PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 35, Fasc. 2 (2015), pp. 285–300

ALMOST SURE CENTRAL LIMIT THEOREMS FOR RANDOM RATIOS AND APPLICATIONS TO LSE FOR FRACTIONAL ORNSTEIN–UHLENBECK PROCESSES

BY

PEGGY CÉNAC (DIJON) AND KHALIFA ES-SEBAIY (MARRAKESH)

Abstract. We will investigate an almost sure central limit theorem (ASCLT) for sequences of random variables having the form of a ratio of two terms such that the numerator satisfies the ASCLT and the denominator is a positive term which converges almost surely to one. This result leads to the ASCLT for least squares estimators for Ornstein–Uhlenbeck process driven by fractional Brownian motion.

2010 AMS Mathematics Subject Classification: Primary: 60F05, 60G15, 60H05, 60H07; Secondary: 62F12.

Key words and phrases: Almost sure central limit theorem, least squares estimator, fractional Ornstein–Uhlenbeck process, multiple stochastic integrals.

1. INTRODUCTION

The almost sure central limit theorem (ASCLT) was simultaneously proved by Brosamler [5] and Schatte [16]. The simplest form of the ASCLT (see Lacey and Philipp [10]) states that if $\{X_n, n \geqslant 1\}$ is a sequence of real-valued independent identically distributed random variables with $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = 1$, and if we denote by $S_n = \frac{1}{\sqrt{n}}(X_1 + \ldots + X_n)$ the normalized partial sums, then, almost surely, for all $z \in \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{S_k \leqslant z\}} \xrightarrow[n\to\infty]{\text{a.s.}} P(N \leqslant z),$$

where N is an $\mathcal{N}(0,1)$ random variable and $\mathbb{1}_{\{A\}}$ denotes the indicator of the set A. Equivalently, for any bounded and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, one has, almost surely,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(S_k) \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{E}\big(\varphi(N)\big).$$

The ASCLT was first stated, without proof, by Lévy [11]. For more discussion about ASCLT see, e.g., Berkes and Csáki [4] and the references in the survey paper by Berkes [3].

Ibragimov and Lifshits [8], [9] give a criterion (see Theorem 2.1) for the ASCLT based on the rate of convergence of the empirical characteristic function. Using this criterion and Malliavin calculus, Bercu et al. [2] provide a criterion for ASCLT for functionals of general Gaussian fields.

Our first aim is to prove an almost sure central limit theorem for a sequence of the form $\{G_n/R_n\}_{n\geqslant 1}$, where $\{G_n\}_{n\geqslant 1}$ satisfies the ASCLT and $\{R_n\}_{n\geqslant 1}$ is a sequence of positive random variables not necessarily independent of $\{G_n\}$ and converging almost surely to one (see Theorem 3.1). We apply our ASCLT to a fractional Ornstein–Uhlenbeck process $X=\{X_t, t\geqslant 0\}$ defined as

$$X_0 = 0, \quad dX_t = -\theta X_t dt + dB_t,\ t\geqslant 0, \tag{1.1}$$

where $B=\{B_t, t\geqslant 0\}$ is a fractional Brownian motion with Hurst parameter $H\in\left(\frac{1}{2},1\right)$, and θ is a real parameter. θ is unknown and estimated with least squares estimators (LSE). Theorem 3.1 leads to the ASCLT for the LSE in this model.

Continuous observations. Recently, the parametric estimation of the continuously observed fractional Ornstein–Uhlenbeck process defined in (1.1) was studied by using the least squares estimator (LSE) defined by

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt}.$$

In the ergodic case, that is, when $\theta>0$, Hu and Nualart [6] proved that the LSE $\widehat{\theta}_T$ of θ is strongly consistent and asymptotically normal. In addition, they also proved that the estimator

$$\overline{\theta}_T = \left(\frac{1}{H\Gamma(2H)T}\int_0^T X_t^2 dt\right)^{-1/(2H)}$$

is strongly consistent and asymptotically normal. In the non-ergodic case $\theta<0$, Belfadli et al. [1] established that the LSE $\widehat{\theta}_T$ of θ is strongly consistent and asymptotically Cauchy.

In this paper, we focus our discussion on the ergodic case $\theta>0$. We shall prove that when $H\in(1/2,3/4)$, the sequence $\{\sqrt{n}(\theta-\widehat{\theta}_n)\}_{n\geqslant 1}$ satisfies the ASCLT (see Theorem 4.2).

Discrete observations. Assume that the process X is observed equidistantly in time with the step size $h>0$, that is, for any $i\in\{0,\dots,n\}$, $t_i=ih$. Hu and Song [7], motivated by the estimator $\overline{\theta}_T$, proved that the estimator

$$\widetilde{\theta}_n = \left(\frac{1}{H\Gamma(2H)n}\sum_{i=1}^n X_{t_{i-1}}^2\right)^{-1/(2H)} \tag{1.2}$$

is strongly consistent and asymptotically normal.

In the present work, we shall also prove that, in the case when $H \in (1/2, 3/4)$, the sequence

$$\left\{ \frac{\sqrt{n}}{\sigma(H,\theta)}(\theta - \widetilde{\theta}_n) \right\}_{n \geqslant 1}$$

satisfies the ASCLT (see Theorem 4.3).

