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*GI/GI/*1 QUEUES WITH INFINITE MEANS OF SERVICE TIME AND INTERARRIVAL TIME

BY

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Abstract. The main results deal with the *GI/GI/*1 queues with infinite means of the service times and interarrival times. Theorem 3.1 gives an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the consecutive customers. Theorem 4.1 gives an asymptotic of stationary waiting times in a heavy traffic situation. In a special case, the asymptotic stationary waiting times have an exponential distribution (Corollary 4.1).

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1. INTRODUCTION

The paper deals with the *GI/GI/*1 queues with infinite means of the service times and interarrival times. An example of such queues is the situation when the distributions of the service times and of the interarrival times belong to the domain of attraction of the stable distributions with stable parameters smaller than one. For such systems we consider an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the consecutive customers and an asymptotic of the stationary waiting times. The subject of an asymptotic of the process of the waiting times and an asymptotic of the stationary waiting times in the heavy traffic situation, when the means of the service times and interarrival times are finite, is well known and is nicely and completely presented in Whitt [9]. As far as I know this subject was not considered in the case when the means of the service times and interarrival times are infinite. Before giving a motivation to consider this subject in the last situation let us recall a description of the *GI/GI/*1 queue with FIFO discipline of the service. Such a system is described by a sequence $\{(v_k, u_k), k \geqslant 1\}$ of nonnegative random variables v_k and u_k such that the sequences $\{v_k\}$ and $\{u_k\}$ are mutually independent and each of them is the sequence of independent and identically distributed random variables. Then v_k represents the service time of the k-th unit, $k \geqslant 1$, and u_k represents the interarrival time between the k-th and $(k+1)$-st units, $k \geqslant 1$.

To give a real model of *GI/GI/*1 queue with infinite means of the service times and interarrival times let us consider two independent processes $\mathcal{W} = \{\mathcal{W}(t), t \geqslant 0\}$ and $X = \{X(t), t \geqslant 0\}$, where $\mathcal{W}$ is a standard Wiener process and X is the stable Lévy process with $EX(t) = 0$, nonpositive jumps and stable parameter $\alpha_0,\ 1 < \alpha_0 \leqslant 2$. Let $t_k,\ k \geqslant 1$, be the first passage time of the level $x = k$ by the Wiener process and let $t_k^{'}$ be the first passage time of the level $x = k$ by the process X. Let

$$v_k \stackrel{\text{df}}{=} t_k^{'} - t_{k-1}^{'}, \quad u_{k+1} \stackrel{\text{df}}{=} t_k - t_{k-1}, \quad k \geqslant 1, \quad t_0 = t_0^{'} = 0.$$

Then $\{v_k\}$ and $\{u_k\}$ are mutually independent sequences of nonnegative random variables and each of them is the sequence of independent and identically distributed random variables. Moreover, u_k and $v_k,\ k \geqslant 1$, are stable distributed with stable parameters $\alpha = 1/2$ and $\alpha = 1/\alpha_0 < 1$, respectively. Of course, $Ev_k = Eu_k = \infty$.

Now, let us imagine that the arrival stream to *GI/GI/*1 queue and the service times of units in this system are managed by an operator in the following way. The operator observes two independent processes $\mathcal{W}$ and X and he sends the k-th unit, $k \geqslant 2$, to the system at the moment t_{k-1} which is the first passage time of the level $x = k,\ k \geqslant 2$, by the Wiener process $\mathcal{W}$. The k-th unit gets the service time $v_k = t_k^{'} - t_{k-1}^{'}$ which is the time between the first passage times of the levels $x = k-1$ and $x = k$, respectively, by the process X. Of course, such a *GI/GI/*1 queue is with infinite means of the service times and interarrival times.

