigma_{\mathbf{e}}(h,l) & \textit{if } k = 0, \\ \sum_{i=1}^{p} c_i \alpha_i \gamma_{k-i}(h,l) & \textit{if } k \geqslant 1 \end{cases} \tag{6.5}$$

$(h, l = 1, \ldots, d)$ *with* $\Gamma(k) = \Gamma(-k)$, $k \geqslant 1$.

(iii) *The autocorrelation matrix function* $R(k) = \{\rho_k(h,l)\}$ *at lag* k, $k \geqslant 1$, *is given by*

$$\rho_k(h,l) = \sum_{i=1}^{p} c_i \alpha_i \rho_{k-i}(h,l) \quad (h, l = 1, \ldots, d), \tag{6.6}$$

where $\rho_0 = 1$, $\rho_k = \rho_{-k}$, *and* c_i *and* α_i $(i = 1, \ldots, p)$ *are as in* (6.1).

Next, we determine the marginal distribution of the innovation sequence of a stationary $\mathcal{F}$-INAR (p) process with a geometric $\mathcal{F}$-stable marginal distribution. The proof is essentially the same as the one given for the univariate case (Proposition 5.5 in Aly and Bouzar [2]) and relies on a decomposition lemma for a specific class of rational functions (see Lemma 5.4 in Aly and Bouzar [2]).

THEOREM 6.1. *If* $\{\mathbf{X}_n\}$ *is a* $\mathbf{Z}_+^d$*-valued stationary* $\mathcal{F}$*-INAR* (p) *process with a geometric* $\mathcal{F}$*-stable marginal distribution with pgf given by* (3.5), *then its innovation sequence* $\{\mathbf{e}_n\}$ *admits the representation*

$$\mathbf{e}_n \stackrel{d}{=} \sum_{j=0}^{p} I(\kappa_n = j) \beta_j \odot_F \mathbf{E}_n, \tag{6.7}$$

where $\{\kappa_n\}$ *is a sequence of iid random variables such that* $P(\kappa_n = j) = c_j'$, $j = 0, \ldots, p$, $\sum_{j=0}^{p} c_j' = 1$, $\{\mathbf{E_n}\}$ *is a sequence of iid random vectors*, *independent of* $\{\kappa_n\}$, *such that for each* n, $\mathbf{E_n}$ *has the same distribution as* $\mathbf{X}_n$.

It is possible to extend Theorem 6.1 to stationary $\mathcal{F}$-INAR (p) processes with a geometric $\mathcal{F}$-semistable marginal distribution by imposing a commensurability assumption on the numbers $\{-\ln \alpha_i, i = 1, \ldots, p\}$. However, the assumption puts a significant restriction on the applicability of the result.

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