The paper is organized as follows. Section 2 contains the basic tools of Malliavin calculus for the fractional Brownian motion needed throughout the paper. In Section 3 we prove the ASCLT for a sequence of random variables having the form of a ratio of two terms such that the numerator satisfies the ASCLT and the denominator is a positive term which converges almost surely to one. In Section 4, we use our ASCLT to study the ASCLT for the estimators $\widehat{\theta}_n$ and $\widetilde{\theta}_n$.

2. PRELIMINARIES

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see Nualart [14].

The fractional Brownian motion $\{B_t, t \geqslant 0\}$ with Hurst parameter $H \in (0,1)$ is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t,s) := \mathbb{E}(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Assume that B is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the sigma-field generated by B. By Kolmogorov's continuity criterion and the equality

$$\mathbb{E}\,(B_t - B_s)^2 = |s-t|^{2H}, \quad s,\ t \geqslant 0,$$

B has Hölder continuous paths of order $H - \varepsilon$ for all $\varepsilon \in (0, H)$.

Fix a time interval $[0,T]$. We denote by $\mathcal{H}$ the canonical Hilbert space associated with the fractional Brownian motion B. That is, $\mathcal{H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator functions $1\!\!1_{\{[0,t]\}},\ t \in [0,T]$, with respect to the scalar product

$$\langle 1\!\!1_{\{[0,t]\}}, 1\!\!1_{\{[0,s]\}} \rangle = R_H(t,s).$$

We denote by $|\cdot|_{\mathcal{H}}$ the associated norm. The mapping $1\!\!1_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B. We denote this isometry by

$$\varphi \mapsto B(\varphi) = \int_0^T \varphi(s)\, dB_s.$$

When $H > \frac{1}{2}$, the elements of $\mathcal{H}$ may be not functions but distributions of negative order (see Pipiras and Taqqu [15]). Therefore, it is of interest to know significant subspaces of functions contained in it.

Let $|\mathcal{H}|$ be the set of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}|} := H(2H-1)\int_0^T\int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2}dudv < \infty.$$

Note that if $\varphi,\ \psi \in |\mathcal{H}|$, then

$$\mathbb{E}\big(B(\varphi)B(\psi)\big) = H(2H-1)\int_0^T\int_0^T \varphi(u)\psi(v)|u-v|^{2H-2}dudv.$$

It follows actually from Pipiras and Taqqu [15] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. Moreover, one has

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{2.1}$$

Let $C_b^\infty(\mathbb{R}^n,\mathbb{R})$ be the class of infinitely differentiable functions $f:\mathbb{R}^n\to\mathbb{R}$ such that f and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of cylindrical random variables F of the form

$$F = f\big(B(\varphi_1),\dots,B(\varphi_n)\big), \tag{2.2}$$

where $n\geqslant 1$, $f\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ and $\varphi_1,\dots,\varphi_n\in\mathcal{H}$. The derivative operator D of a cylindrical random variable F of the form (2.2) is defined as the $\mathcal{H}$-valued random variable

$$D_tF = \sum_{i=1}^N \frac{\partial f}{\partial x_i}\big(B(\varphi_1),\dots,B(\varphi_n)\big)\varphi_i(t).$$

In this way the derivative DF is an element of $L^2(\Omega;\mathcal{H})$. For $p\geqslant 1$, let $D^{1,p}$ be the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|^p_{1,p} = \mathbb{E}(\|F\|^p) + \mathbb{E}(\|DF\|^p_{\mathcal{H}}).$$

The divergence operator δ is the adjoint of the derivative operator D. Concretely, a random variable $u\in L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator $\mathrm{Dom}(\delta)$ if, for every $F\in\mathcal{S}$,

$$\mathbb{E}\,|\langle DF,u\rangle_{\mathcal{H}}| \leqslant c\|F\|_{L^2(\Omega)}.$$

In this case $\delta(u)$ is given by the duality relationship

$$\mathbb{E}\big(F\delta(u)\big) = \mathbb{E}\,\langle DF,u\rangle_{\mathcal{H}}$$

for any $F\in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s dB_s, \quad u\in\mathrm{Dom}(\delta).$$

In particular, for $h\in\mathcal{H}$, $B(h)=\delta(h)=\int_0^T h_s dB_s$.

For every $n \geqslant 1$, let $\mathcal{H}_n$ be the n-th Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}}=1\}$, where H_n is the n-th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\odot n}} = \frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H}^{\odot n}$ the following multiplication formula holds:

$$\mathbb{E}\big(I_n(f)I_n(g)\big) = n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}}.$$

On the other hand, it is well known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where $f_n \in \mathcal{H}^{\odot n}$ are uniquely determined by F.