In the paper, we consider *GI/GI/*1 queues for which the distributions of the service times and the interarrival times belong to the domain of attraction of the stable distributions with stable parameter $\alpha < 1$. For such queues we give, in the next section, the sufficient conditions for existing the stationary waiting time (Theorem 2.1). In the third section, we give an asymptotic, in a heavy traffic situation, of the sequence of waiting times of the subsequent customers (Theorem 3.1). In the fourth section, we give an asymptotic of the stationary waiting times in the heavy traffic situation (Theorem 4.1).

In the paper we use the following notation: $(x)_+ = \max(0, x)$, $\stackrel{D}{=}$ stands for the equality of distributions of random variables, $\Rightarrow$ means the weak convergence of distributions of random variables or the weak convergence of distributions of stochastic processes with sample paths in $D[0,\infty)$, while $D[0,\infty)$ denotes the space of real-valued functions on $[0,\infty)$, right continuous and with limits from the left. It is equipped with the Skorokhod J_1 topology. The weak convergence of distributions of stochastic processes with sample paths in $D[0,\infty)$ is equipped with the Skorokhod J_1 topology. Furthermore, for a nonnegative random variable Y we use the notation $Y \sim S_\alpha(\sigma,\gamma)$, which means that Y is stable distributed with stable parameter $\alpha,\ 0 < \alpha < 1$, scale parameter $\sigma,\ 0 < \sigma < \infty$, and location parameter $\gamma,\ 0 \leqslant \gamma < \infty$. In this case the Laplace transform of Y is of the form

$$E\exp(-sY) = \exp(-as^{\alpha} - s\gamma), \quad s \geqslant 0, \tag{1.1}$$

where $a = \sigma^{\alpha}\big(\cos(\pi\alpha/2)\big)^{-1}$.

2. STATIONARY WAITING TIME

Let w_k denote the waiting time for the service of the k-th unit in the *GI/GI/1* queue, with $w_1 = 0$. Then

$$w_{k+1} = S_k - \min_{0 \leqslant j \leqslant k} S_j, \quad k \geqslant 1,$$

where

$$S_0 = 0, \quad S_k = \sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j = \sum_{j=1}^{k} (v_j - u_j), \quad k \geqslant 1.$$

Let ω be a random variable such that $w_k \Rightarrow \omega$ if this limit exists. It means that ω is a random variable with distribution being the limiting distribution of w_k in the sense of the weak convergence. If this limit exists, then ω is called the *stationary waiting time* for the *GI/GI/1* queue. Since

$$w_{k+1} = S_k - \min_{0 \leqslant j \leqslant k} S_j \stackrel{D}{=} \max_{0 \leqslant j \leqslant k} S_j \equiv \tilde{w}_k, \quad k \geqslant 1,$$

and $\tilde{w}_k \leqslant \tilde{w}_{k+1}$, the stationary waiting time exists iff

$$S_k \to -\infty \text{ a.e.} \tag{2.1}$$

We know that if $Ev_1 < Eu_1$, then (2.1) holds and

$$\omega = \sup_{0 \leqslant k < \infty} S_k. \tag{2.2}$$

From Bingham [3] we know that in the general case (without the assumption $Ev_1 < Eu_1$) the convergence (2.1) holds iff

$$\sum_{k=1}^{\infty} \frac{1}{k} P(S_k > 0) < \infty, \tag{2.3}$$

and then we have (2.2) and the characteristic function of the stationary waiting time ω is of the form

$$\varphi(z) \stackrel{\text{df}}{=} E\exp(iz\omega) = \exp\Big\{\sum_{k=1}^{\infty} \frac{1}{k}\Big(E\exp\big(iz(S_k)_+\big) - 1\Big)\Big\}, \quad z \in R. \tag{2.4}$$

The following theorem gives sufficient conditions for (2.1) in the case when the distributions of the service times and interarrival times belong to the domain of attraction of the stable distributions.

