# A CONSTANT REGRESSION CHARACTERIZATION OF THE MARCHENKO–PASTUR LAW

BY

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*Abstract.* In this paper, Lukacs type characterization of Marchenko–Pastur distribution in free probability is studied. We prove that for free $\mathbb{X}$ and $\mathbb{Y}$, if conditional moments of order $1$ and $-1$ of $(\mathbb{X} + \mathbb{Y})^{-1/2}\mathbb{X}(\mathbb{X} + \mathbb{Y})^{-1/2}$ given $\mathbb{X} + \mathbb{Y}$ are constant, then $\mathbb{X}$ and $\mathbb{Y}$ follow the Marchenko–Pastur distribution.

**2010 AMS Mathematics Subject Classification:** Primary: 46L54; Secondary: 62E10.

**Key words and phrases:** Lukacs characterization, free Poisson distribution, free cumulants.

## 1. INTRODUCTION

Since the publication of the paper [13] free probability theory has been developed in many various directions. It turns out that many classical results for independent random variables such as, for example, the Central Limit Theorem have their free analogues. One of the deepest relations between classical and free probability is constituted by so-called Bercovici–Pata bijections which give bijection between infinitely divisible distributions with respect to free and classical convolution.

In this paper we are interested in characterization problems in free probability. This seems to be another field which gives some interesting connections between classical and free probability. Our result is a new example of known, but not completely well understood phenomena of analogies between characterizations in classical and free probability. A basic example of such analogy is Bernstein's theorem which characterizes the Gaussian distribution by independence of $X + Y$ and $X - Y$ for independent $X$ and $Y$. In [7] it is proved that a similar result holds for the Wigner semicircle law when independence is replaced by freeness assumption.

The main result of this paper is closely related to the Lukacs theorem which provides a characterization of the gamma distribution by independence of $V = X + Y$ and $U = X/(X + Y)$ for independent $X$ and $Y$ (see [6]). It is known

that the assumption of independence of $U$ and $V$ can be replaced by a weaker assumption, i.e. the assumption of constancy of regressions $\mathbb{E}(U|V)$ and $\mathbb{E}(U^2|V)$ (see [5]). In [15] it is proved that constancy of regressions $\mathbb{E}(U|V)$ and $\mathbb{E}(U^{-1}|V)$ also characterizes the gamma distribution.

The Lukacs property was also studied in a context of free probability in [1] where the Laha–Lukacs regression of free Meixner family is considered (see also [3] and [4]). Theorem 3.1 from [1] contains, as a special case, a free analogue of the Lukacs regressions in the case of constancy of regressions of $\mathbb{U}$ and $\mathbb{U}^2$ given by $\mathbb{V}$. It turns out that such conditions characterize the Marchenko–Pastur (free Poisson) distribution. The proof that the Marchenko–Pastur distributed, free $\mathbb{X}$ and $\mathbb{Y}$ have the property that $\mathbb{V} = \mathbb{X} + \mathbb{Y}$ and $\mathbb{U} = (\mathbb{X} + \mathbb{Y})^{-1/2}\mathbb{X}(\mathbb{X} + \mathbb{Y})^{-1/2}$ are free can be found in [10]. The aim of this paper is to complete the picture of the analogy of the Lukacs independence property in classical and free probability. We will prove a free analogue of the result from [15]. The proof of the main result relies mainly on the technique developed in our previous papers [11], [9].

The paper is organized as follows: in Section 2 we briefly introduce basic notions of free probability and known facts which are needed to prove the main result. In Section 3 we state and prove the main result of the paper.

## 2. PRELIMINARIES

In this section we give a collection of facts which we need in this paper. For a more detailed introduction we refer to our previous papers [11], [9]. A comprehensive introduction to free probability can be found in [8] or [14].

By a *non-commutative probability space* we understand a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra, and $\varphi$ is a faithful, normal, tracial state; elements of $\mathcal{A}$ are called *random variables*.