Let $\{e_n, n \geqslant 1\}$ be a complete orthonormal system in $\mathcal{H}$. Given $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the r-th contraction of f and g is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{i_1=1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathcal{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathcal{H}^{\otimes r}}.$$

In particular, note that $f \otimes_0 g = f \otimes g$ and, when $p = q$, $f \otimes_p g = \langle f, g\rangle_{\mathcal{H}^{\otimes p}}$. Since, in general, the contraction $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathcal{H}^{\odot(p+q-2r)}$. When $f \in \mathcal{H}^{\odot q}$, we write $I_q(f)$ to indicate its q-th multiple integral with respect to X. The following formula is useful to compute the product of such multiple integrals: if $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{2.3}$$

Let us now recall the criterion of Ibragimov and Lifshits [9], which plays a crucial role in Bercu et al. [2] to study ASCLTs for sequences of functionals of general Gaussian fields.

THEOREM 2.1 (Ibragimov and Lifshits [9]). *Let $\{G_n\}$ be a sequence of random variables converging in distribution towards a random variable G_∞, and set*

$$\Delta_n(t) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\big(e^{itG_k} - \mathbb{E}(e^{itG_\infty})\big).$$

Assume that, for all $r > 0$,

$$\sup_{|t|\leqslant r} \sum_n \frac{E|\Delta_n(t)|^2}{n \log n} < \infty.$$

Then, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$, *one has*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\big(\varphi(G_\infty)\big).$$

For the rest of the paper, we will use the standard notation $\phi(z) := P(N \leqslant z)$, where N is an $\mathcal{N}(0,1)$ random variable. We will denote by $C(\theta, H)$ a generic positive constant which depends only on θ and H.

3. ALMOST SURE CENTRAL LIMIT THEOREMS

In this section we shall state and prove our results concerning the ASCLT for the sequences of $\mathbb{R}$-valued random variables of the form $\{G_n/R_n\}_{n\geqslant 1}$ and $\{G_n + R_n\}_{n\geqslant 1}$.

THEOREM 3.1. *Let* $\{G_n\}_{n\geqslant 1}$ *be a sequence of* $\mathbb{R}$*-valued random variables satisfying the ASCLT. Let* $\{R_n\}_{n\geqslant 1}$ *be a sequence of positive random variables converging almost surely to one. Then* $\{G_n/R_n\}_{n\geqslant 1}$ *satisfies the ASCLT. In other words, if* N *is an* $\mathcal{N}(0,1)$ *random variable, then, almost surely, for all* $z \in \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{G_k \leqslant z R_k\}} \xrightarrow[n\to\infty]{a.s.} \phi(z).$$

THEOREM 3.2. *Let* $\{G_n\}_{n\geqslant 1}$ *be a sequence of* $\mathbb{R}$*-valued random variables satisfying the ASCLT. Let* $\{R_n\}_{n\geqslant 1}$ *be a sequence of* $\mathbb{R}$*-valued random variables converging almost surely to zero. Then* $\{G_n + R_n\}_{n\geqslant 1}$ *satisfies the ASCLT. In other words, almost surely, for all* $z \in \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{G_k + R_k \leqslant z\}} \xrightarrow[n\to\infty]{a.s.} \phi(z).$$

REMARK 3.1. *A similar result to Theorem* 3.2 *for the ASCLT of the sequence* $\{G_n + R_n\}_{n\geqslant 1}$, *where* $\{R_n\}_{n\geqslant 1}$ *converges in* $L^2(\Omega)$ *to zero, and such that*

$$\sum_{n\geqslant 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}|R_k|^2 < \infty$$

was established by Nourdin and Peccati in [13].

The proofs of Theorems 3.1 and 3.2 are respectively direct consequences of the following two lemmas:

LEMMA 3.1. *Let $\{G_n\}_{n\geqslant 1}$ and $\{R_n\}_{n\geqslant 1}$ be two sequences of real-valued random variables. Define*

$$U_{n,\varepsilon} := \left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z(1-\varepsilon)\}} - \phi\big(z(1-\varepsilon)\big)\right|, \tag{3.1}$$

$$V_{n,\varepsilon} := \left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z(1+\varepsilon)\}} - \phi\big(z(1+\varepsilon)\big)\right|. \tag{3.2}$$

Then, for all $z\in\mathbb{R}$ and $\varepsilon>0$,

$$\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant zR_k\}} - \phi(z)\right| \leqslant \max(U_{n,\varepsilon}, V_{n,\varepsilon}) + \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{|R_k-1|\geqslant\varepsilon\}} + \varepsilon.$$

LEMMA 3.2. *Let $\{S_n\}_{n\geqslant 1}$ and $\{R_n\}_{n\geqslant 1}$ be two sequences of real-valued random variables. Define*

$$T_{n,\eta} := \left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z+\eta\}} - \phi(z+\eta)\right|, \tag{3.3}$$

$$W_{n,\eta} := \left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z-\eta\}} - \phi(z-\eta)\right|. \tag{3.4}$$

Then, for all $z\in\mathbb{R}$ and $\eta>0$,

$$\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k+R_k\leqslant z\}} - \phi(z)\right| \leqslant \max(T_{n,\eta}, W_{n,\eta}) + \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{|R_k|>\eta\}} + \frac{\eta}{\sqrt{2\pi}}.$$