THEOREM 2.1. *Let the generating sequence* $\{(v_k, u_k), k \geqslant 1\}$ *for the GI/GI/*1 *queue be such that*

$$\frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{k} v_j \Rightarrow \tilde{v}, \qquad \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{k} u_j \Rightarrow \tilde{u}, \tag{2.5}$$

where the random variables $\tilde{v}$ *and* $\tilde{u}$ *are stable distributed, with stable parameters* α_1 *and* α_2, *respectively, and* $0 < \alpha_2 < \alpha_1 < 1$. *Then*

$$\tilde{w}_k \overset{a.s.}{\to} \sup_{0 \leqslant k < \infty} \big(\sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \big) \equiv \omega. \tag{2.6}$$

Proof. Because $\{v_k\}$ and $\{u_k\}$ are mutually independent and each of them is the sequence of independent and identically distributed random variables, by the convergences in (2.5) we get

$$\Big(\frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{[kt]} v_j, \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j \Big) \Rightarrow \big(L^v(t), L^u(t) \big), \tag{2.7}$$

where L^v and L^u are mutually independent α_1- and α_2-stable Lévy processes, respectively. Hence we get the convergence

$$\begin{aligned} \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} v_j - \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j &= \frac{k^{1/\alpha_1}}{k^{1/\alpha_2}} \frac{1}{k^{1/\alpha_1}} \sum_{j=1}^{[kt]} v_j - \frac{1}{k^{1/\alpha_2}} \sum_{j=1}^{[kt]} u_j \\ &\Rightarrow 0 \cdot L^v(t) - L^u(t) = -L^u(t), \end{aligned} \tag{2.8}$$

which, in turn, implies

$$\sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \overset{p}{\to} -\infty. \tag{2.9}$$

Since the convergence in probability of the series of mutually independent random variables implies almost sure convergence, we have the following almost sure convergence: $\sum_{j=1}^{k} v_j - \sum_{j=1}^{k} u_j \to -\infty$. This, in turn, gives the assertion of the theorem. ■

From the above theorem we get immediately the following corollary:

COROLLARY 2.1. *If the generating sequence* $\{(v_k, u_k), k \geqslant 1\}$ *for the GI/GI/*1 *queue is such that*

$$v_k \sim S_{\alpha_1}(\sigma_1, \bar{v}) \quad \textit{and} \quad u_k \sim S_{\alpha_2}(\sigma_2, \bar{u}),$$

where $0 < \alpha_2 < \alpha_1 < 1$ *and* $\bar{v}, \bar{u} \geqslant 0$, *then the convergences* (2.5) *and* (2.6) *hold.*

3. AN ASYMPTOTIC OF WAITING TIMES IN HEAVY TRAFFIC

Let $\{Q_n, n \geqslant 1\}$ be a sequence of *GI/GI/*1 queues, where Q_n is generated by $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}$ and let $w_{n,k}$ be the waiting time for the service of the k-th customer in the Q_n system. In this section we consider an asymptotic of $w_{n,k}$, as $k, n \to \infty$, in a heavy traffic situation. An example of the heavy traffic situation is the case when the stability parameters of the interarrival time distributions are greater than or close to the stability parameters of the service time distributions. Let

$$S_{n,0} = 0, \quad S_{n,k} = \sum_{j=1}^{k} v_{n,j} - \sum_{j=1}^{k} u_{n,j} = \sum_{j=1}^{k} (v_{n,j} - u_{n,j}), \quad k \geqslant 1, n \geqslant 1.$$

Then

$$w_{n,k+1} = S_{n,k} - \min_{0 \leqslant j \leqslant k} S_{n,j}.$$

Let

$$W_n(t) = k_n^{-1/\alpha_{n,2}} w_{n,[k_n t]+1}, \quad t \geqslant 0,$$

(3.1)

$$V_n(t) = \frac{1}{k_n^{1/\alpha_{n,1}}} \sum_{j=1}^{[k_n t]} (v_{n,j} - \bar{v}_n), \quad U_n(t) = \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} (u_{n,j} - \bar{u}_n), \quad t \geqslant 0,$$

where $\bar{v}_n$ and $\bar{u}_n, n \geqslant 1$, are nonnegative numbers, $\{k_n\}$ is a sequence of positive integers tending to infinity and $0 < \alpha_{n,1},\ \alpha_{n,2} < 1$.