We say that probability measure $\mu$ is the *distribution of a self-adjoint random variable* $\mathbb{X} \in \mathcal{A}$ if $\varphi(\mathbb{X}^n) = \int_{\mathbb{R}} t^n d\mu(t)$ for $n = 1, 2, \ldots$

Let $\chi = \{B_1, B_2, \ldots\}$ be a partition of the set of numbers $\{1, \ldots, k\}$. The partition $\chi$ is a *crossing partition* if there exist distinct blocks $B_r, B_s \in \chi$ and numbers $i_1, i_2 \in B_r$, $j_1, j_2 \in B_s$ such that $i_1 < j_1 < i_2 < j_2$. Otherwise, $\chi$ is called a *non-crossing partition*. The set of all non-crossing partitions of $\{1, \ldots, k\}$ is denoted by $NC(k)$.

For any $k = 1, 2, \ldots$, (joint) cumulants of order $k$ of non-commutative random variables $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are defined recursively as $k$-linear maps $\mathcal{R}_k : \mathcal{A} \to \mathbb{C}$ by the equations

$$\varphi(\mathbb{Y}_1 \cdot \ldots \cdot \mathbb{Y}_m) = \sum_{\chi \in NC(m)} \prod_{B \in \chi} \mathcal{R}_{|B|}(\mathbb{Y}_i, \, i \in B)$$

which are satisfied for any $\mathbb{Y}_i \in \{\mathbb{X}_1, \ldots, \mathbb{X}_n\}$, $i = 1, \ldots, m$, and any $m = 1, 2, \ldots$, with $|B|$ denoting the number of elements in the block $B$.

The notion of *freeness* can be characterized in terms of behaviour of cumulants in the following way. Consider unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of an algebra $\mathcal{A}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$. Subalgebras $(\mathcal{A}_i)_{i \in I}$ are freely independent iff for any $n = 2, 3, \ldots$ and for any $\mathbb{X}_j \in \mathcal{A}_{i(j)}$ with $i(j) \in I$, $j = 1, \ldots, n$, any $n$-cumulant

$$\mathcal{R}_n(\mathbb{X}_1, \ldots, \mathbb{X}_n) = 0$$

if there exists a pair $k, l \in \{1, \ldots, n\}$ such that $i(k) \neq i(l)$.

In the sequel we will use the following formula from [2] which connects cumulants and moments for non-commutative random variables:

(2.1)
$$\varphi(\mathbb{X}_1 \ldots \mathbb{X}_n) = \sum_{k=1}^{n} \sum_{1 < i_2 < \ldots < i_k \leqslant n} \mathcal{R}_k(\mathbb{X}_1, \mathbb{X}_{i_2}, \ldots, \mathbb{X}_{i_k}) \prod_{j=1}^{k} \varphi(\mathbb{X}_{i_j+1} \ldots \mathbb{X}_{i_{j+1}-1})$$

with $i_1 = 1$ and $i_{k+1} = n + 1$ (empty products are equal to one).

Non-commutative conditional expectation is well defined in so-called $W^*$-probability spaces, i.e. non-commutative probability spaces where the algebra $\mathcal{A}$ is a von Neumann algebra. Non-commutative conditional expectation has many properties analogous to those of classical conditional expectation. For more details one can consult, e.g., [12]. Here we state two of them which we need in the sequel.

LEMMA 2.1.  *Consider a $W^*$-probability space $(\mathcal{A}, \varphi)$.*
• *If $\mathbb{X} \in \mathcal{A}$ and $\mathbb{Y} \in \mathcal{B}$, where $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$, then*

(2.2)
$$\varphi(\mathbb{X}\,\mathbb{Y}) = \varphi\big(\varphi(\mathbb{X}|\mathcal{B})\,\mathbb{Y}\big).$$

• *If $\mathbb{X}, \mathbb{Z} \in \mathcal{A}$ are free, then*

(2.3)
$$\varphi(\mathbb{X}|\mathbb{Z}) = \varphi(\mathbb{X})\,\mathbb{I}.$$

Now we give some basic analytical tools used to deal with non-commutative random variables and their distributions.