Proof of Lemma 3.1. It is inspired by Lemma 1 from Michael and Pfanzagl [12], p. 78. The case $\varepsilon\geqslant 1$ is easy. We now assume that $\varepsilon\in(0,1)$. When $z\geqslant 0$, using the inclusion

$$\{G_k\leqslant(1-\varepsilon)z\}\subset\{G_k\leqslant zR_k\}\cup\{R_k\leqslant 1-\varepsilon\},$$

we have

$$\mathbb{1}_{\{G_k\leqslant z(1-\varepsilon)\}} \leqslant \mathbb{1}_{\{G_k\leqslant zR_k\}} + \mathbb{1}_{\{|R_k-1|\geqslant\varepsilon\}}. \tag{3.5}$$

Since, for every $x\geqslant 0$, $xe^{-x^2/2}\leqslant e^{-1/2}$, we get

$$\big|\phi(z)-\phi\big(z(1-\varepsilon)\big)\big| \leqslant \min\left(\frac{1}{2}, \frac{z\varepsilon}{\sqrt{2\pi}}\exp\left(\frac{-z^2(1-\varepsilon)^2}{2}\right)\right) \leqslant \varepsilon. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k\leqslant zR_k\}}-\phi(z)\geqslant -U_{n,\varepsilon}-\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k-1|\geqslant\varepsilon\}}-\varepsilon.$$

Now, when $z\leqslant 0$, the inclusion $\{G_k\leqslant(1+\varepsilon)z\}\subset\{G_k\leqslant zR_k\}\cup\{R_k\geqslant 1+\varepsilon\}$ leads to

$$1\!\!1_{\{G_k\leqslant z(1+\varepsilon)\}}\leqslant 1\!\!1_{\{G_k\leqslant zR_k\}}+1\!\!1_{\{|R_k-1|\geqslant\varepsilon\}}.$$

Moreover, since

$$\big|\phi(z)-\phi\big(z(1+\varepsilon)\big)\big|\leqslant\frac{|z|\varepsilon}{\sqrt{2\pi}}\exp\left(\frac{-z^2(1+\varepsilon)^2}{2}\right)\leqslant\varepsilon,$$

we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k\leqslant zR_k\}}-\phi(z)\geqslant -V_{n,\varepsilon}-\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k-1|\geqslant\varepsilon\}}-\varepsilon.$$

Thus, for every $z\in\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k\leqslant zR_k\}}-\phi(z)\geqslant-\max(U_{n,\varepsilon},V_{n,\varepsilon})-\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k-1|\geqslant\varepsilon\}}-\varepsilon.$$

Following the same guidelines as above and using

$$\begin{aligned}&\{G_k\leqslant zR_k\}\subset\{G_k\leqslant(1+\varepsilon)z\}\cup\{R_k\geqslant 1+\varepsilon\}\quad\text{for } z\geqslant 0,\\&\{G_k\leqslant zR_k\}\subset\{G_k\leqslant(1-\varepsilon)z\}\cup\{R_k\leqslant 1-\varepsilon\}\quad\text{for } z\leqslant 0\end{aligned}$$

we get, for every $z\in\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k\leqslant zR_k\}}-\phi(z)\leqslant\max(U_{n,\varepsilon},V_{n,\varepsilon})+\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k-1|\geqslant\varepsilon\}}+\varepsilon.$$

This completes the proof of Lemma 3.1. ■

Proof of Lemma 3.2. Fix $z\in\mathbb{R}$ and $\eta>0$. Remark that

$$\{G_k+R_k\leqslant z\}\subset\{G_k\leqslant z+\eta\}\cup\{|R_k|>\eta\}.$$

Thus we obtain

$$\begin{aligned}&\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k+R_k\leqslant z\}}-\phi(z)\\&\leqslant\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{G_k\leqslant z+\eta\}}-\phi(z+\eta)+\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k|>\eta\}}+\phi(z+\eta)-\phi(z)\\&\leqslant T_{n,\eta}+\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1\!\!1_{\{|R_k|>\eta\}}+\frac{\eta}{\sqrt{2\pi}}.\end{aligned}$$

On the other hand, it follows from the inclusion

$$\{G_k \leqslant z-\eta\} \subset \{G_k + R_k \leqslant z\} \cup \{|R_k| > \eta\}$$

that

$$\begin{aligned}
&\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k+R_k\leqslant z\}} - \phi(z)\\
&\geqslant \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z+\eta\}} - \phi(z-\eta) - \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{|R_k|>\eta\}} + \phi(z-\eta) - \phi(z)\\
&\geqslant -W_{n,\eta} - \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{|R_k|>\eta\}} - \frac{\eta}{\sqrt{2\pi}}.
\end{aligned}$$

The desired conclusion follows. ■

4. APPLICATION TO LSE FOR FRACTIONAL ORNSTEIN–UHLENBECK PROCESS

First we recall a result of [2] concerning the ASCLT for multiple stochastic integrals.