THEOREM 3.1. *Let the sequences* $\{\alpha_{n,1}\}, \{\alpha_{n,2}\}, \{\bar{v}_n\}, \{\bar{u}_n\}$ *and* $\{k_n\}$ *be such that*

$$\alpha_{n,1} \to \alpha_1, \quad \alpha_{n,2} \to \alpha_2, \quad 0 < \alpha_1, \alpha_2 < 1, \tag{3.2}$$

$$k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} \to a, \quad 0 \leqslant a < \infty, \tag{3.3}$$

$$k_n^{(\alpha_{n,2}-1)/\alpha_{n,2}} (\bar{v}_n - \bar{u}_n) \equiv \mu_n \to \mu, \quad |\mu| < \infty. \tag{3.4}$$

Furthermore, let

$$V_n \Rightarrow V, \quad U_n \Rightarrow U \tag{3.5}$$

in $D[0,\infty)$ *with Skorokhod* J_1 *topology, where* V *and* U *are* α_1- *and* α_2-*stable Lévy processes, respectively. Then*

$$W_n \Rightarrow W \tag{3.6}$$

in $D[0,\infty)$ *with Skorokhod* J_1 *topology, where for each* $t \geqslant 0$

$$\begin{aligned} W(t) &= aV(t) - U(t) + \mu t - \inf_{0 \leqslant s \leqslant t} \big(aV(s) - U(s) + \mu s\big) \\ &\overset{D}{=} \sup_{0 \leqslant s \leqslant t} \big(aV(s) - U(s) + \mu s\big). \end{aligned} \tag{3.7}$$

P r o o f. Let f be the mapping of $D[0,\infty)$ into $D[0,\infty)$ defined as $f(x)(t) = x(t) - \inf_{0\leqslant s\leqslant t} x(s)$. The mapping f is continuous in $D[0,\infty)$ with Skorokhod J_1 topology. Notice that

$$W_n(t) = \frac{1}{k_n^{1/\alpha_{n,2}}} w_{[k_n t]+1} = f(X_n)(t),$$

where

$$X_n(t) \stackrel{\text{df}}{=} \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} v_{n,j} - \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} u_{n,j}, \quad t \geqslant 0,\ n \geqslant 1.$$

But

$$X_n = k_n^{1/\alpha_{n,1}-1/\alpha_{n,2}} V_n - U_n + \eta_n,$$

where

$$\eta_n(t) = (\bar{v}_n - \bar{u}_n) \frac{[k_n t]}{k_n} k_n^{1-1/\alpha_{n,2}}, \quad t \geqslant 0,\ n \geqslant 1.$$

Using the mutual independence of processes V_n, U_n and η_n and the convergences in (3.5) and (3.4), we get

$$(V_n, U_n, \eta_n) \Rightarrow (V, U, \mu e),$$

in $D[0,\infty) \times D[0,\infty) \times D[0,\infty)$ with product Skorokhod J_1 topology, where $e(t) = t, t \geqslant 0$. This and the mutual independence of processes V, U and e give the convergence

$$k_n^{1/\alpha_{n,1}-1/\alpha_{n,2}} V_n - U_n + \eta_n \Rightarrow aV - U + \mu e$$

in $D[0,\infty)$ with Skorokhod J_1 topology. Now, using the continuity of the mapping f in $D[0,\infty)$ with Skorokhod J_1 topology and Theorem 5.1 from Billingsley [2], we get the assertion of the theorem. ∎

The following remark describes the situations in which $a = 1$ or $0 < a < \infty$ or $a = 0$ in condition (3.3).