For a non-commutative random variable $\mathbb{X}$ its $r$-transform is defined as

$$r_{\mathbb{X}}(z) = \sum_{n=0}^{\infty} \mathcal{R}_{n+1}(\mathbb{X})\,z^n.$$

In [13] it is proved that $r$-transform of a random variable with compact support is analytic in a neighbourhood of zero. From properties of cumulants it is immediate that for $\mathbb{X}$ and $\mathbb{Y}$ which are freely independent

(2.4)
$$r_{\mathbb{X}+\mathbb{Y}} = r_{\mathbb{X}} + r_{\mathbb{Y}}.$$

If $\mathbb{X}$ has the distribution $\mu$, then we will often write $r_\mu$ instead of $r_{\mathbb{X}}$. The Cauchy

transform of a probability measure $\nu$ is defined as

$$G_\nu(z) = \int_{\mathbb{R}} \frac{\nu(dx)}{z - x}, \quad \Im(z) > 0.$$

Cauchy transforms and $r$-transforms are related by

(2.5) $$G_\nu\left(r_\nu(z) + \frac{1}{z}\right) = z.$$

Finally, we define a moment generating function $M_{\mathbb{X}}$ of a random variable $\mathbb{X}$ by

$$M_{\mathbb{X}}(z) = \sum_{n=0}^{\infty} \varphi(\mathbb{X}^n)\, z^n.$$

It is easy to see that

(2.6) $$M_{\mathbb{X}}(z) = \frac{1}{z} G_{\mathbb{X}}\left(\frac{1}{z}\right).$$

We will need the following lemma proved in [9].

LEMMA 2.2. *Let $\mathbb{V}$ be a compactly supported, invertible non-commutative random variable. Define $C_n = \mathcal{R}_n(\mathbb{V}^{-1}, \mathbb{V}, \dots, \mathbb{V})$, and $C(z) = \sum_{i=1}^{\infty} C_i z^{i-1}$. Then, for $z$ in a neighbourhood of zero, we have*

(2.7) $$C(z) = \frac{z + C_1}{1 + zr(z)},$$

*where $r(z)$ is the R-transform of $\mathbb{V}$. In particular,*

(2.8) $$C_2 = 1 - C_1 \mathcal{R}_1(\mathbb{V}), \quad C_n = -\sum_{i=1}^{n-1} C_i \mathcal{R}_{n-i}(\mathbb{V}), \, n \geqslant 2.$$

The main result of this paper is a characterization of the Marchenko–Pastur distribution. A random variable $\mathbb{X}$ is said to be *Marchenko–Pastur* (or *free Poisson*) *distributed* if it has the distribution $\nu = \nu(\lambda, \alpha)$, $\lambda, \alpha > 0$, defined by the formula

(2.9) $$\nu = \max\{0,\, 1 - \lambda\}\, \delta_0 + \lambda\tilde{\nu},$$

where the measure $\tilde{\nu}$ is supported on the interval $\left(\alpha(1 - \sqrt{\lambda})^2,\, \alpha(1 + \sqrt{\lambda})^2\right)$ and has the density (with respect to the Lebesgue measure)

$$\tilde{\nu}(dx) = \frac{1}{2\pi\alpha x} \sqrt{4\lambda\alpha^2 - \left(x - \alpha(1 + \lambda)\right)^2}\, dx.$$

The parameters $\lambda$ and $\alpha$ are called the *rate* and the *jump size*, respectively.