THEOREM 4.1 (Bercu et al. [2]). *Let $q \geqslant 2$ be an integer and let $\{G_n\}_{n\geqslant 1}$ be a sequence of the form $G_n = I_q(f_n)$ with $f_n \in \mathcal{H}^{\odot q}$. Assume that $\mathbb{E}[G_n^2] = q!\|f_n\|^2_{\mathcal{H}^{\otimes q}} = 1$ for all n and that G_n converges in distribution towards a standard Gaussian. Moreover, assume that*

$$\sum_{n=2}^{\infty}\frac{1}{n\log^2 n}\sum_{k=1}^{n}\frac{1}{k}\|f_k\otimes_r f_k\|_{\mathcal{H}^{\otimes 2(q-r)}} < \infty \text{ for every } 1\leqslant r\leqslant q-1, \tag{4.1}$$

$$\sum_{n=2}^{\infty}\frac{1}{n\log^3 n}\sum_{k,l=1}^{n}\frac{|\langle f_k, f_l\rangle_{\mathcal{H}^{\otimes q}}|}{kl} < \infty. \tag{4.2}$$

Then $\{G_n\}_{n\geqslant 1}$ satisfies an ASCLT. In other words, almost surely, for all $z\in\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{1}_{\{G_k\leqslant z\}} \xrightarrow[n\to\infty]{a.s.} \phi(z)$$

or, equivalently, almost surely, for any bounded and continuous function $\varphi:\mathbb{R}\to\mathbb{R}$, we have

$$\frac{1}{\log(n)}\sum_{k=1}^{n}\frac{1}{k}\varphi(G_k) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\varphi(N).$$

4.1. Continuous case. In this section we apply Theorem 3.1 to a least squares estimator for fractional Ornstein–Uhlenbeck processes based on continuous-time observations.

Let us consider the fractional Ornstein–Uhlenbeck process $X = \{X_t, t \geqslant 0\}$ given by the linear stochastic differential equation

$$X_0 = 0 \quad \text{and} \quad dX_t = -\theta X_t dt + dB_t,\ t \geqslant 0, \tag{4.3}$$

where $B = \{B_t, t \geqslant 0\}$ is a fractional Brownian motion of Hurst index $H \in \left(\frac{1}{2}, 1\right)$ and θ is a real unknown parameter. Let $\widehat{\theta}_t$ be a least squares estimator (LSE) of θ given by

$$\widehat{\theta}_t = -\frac{\int_0^t X_s\, \delta X_s}{\int_0^t X_s^2 ds}, \quad t > 0. \tag{4.4}$$

This LSE is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \mapsto \int_0^t |\dot{X}_s + \theta X_s|^2 ds.$$

The linear equation (4.3) has the following explicit solution:

$$X_t = e^{-\theta t} \int_0^t e^{\theta s} dB_s, \quad t > 0. \tag{4.5}$$

Using the equations (4.3) and (4.5) we can write the LSE $\{\widehat{\theta}_t\}$ defined in (4.4) as follows:

$$\widehat{\theta}_t - \theta = -\frac{\int_0^t X_s \delta B_s}{\int_0^t X_s^2 ds} = -\frac{\int_0^t \delta B_s e^{\theta s} \int_0^s \delta B_r e^{-\theta r}}{\int_0^t X_s^2 ds}. \tag{4.6}$$

Thus, we have

$$\sqrt{t}(\theta - \widehat{\theta}_t) = \frac{F_t}{t^{-1} \int_0^t X_s^2 ds}, \quad t > 0, \tag{4.7}$$

where

$$F_t := I_2(f_t)$$

is a multiple integral of f_t with

$$f_t(u, v) = \frac{1}{2\sqrt{t}} e^{-\theta|u-v|} \mathbb{1}^{\otimes 2}_{\{[0,t]\}}(u, v).$$

Until the end of this paper we will use the following notation for all $t > 0$:

$$\sigma_t = \lambda(\theta, H)\sqrt{\mathbb{E}(F_t^2)} \quad \text{with } \lambda(\theta, H) := \theta^{-2H} H\Gamma(2H). \tag{4.8}$$

We are now ready to state the main result of this subsection. First we recall some results of Hu and Nualart [6] needed throughout the paper:

$$\mathbb{E}(F_t^2) \xrightarrow[t\to\infty]{} A(\theta, H), \tag{4.9}$$

where

$$A(\theta, H) = \theta^{1-4H}\left(H^2(4H-1)\left[\Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)}\right]\right).$$

Moreover, for every $t \geqslant 0$

$$\mathbb{E}[(\|DF_t\|^2_{\mathcal{H}} - \mathbb{E}\|DF_t\|^2_{\mathcal{H}})^2] \leqslant C(\theta, H)t^{8H-6}, \tag{4.10}$$

and as $t \to \infty$

$$F_t \xrightarrow{\mathrm{d}} N \sim \mathcal{N}\big(0, A(\theta, H)\big) \tag{4.11}$$

(where $\xrightarrow{\mathrm{d}}$ means convergence in distribution). At last, we have the convergence

$$\frac{1}{t}\int_0^t X_s^2 ds \xrightarrow[t\to\infty]{\text{a.s.}} \lambda(\theta, H) \tag{4.12}$$

as $t \to \infty$.