REMARK 3.1. *The following implications hold*:

(3.8) *If* $\alpha_{n,2} - \alpha_{n,1} \to 0$ *and* $k_n \dfrac{\alpha_{n,2} - \alpha_{n,1}}{\alpha_{n,1}\alpha_{n,2}} \to c,\ \ c \in R,$ *then* $a = 1.$

(3.9) *If* $\ln(k_n) \dfrac{\alpha_{n,1} - \alpha_{n,2}}{\alpha_{n,1}\alpha_{n,2}} \to c$ *and* $c = \ln a,$ *then* $0 < a < \infty.$

(3.10) *If* $\alpha_{n,2} < \alpha_{n,1},$ *then* $a = 0.$

P r o o f. Let
$$\beta_n = \frac{1}{\alpha_{n,2}} - \frac{1}{\alpha_{n,1}} = \frac{\alpha_{n,1} - \alpha_{n,2}}{\alpha_{n,1}\alpha_{n,2}}.$$
The implication in (3.8) follows from the equality

$$k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{(1/k_n)\cdot k_n(-\beta_n)}$$

and from the convergences $k_n^{1/k_n} \to 1$ and $k_n\beta_n \to -c,\ c \in R$. The implication in (3.9) follows from the representation

$$k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{-\beta_n} = e^{\ln(k_n)(-\beta_n)}$$

and the assumed convergence. The implication in (3.10) follows from the representation
$$k_n^{1/\alpha_{n,1}} / k_n^{1/\alpha_{n,2}} = k_n^{-\beta_n},$$

where $\beta_n > 0$. This completes the proof of the remark. ∎

COROLLARY 3.1. *If* $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}, n \geqslant 1$, *are such that*

$$v_{n,k} \sim S_{\alpha_{n,1}}(\sigma_{n,1}^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_{n,2}}(\sigma_{n,2}^2, \bar{u}_n),$$

where $\alpha_{n,1} \to \alpha$, $\alpha_{n,2} \to \alpha$, $0 < \alpha < 1$, *and the convergences* (3.3)–(3.5) *hold*, *then the convergence* (3.6) *holds*, *where* V *and* U *are* α*-stable Lévy processes.*

COROLLARY 3.2. *If* $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}, n \geqslant 1$, *are such that*

$$v_{n,k} \sim S_{\alpha}(\sigma_1^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha}(\sigma_2^2, \bar{u}_n),$$

then condition (3.5) *holds. Furthermore*, *if condition* (3.4) *holds, then the convergence* (3.6) *holds with* $a = 1$, *and* V *and* U *are* α*-stable Lévy processes.*

COROLLARY 3.3. *If* $\{(v_{n,k}, u_{n,k}), k \geqslant 1\}, n \geqslant 1$, *are such that*

$$v_{n,k} \sim S_{\alpha_1}(\sigma_1^2, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_2}(\sigma_2^2, \bar{u}_n)$$

for each $t \geqslant 0$, *where* $0 < \alpha_2 < \alpha_1 < 1$, *then condition* (3.5) *holds. Furthermore*, *if condition* (3.4) *holds with* $\mu > 0$, *then* (3.6) *holds with*

$$W(t) = \mu t - U(t) - \inf_{0 \leqslant s \leqslant t} \big(\mu s - U(s)\big) \stackrel{D}{=} \sup_{0 \leqslant s \leqslant t} \big(\mu s - U(s)\big), \tag{3.11}$$

where U *is an* α*-stable Lévy process.*

P r o o f. The corollary follows from Theorem 3.1 with $a = 0$. ∎

REMARK 3.2. *If U is an α-stable Lévy process with $\alpha = 1/2$ and such that $U(1) \sim S_{1/2}(\sigma, 1)$, then*

$$\sup_{0 \leqslant s \leqslant t} \big(\mu s - U(s)\big) \stackrel{D}{=} \frac{\mu}{\sqrt{\sigma}} \sup_{0 \leqslant s \leqslant U(t)} \left(\mathcal{W}(s) - \frac{\sqrt{\sigma}}{\mu} s\right),$$