It is easy to see that if the distribution of $\mathbb{X}$ is free Poisson $\nu(\lambda, \alpha)$, then $\mathcal{R}_n(\mathbb{X}) = \alpha^n \lambda$, $n = 1, 2, \dots$ Therefore, its $r$-transform has the form

$$r_{\nu(\lambda,\alpha)}(z) = \frac{\lambda\alpha}{1 - \alpha z}.$$

## 3. MAIN RESULT

In this section we study a regressive characterization of the Marchenko–Pastur distribution which is a free counterpart of the characterization of the gamma distribution proved in [15].

THEOREM 3.1. *Let* $(\mathcal{A}, \varphi)$ *be a* $W^*$-*probability space*, *let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be non-commutative random variables in* $(\mathcal{A}, \varphi)$. *Assume that* $\mathbb{X}$ *and* $\mathbb{Y}$ *are free*, $\mathbb{X}$ *is strictly positive*, $\mathbb{Y}$ *is positive*, *and there exist real numbers* $c$ *and* $d$ *such that*

$$\text{(3.1)} \qquad \varphi\left(\mathbb{X}|\,\mathbb{X}+\mathbb{Y}\right) = c(\mathbb{X}+\mathbb{Y})$$

*and*

$$\text{(3.2)} \qquad \varphi(\mathbb{X}^{-1}|\mathbb{X}+\mathbb{Y}) = d(\mathbb{X}+\mathbb{Y})^{-1}.$$

*Then* $\mathbb{X}$ *and* $\mathbb{Y}$ *have the free Poisson distributions* $\nu(c\theta, \alpha)$ *and* $\nu\left((1-c)\theta, \alpha\right)$, *respectively*, *where* $\theta = (d-1)/(cd-1)$, $c\theta > 1$, *and* $\alpha = (cd-1)/\left(C_1(1-c)\right)$ *for some* $C_1 > 0$.

REMARK 3.1. *Note that since* $(\mathbb{X}+\mathbb{Y})^{1/2}$ *and* $(\mathbb{X}+\mathbb{Y})^{-1/2}$ *belong to the von Neumann algebra generated by* $\mathbb{X}+\mathbb{Y}$ *and* $\mathbb{I}$, *it follows that by the properties of conditional expectation the above statement can easily be rewritten to have constant right-hand sides of equations* (3.1) *and* (3.2). *Therefore*, *we call the above result a* constant regression characterization.

P r o o f   o f   T h e o r e m   3.1. Multiplying (3.1) and (3.2) by $(\mathbb{X}+\mathbb{Y})^n$ and applying the state to the both sides of the equations, we obtain for $n \geqslant 0$

$$\text{(3.3)} \qquad \varphi\left(\mathbb{X}(\mathbb{X}+\mathbb{Y})^n\right) = c\,\varphi\left((\mathbb{X}+\mathbb{Y})^{n+1}\right),$$

$$\text{(3.4)} \qquad \varphi\left(\mathbb{X}^{-1}(\mathbb{X}+\mathbb{Y})^n\right) = d\,\varphi\left((\mathbb{X}+\mathbb{Y})^{n-1}\right).$$

Let us define three sequences $(\alpha_n)_{n\geqslant-1}$, $(\beta_n)_{n\geqslant0}$ and $(\delta_n)_{n\geqslant0}$ as follows:

$$\alpha_n = \varphi\left((\mathbb{X}+\mathbb{Y})^n\right),$$
$$\beta_n = \varphi\left(\mathbb{X}\,(\mathbb{X}+\mathbb{Y})^n\right),$$
$$\delta_n = \varphi\left(\mathbb{X}^{-1}\,(\mathbb{X}+\mathbb{Y})^n\right).$$