THEOREM 4.2. *Assume $H \in (1/2, 3/4)$. Then, almost surely, for all $z \in \mathbb{R}$,*

$$\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}\mathbb{1}_{\{(\sqrt{k}/\sigma_k)(\theta-\widehat{\theta}_k)\leqslant z\}} \xrightarrow[n\to\infty]{} \phi(z)$$

or, equivalently, almost surely, for any bounded and continuous function φ

$$\frac{1}{\log n}\sum_{k=1}^n \frac{1}{k}\varphi\left(\frac{\sqrt{k}}{\sigma_k}(\theta-\widehat{\theta}_k)\right) \xrightarrow[n\to\infty]{} \mathbb{E}\big(\varphi(N)\big).$$

Proof. Let us consider, for each $t > 0$,

$$G_t = \frac{1}{\sqrt{\mathbb{E}(F_t^2)}}F_t = \frac{1}{\sqrt{\mathbb{E}(F_t^2)}}I_2(f_t)$$

and

$$R_t = \frac{1}{\lambda(\theta, H)t}\int_0^t X_s^2 ds.$$

Thus, (4.7) leads to

$$\frac{\sqrt{n}}{\sigma_n}(\theta - \widehat{\theta}_n) = G_n/R_n, \quad n \geqslant 1.$$

It follows from (4.12) that R_n converges almost surely to one as n tends to infinity. Then, using Theorem 3.1 it suffices to show that $\{G_n\}_{n\geqslant 1}$ satisfies the ASCLT. To do that, it is sufficient to prove that $\{G_n\}_{n\geqslant 1}$ satisfies the conditions of Theorem 4.1.

We have $E(G_n^2) = 1$. In addition, the convergence of G_n towards the standard Gaussian is a straightforward consequence of (4.9) and (4.11). It remains to fulfill the conditions (4.1) and (4.2). Hence, we shall prove that

$$I = \sum_{n\geqslant 2} \frac{1}{n\log^2(n)} \sum_{k=1}^{n} \frac{1}{k}\|f_k \otimes_1 f_k\|_{\mathcal{H}^{\otimes 2}} < \infty, \tag{4.13}$$

and

$$J = \sum_{n\geqslant 2} \frac{1}{n\log^3(n)} \sum_{k,l=1}^{n} \frac{|\langle f_k, f_l\rangle_{\mathcal{H}^{\otimes 2}}|}{kl} < \infty. \tag{4.14}$$

Let us deal with the first convergence (4.13). For every $t > 0$, we have

$$\mathbb{E}[(\|DF_t\|^2_{\mathcal{H}} - \mathbb{E}\|DF_t\|^2_{\mathcal{H}})^2] = 16\|f_t \otimes_1 f_t\|^2_{\mathcal{H}^{\otimes 2}}. \tag{4.15}$$

Combining (4.10) and (4.15) we obtain

$$I \leqslant C(\theta, H) \sum_{n\geqslant 2} \frac{1}{n\log^2(n)} \sum_{k=1}^{n} \frac{1}{k^{4-4H}} \tag{4.16}$$

and, consequently,

$$I \leqslant C(\theta, H) \sum_{n\geqslant 2} \frac{1}{n^{4-4H}} < \infty, \tag{4.17}$$

since $H < 3/4$, where $C(\theta, H)$ is a generic constant depending only on θ, H.

Now, we prove (4.14). Let $k < l$. Then for some $k^* \in [0, k]$ we have

$$\begin{aligned}
|\langle f_k, f_l\rangle_{\mathcal{H}}| &= H^2(2H-1)^2 \frac{1}{\sqrt{kl}} \\
&\quad \times \int_{[0,k]^2} dxdu\, e^{-\theta|x-u|} \int_{[0,l]^2} dydv\, e^{-\theta|y-v|} |x-y|^{2H-2}|u-v|^{2H-2} \\
&= 2H^2(2H-1)^2 \sqrt{\frac{k}{l}} \int_{[0,k^*]} du\, e^{-\theta|k^*-u|} \\
&\quad \times \int_{[0,l]^2} dydv\, e^{-\theta|y-v|}|k^* - y|^{2H-2}|u-v|^{2H-2} \\
&:= 2H^2(2H-1)^2 \sqrt{\frac{k}{l}}(D^{(1)} + D^{(2)} + D^{(3)} + D^{(4)}).
\end{aligned}$$

Moreover, the first term can be bounded above by

$$\begin{aligned} D^{(1)} &= \int\limits_{[0,k^*]} du\, e^{-\theta(k^*-u)} \int\limits_{[0,k^*]^2} dydv\, e^{-\theta|y-v|}(k^*-y)^{2H-2}|u-v|^{2H-2} \\ &= \int\limits_{[0,k^*]^3} e^{-\theta u}\, e^{-\theta|y-v|} y^{2H-2}|u-v|^{2H-2} dudvdy \\ &\leqslant \int\limits_{[0,\infty)^3} e^{-\theta u}\, e^{-\theta|y-v|} y^{2H-2}|u-v|^{2H-2} dudvdy < \infty. \end{aligned}$$