We can rewrite (3.3) and (3.4) as

$$\text{(3.5)} \qquad \beta_n = c\,\alpha_{n+1},$$

$$\text{(3.6)} \qquad \delta_n = d\,\alpha_{n-1}.$$

Multiplying both sides of the above equations by $z^n$ and summing over $n = 0, 1, \ldots$ we get

$$(3.7) \qquad B(z) = c \frac{1}{z} \big( A(z) - 1 \big),$$

$$(3.8) \qquad D(z) = d\, z \bigg( A(z) + \frac{\alpha_{-1}}{z} \bigg),$$

where

$$A(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \quad B(z) = \sum_{n=0}^{\infty} \beta_n z^n, \quad D(z) = \sum_{n=0}^{\infty} \delta_n z^n.$$

Using formula (2.1) and freeness of $\mathbb{X}$ and $\mathbb{Y}$, for a sequence $\beta_n$ we get

$$\begin{aligned}
\beta_n = {} & \mathcal{R}_1 \alpha_n \\
& + \mathcal{R}_2 \left( \alpha_{n-1} + \alpha_{n-2}\alpha_1 + \alpha_{n-3}\alpha_2 + \ldots + \alpha_1 \alpha_{n-2} + \alpha_{n-1} \right) \\
& + \mathcal{R}_3 \left( \alpha_{n-2} + \alpha_{n-3}\alpha_1 + \alpha_{n-4}\alpha_1\alpha_1 + \ldots \right) \\
& + \ldots + \mathcal{R}_{n+1},
\end{aligned}$$

where $\mathcal{R}_n = \mathcal{R}_n(\mathbb{X})$. This gives for $n \geqslant 0$

$$\beta_n = \sum_{k=1}^{n+1} \mathcal{R}_k \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} \ldots \alpha_{i_k}.$$

Using the above equations we get

$$\begin{aligned}
B(z) &= \sum_{n=0}^{\infty} z^n \beta_n = \sum_{n=0}^{\infty} z^n \sum_{k=1}^{n+1} \mathcal{R}_k \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} \ldots \alpha_{i_k} \\
&= \sum_{k=1}^{\infty} z^{k-1} \mathcal{R}_k \sum_{n=k-1}^{\infty} \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} z^{i_1} \ldots \alpha_{i_k} z^{i_k} \\
&= \sum_{k=1}^{\infty} z^{k-1} \mathcal{R}_k \sum_{m=0}^{\infty} \sum_{i_1 + \ldots + i_k = m} \alpha_{i_1} z^{i_1} \ldots \alpha_{i_k} z^{i_k} = \sum_{k=1}^{\infty} z^{k-1} A^k(z) \mathcal{R}_k.
\end{aligned}$$

This implies that

$$(3.9) \qquad B(z) = A(z) r_{\mathbb{X}} \big( z A(z) \big),$$

where $r_{\mathbb{X}}(z) = \sum_{n=0}^{\infty} \mathcal{R}_{n+1} z^n$. Note that $r_{\mathbb{X}}$ is the $r$-transform of $\mathbb{X}$.

Next we proceed similarly with the sequence $(\delta_n)_n$ and we obtain

$$\begin{aligned}
\delta_n = {} & C_1 \alpha_n \\
& + C_2 \left( \alpha_{n-1} + \alpha_{n-2}\alpha_1 + \alpha_{n-3}\alpha_2 + \ldots + \alpha_1 \alpha_{n-2} + \alpha_{n-1} \right) \\
& + C_3 \left( \alpha_{n-2} + \alpha_{n-3}\alpha_1 + \alpha_{n-4}\alpha_1\alpha_1 + \ldots \right) \\
& + \ldots + C_{n+1},
\end{aligned}$$

where $C_n = \mathcal{R}_n\big(\mathbb{X}^{-1}, \underbrace{\mathbb{X}, \ldots, \mathbb{X}}_{n-1}\big)$ for $n \geqslant 0$. Thus for $n \geqslant 0$ we have

$$\delta_n = \sum_{k=1}^{n+1} C_k \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} \ldots \alpha_{i_k}.$$

The above equation gives us

$$
\begin{aligned}
D(z) &= \sum_{n=0}^{\infty} z^n \delta_n = \sum_{n=0}^{\infty} z^n \sum_{k=1}^{n+1} C_k \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} \ldots \alpha_{i_k} \\
&= \sum_{k=1}^{\infty} z^{k-1} C_k \sum_{n=k-1}^{\infty} \sum_{i_1 + \ldots + i_k = n+1-k} \alpha_{i_1} z^{i_1} \ldots \alpha_{i_k} z^{i_k} \\
&= \sum_{k=1}^{\infty} z^{k-1} C_k \sum_{m=0}^{\infty} \sum_{i_1 + \ldots + i_k = m} \alpha_{i_1} z^{i_1} \ldots \alpha_{i_k} z^{i_k} = \sum_{k=1}^{\infty} z^{k-1} A^k(z) C_k.
\end{aligned}
$$

This implies that

$$D(z) = A(z) C\big(zA(z)\big),$$

where $C(z) = \sum_{n=0}^{\infty} C_{n+1} z^n$. Using Lemma 2.2 we get

$$(3.10) \qquad D(z) = A(z) \frac{zA(z) + C_1}{1 + zA(z) r_{\mathbb{X}}\big(zA(z)\big)}.$$

Using equations (3.9) and (3.10), we can rewrite (3.7) and (3.8) as

$$A(z) r_{\mathbb{X}}\big(zA(z)\big) = c\,\frac{1}{z}\big(A(z) - 1\big),$$

$$A(z) \frac{zA(z) + C_1}{1 + zA(z) r_{\mathbb{X}}\big(zA(z)\big)} = d\,z\left(A(z) + \frac{\alpha_{-1}}{z}\right).$$

Let us define an auxiliary function $h(z) = zA(z) r_{\mathbb{X}}\big(zA(z)\big)$. Then we can rewrite the above equations as

$$(3.11) \qquad h(z) = c\big(A(z) - 1\big),$$

$$(3.12) \qquad A(z) \frac{zA(z) + C_1}{1 + h(z)} = d\,z\left(A(z) + \frac{\alpha_{-1}}{z}\right).$$

Since $h(0) = 0$, in some neighbourhood of zero we can multiply (3.12) by $1 + h$. Taking into account that equation (3.2) implies $C_1 = d\,\alpha_{-1}$, we get

$$zA^2(z) + A(z)C_1 - zA(z)d\big(1 + h(z)\big) - C_1\big(1 + h(z)\big) = 0.$$

In the above equation we can replace one function $A$ in the first and second terms by $(h + c)/c$, which follows from (3.11). After simple transformations we obtain

$$(3.13) \qquad \frac{h(z)}{zA(z)} = \frac{c(d-1)}{C_1(1-c) - zA(z)(cd-1)}.$$

Recall that $h(z) = zA(z)r_{\mathbb{X}}\big(zA(z)\big)$. Since $r$ is analytic in a neighbourhood of zero and $\lim_{z \to 0} zA(z) = 0$, we get

$$(3.14) \qquad r_{\mathbb{X}}(z) = \frac{c(d-1)}{C_1(1-c) - z(cd-1)}.$$

From equation (3.1) and the assumption that $\mathbb{X}$ and $\mathbb{Y}$ are positive we get

$$c = \varphi(\mathbb{X})/\varphi\left(\mathbb{X} + \mathbb{Y}\right) \in (0,1).$$

Similarly, freeness of $\mathbb{X}$ and $\mathbb{Y}$ gives us

$$d = \varphi\left(\mathbb{X}^{-1}(\mathbb{X} + \mathbb{Y})\right) = 1 + \varphi(\mathbb{X}^{-1})\varphi(\mathbb{Y}) > 1.$$

The Cauchy–Schwarz inequality implies $cd > 1$. This means that $\mathbb{X}$ has free Poisson distribution with parameters

$$\lambda = c\theta = \frac{c(d-1)}{cd-1} \quad \text{and} \quad \alpha = \frac{cd-1}{C_1(1-c)}.$$