The last inequality is a consequence of the proof of Lemma 5.3 (see only web Appendix) in [6]. Following the same guidelines, we get for the other terms:

$$\begin{aligned} D^{(2)} &= \int\limits_{[0,k^*]} du\, e^{-\theta(k^*-u)} \int\limits_{[k^*,l]^2} dydv\, e^{-\theta|y-v|}(y-k^*)^{2H-2}|u-v|^{2H-2} \\ &= \int\limits_{[0,k^*]} du\, e^{-\theta u} \int\limits_{[0,l-k^*]^2} dydv\, e^{-\theta|y-v|} y^{2H-2}(u+v)^{2H-2} \\ &\leqslant \int\limits_{[0,\infty)^3} e^{-\theta u}\, e^{-\theta|y-v|} y^{2H-2}|u-v|^{2H-2} dudvdy < \infty, \end{aligned}$$

$$\begin{aligned} D^{(3)} &= \int\limits_{[0,k^*]} du\, e^{-\theta(k^*-u)} \int\limits_{[0,k^*]} dy \int\limits_{[k^*,l]} dv\, e^{-\theta|y-v|}(k^*-y)^{2H-2}|u-v|^{2H-2} \\ &= \int\limits_{[0,k^*]} du\, e^{-\theta u} \int\limits_{[0,k^*]} dy \int\limits_{[0,l-k^*]} dv\, e^{-\theta(y+v)} y^{2H-2}(u+v)^{2H-2} \\ &\leqslant \int\limits_{[0,\infty)^3} e^{-\theta u}\, e^{-\theta|y-v|} y^{2H-2}|u-v|^{2H-2} dudvdy < \infty, \end{aligned}$$

and

$$\begin{aligned} D^{(4)} &= \int\limits_{[0,k^*]} du\, e^{-\theta(k^*-u)} \int\limits_{[k^*,l]} dy \int\limits_{[0,k^*]} dv\, e^{-\theta|y-v|}(y-k^*)^{2H-2}|u-v|^{2H-2} \\ &= \int\limits_{[0,k^*]} du\, e^{-\theta u} \int\limits_{[0,l-k^*]} dy \int\limits_{[0,k^*]} dv\, e^{-\theta(y+v)} y^{2H-2}|u-v|^{2H-2} \\ &\leqslant \int\limits_{[0,\infty)^3} e^{-\theta u}\, e^{-\theta|y-v|} y^{2H-2}|u-v|^{2H-2} dudvdy < \infty. \end{aligned}$$

Thus, we deduce that, for every $k < l$,

$$|\langle f_k, f_l \rangle_{\mathcal{H}}| = C(\theta, H)\sqrt{\frac{k}{l}}.$$

Consequently, we obtain

$$J \leqslant C(\theta, H) \sum_{n\geqslant 2} \frac{1}{n\log^3(n)} \sum_{l=1}^{n} \frac{1}{l^{3/2}} \sum_{k=1}^{l} \frac{1}{\sqrt{k}} \leqslant C(\theta, H) \sum_{n\geqslant 2} \frac{1}{n\log^3(n)} \sum_{l=1}^{n} \frac{1}{l} \leqslant C(\theta, H) \sum_{n\geqslant 2} \frac{1}{n\log^2(n)} < \infty, \tag{4.18}$$

which concludes the proof. ∎

4.2. Discrete case. Consider the fractional Ornstein–Uhlenbeck process $X = \{X_t, t \geqslant 0\}$ defined in (4.3). Assume that the process X is observed equidistantly in time with the step size $h > 0$: $t_i = ih, i = 0, \dots, n$.

THEOREM 4.3. *Assume $H \in (1/2, 3/4)$. Let $\widetilde{\theta}_n$ be the estimator of θ defined in* (1.2). *Then, almost surely, for all $z \in \mathbb{R}$,*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{(\sqrt{n}/\sigma(H,\theta))(\theta - \widetilde{\theta}_k) \leqslant z\}} \xrightarrow[n\to\infty]{a.s.} \phi(z),$$

or, equivalently, for any bounded and continuous function φ,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi\left(\frac{\sqrt{n}}{\sigma(H,\theta)}(\theta - \widetilde{\theta}_k)\right) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\big(\varphi(N)\big),$$

where $\sigma(H,\theta) > 0$ is a constant depending only on H and θ.

Proof. Setting

$$Q_n := \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}^2,$$

we can write

$$\widetilde{\theta}_n = \left(\frac{Q_n}{H\Gamma(2H)}\right)^{-1/(2H)}. \tag{4.19}$$

Let us recall that (see [7]), as $n \to \infty$,

$$\widetilde{\theta}_n \xrightarrow{\text{a.s.}} \theta \tag{4.20}$$

and

$$\frac{\sqrt{n}}{\sigma(H,\theta)}(\theta - \widetilde{\theta}_n) \xrightarrow{\text{d}} \mathcal{N}(0,1). \tag{4.21}$$