Since $c \in (0,1)$ and $cd > 1$, we have $\lambda > 1$

Next we shall determine the distribution of $\mathbb{Y}$. Substituting in equation (3.11) $h$ from (3.13), we get

$$A^2(z)z(cd-1) + A(z)\big(zd(1-c) - C_1(1-c)\big) + C_1(1-c) = 0.$$

Since $A$ is the moment transform of $\mathbb{X} + \mathbb{Y}$, we can use the connection between moment and Cauchy transforms, and after substituting $z := 1/z$ we obtain

$$G_{\mathbb{X}+\mathbb{Y}}^2(z)z(cd-1) + G_{\mathbb{X}+\mathbb{Y}}(z)d(1-c) - G_{\mathbb{X}+\mathbb{Y}}(z)zC_1(1-c) + C_1(1-c) = 0.$$

Now, using equation (2.5) we get the $r$-transform of $\mathbb{X} + \mathbb{Y}$ in the form

$$r_{\mathbb{X}+\mathbb{Y}}(z) = \frac{d-1}{C_1(1-c) - (cd-1)z}.$$

Equation (2.4) gives

$$r_{\mathbb{Y}}(z) = \frac{(1-c)(d-1)}{C_1(1-c) - (cd-1)z}.$$

This implies that $\mathbb{Y}$ has the free Poisson distribution with parameters

$$\lambda = \frac{(1-c)(d-1)}{cd-1} = (1-c)\theta \quad \text{and} \quad \alpha = \frac{cd-1}{C_1(1-c)},$$

which completes the proof. ∎

### REFERENCES

[1] M. Bożejko and W. Bryc, *On a class of free Lévy laws related to a regression problem*, J. Funct. Anal. 236 (1) (2006), pp. 59–77.

[2] M. Bożejko, M. Leinert, and R. Speicher, *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math. 175 (2) (1996), pp. 357–388.

[3] W. Ejsmont, *Laha–Lukacs properties of some free processes*, Electron. Comm. Probab. 17 (13) (2012), pp. 1–8.

[4] W. Ejsmont, *Characterizations of some free random variables by properties of conditional moments of third degree polynomials*, J. Theoret. Probab. 27 (3) (2014), pp. 915–931.

[5] R. G. Laha and E. Lukacs, *On a problem connected with quadratic regression*, Biometrika 47 (1960), pp. 335–343.

[6] E. Lukacs, *A characterization of the gamma distribution*, Ann. Math. Statist. 26 (1955), pp. 319–324.

[7] A. Nica, *R-transforms of free joint distributions and non-crossing partitions*, J. Funct. Anal. 135 (2) (1996), pp. 271–296.

[8] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Math. Soc. Lecture Note Ser., Vol. 335, Cambridge University Press, Cambridge 2006.

[9] K. Szpojankowski, *Dual Lukacs regressions of negative orders for non-commutative variables*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 17 (3) (2014), 1450021.

[10] K. Szpojankowski, *On the Lukacs property for free random variables*, http://arxiv.org/abs/1403.5300, 2014.

[11] K. Szpojankowski and J. Wesołowski, *Dual Lukacs regressions for non-commutative variables*, J. Funct. Anal. 266 (1) (2014), pp. 36–54.

[12] M. Takesaki, *Theory of Operator Algebras. I*, Encyclopaedia Math. Sci., Vol 124, Springer, Berlin 2002.

[13] D. V. Voiculescu, *Addition of certain noncommuting random variables*, J. Funct. Anal. 66 (3) (1986), pp. 323–346.

[14] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables*, CRM Monogr. Ser., Vol. 1, American Mathematical Society, Providence, RI, 1992.

[15] J. Wesołowski, *A constant regression characterization of the gamma law*, Adv. in Appl. Probab. 22 (2) (1990), pp. 488–490.

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