We have

$$\frac{\sqrt{n}}{\sigma(H,\theta)}(\theta-\widetilde{\theta}_n)=\xi_n^{-1/(2H)-1}\frac{\sqrt{n}}{2H\sigma(H,\theta)}\left(\frac{Q_n}{H\Gamma(2H)}-\theta^{-2H}\right),$$

where ξ_n is a random variable between $Q_n/\big(H\Gamma(2H)\big)$ and θ^{-2H}. The convergence (4.20) leads to $\theta^{-2H-1}\xi_n^{-1/(2H)-1}\to 1$ almost surely as $n\to\infty$. Then, using Theorem 3.1 it suffices to show that

$$\left\{\frac{\theta^{2H+1}\sqrt{n}}{2H\sigma(H,\theta)}\left(\frac{Q_n}{H\Gamma(2H)}-\theta^{-2H}\right)\right\}_{n\geqslant 1}$$

satisfies the ASCLT. On the other hand,

$$\frac{\theta^{2H+1}\sqrt{n}}{2H\sigma(H,\theta)}\left(\frac{Q_n}{H\Gamma(2H)}-\theta^{-2H}\right):=\overline{G}_n+\overline{R}_n,$$

where

$$\overline{G}_n=\frac{\theta^{2H+1}\sqrt{n}}{2H\sigma(H,\theta)}\left(\frac{Q_n-\mathbb{E}Q_n}{H\Gamma(2H)}\right)\in\mathcal{H}_2,$$

and from [7] it follows that

$$\overline{R}_n=\frac{\theta^{2H+1}\sqrt{n}}{2H\sigma(H,\theta)}\left(\frac{\mathbb{E}Q_n}{H\Gamma(2H)}-\theta^{-2H}\right)$$

converges to zero as $n\to\infty$. Hence, using Theorem 3.2 it remains to prove that $\{\overline{G}_n\}_{n\geqslant 1}$ satisfies the ASCLT. The conditions (4.1) and (4.2) are satisfied by using the following estimates inspired by Hu and Song [7]:

$$\mathbb{E}[(\|D\overline{G}_n\|_{\mathcal{H}}^2-\mathbb{E}\|D\overline{G}_n\|_{\mathcal{H}}^2)^2]\leqslant C(\theta,H)\frac{1}{n^{8H-6}},$$

and for all $k\leqslant l$

$$|\mathbb{E}[\overline{G}_k\overline{G}_l]|\leqslant C(\theta,H)\sqrt{\frac{k}{l}}.$$

Thus the proof of Theorem 4.3 is completed. ■

REFERENCES

[1] R. Belfadli, K. Es-Sebaiy, and Y. Ouknine, *Parameter estimation for fractional Ornstein–Uhlenbeck processes: Non-ergodic case*, Frontiers in Science and Engineering. An International Journal Edited by Hassan II Academy of Science and Technology 1 (1) (2011), pp. 1–16.

[2] B. Bercu, I. Nourdin, and M. Taqqu, *Almost sure central limit theorem on the Wiener space*, Stochastic Process. Appl. 120 (2010), pp. 1607–1628.

[3] I. Berkes, *Results and problems related to the pointwise central limit theorem*, in: *Asymptotic Methods in Probability and Statistics: A Volume in Honour of Miklós Csörgő*, B. Szyszkowicz (Ed.), Elsevier, Amsterdam 1998, pp. 59–96.

[4] I. Berkes and E. Csáki, *A universal result in almost sure central limit theory*, Stochastic Process. Appl. 94 (1) (2001), pp. 105–134.

[5] G. A. Brosamler, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. 104 (3) (1988), pp. 561–574.

[6] Y. Hu and D. Nualart, *Parameter estimation for fractional Ornstein–Uhlenbeck processes*, Statist. Probab. Lett. 80 (2010), pp. 1030–1038.

[7] Y. Hu and J. Song, *Parameter estimation for fractional Ornstein–Uhlenbeck processes with discrete observations*, in: *Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart*, F. Viens et al. (Eds.), Springer Proc. Math. Statist. 34 (2013), pp. 427–442.

[8] I. A. Ibragimov and M. A. Lifshits, *On the convergence of generalized moments in almost sure central limit theorem*, Statist. Probab. Lett. 40 (4) (1998), pp. 343–351.

[9] I. A. Ibragimov and M. A. Lifshits, *On limit theorems of almost sure type*, Theory Probab. Appl. 44 (2) (2000), pp. 254–272.

[10] M. T. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Lett. 9 (1990), pp. 201–205.

[11] P. Lévy, *Théorie de l'addition des variables aléatoires. Monographies des probabilites*, Gauthier-Villars, Paris 1937.

[12] R. Michael and J. Pfanzagl, *The accuracy of the normal approximation for minimum contrast estimate*, Z. Wahrsch. Verw. Gebiete 18 (1971), pp. 37–84.

[13] I. Nourdin and G. Peccati, *Universal Gaussian fluctuations of non-Hermitian matrix ensembles: From weak convergence to almost sure CLTs*, ALEA 7 (2010), pp. 341–375.

[14] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, second edition, 2006.

[15] V. Pipiras and M. S. Taqqu, *Integration questions related to fractional Brownian motion*, Probab. Theory Related Fields 118 (2) (2000), pp. 251–291.

[16] P. Schatte, *On strong versions of the central limit theorem*, Math. Nachr. 137 (1988), pp. 249–256.

Peggy Cénac
Institut de Mathématiques de Bourgogne
Université de Bourgogne
9 Rue Alain Savary, 21078 Dijon, France
E-mail: peggy.cenac@u-bourgogne.fr

Khalifa Es-Sebaiy
National School of Applied Sciences – Marrakesh
Cadi Ayyad University
Av. Abdelkarim Khattabi, Gueliz, Marrakesh
Morocco
E-mail: k.essebaiy@uca.ma

Received on 4.4.2014;
revised version on 18.12.2014