where $\mathcal{W}$ is a Wiener process. Moreover,

$$\sup_{0 \leqslant s < \infty} \big(\mu s - U(s)\big) \stackrel{D}{=} \frac{\mu}{\sqrt{\sigma}} \sup_{0 \leqslant s < \infty} \left(\mathcal{W}(s) - \frac{\sqrt{\sigma}}{\mu} s\right) \equiv M,$$

and

$$P(M > x) = \exp\big(-2(\sigma/\mu^2)x\big), \quad x \geqslant 0.$$

Proof. Using Proposition 1.2.11 from [6], p. 14, we get

$$E \exp\big(-sU(1)\big) = \exp(-\sqrt{as}), \ s \geqslant 0, \quad \text{with } a = 2\sigma. \tag{3.12}$$

Now, by Example 1.3.21 in [1], p. 51, we get

$$E \exp\big(-sT(t)\big) = \exp(-t\delta\sqrt{2s}), \quad s \geqslant 0, \tag{3.13}$$

where

$$T(t) = \inf\{u : \mathcal{W} = \delta t\}, \quad 0 < \delta < \infty.$$

Consequently, from (3.12) and (3.13) we get $\delta\sqrt{2} = \sqrt{a}$, which implies $\delta = \sqrt{\sigma}$. But

$$\sup_{0 \leqslant t < \infty} \big(\mu t - T(t)\big) = \sup_{0 \leqslant t < \infty} \big(\mu t - U(t)\big).$$

Putting $U(t) = s$ if $\mathcal{W}(s) = \delta t$, we get

$$\sup_{0 \leqslant t < \infty} \big(\mu t - U(t)\big) = \sup_{0 \leqslant t < \infty} \left(\frac{\mu}{\delta}\mathcal{W}(s) - s\right) = \frac{\mu}{\delta} \sup_{0 \leqslant t < \infty} \left(\mathcal{W}(s) - \frac{\delta}{\mu}\right) \equiv M.$$

Using Corollary 5.1 from [5], p. 361, we get

$$P(M > x) = \exp\left(-2\frac{\sigma}{\mu^2} x\right), \quad x \geqslant 0.$$

This completes the proof of the remark. ■

4. AN ASYMPTOTIC OF STATIONARY WAITING TIME IN HEAVY TRAFFIC

Let ω_n be the stationary waiting time for the *GI/GI/*1 queue Q_n generated by $\{(v_{n,k}, u_{n,k}), k \geqslant 1\},\ n \geqslant 1$, and let

$$v_{n,k} \sim S_{\alpha_{n,1}}(\sigma_{n,1}, \bar{v}_n), \quad u_{n,k} \sim S_{\alpha_{n,2}}(\sigma_{n,2}, \bar{u}_n), \tag{4.1}$$

$$0 < \alpha_{n,2} < \alpha_{n,1} < 1. \tag{4.2}$$

Let F_n and F be the distribution functions of $u_{n,k} - \bar{u}_n$ and of $u \sim S_{\alpha_2}(\sigma, 0)$, respectively. Below, $\{k_n\}$ represents a sequence of positive integers tending to infinity.

The following theorem gives the asymptotic of the stationary waiting time in a heavy traffic situation defined by conditions (4.3)–(4.5).

THEOREM 4.1. *Let $\{\alpha_{n,1}\}, \{\alpha_{n,2}\}, \{\bar{v}_n\}, \{\bar{u}_n\}$ and $\{k_n\}$ be the sequences such that*

$$0 < \alpha_{n,2} \leqslant \alpha_2 < \alpha_1 \leqslant \alpha_{n,1} < 1, \quad \alpha_{n,2} \uparrow \alpha_2, \quad \alpha_{n,1} \downarrow \alpha_1, \tag{4.3}$$

$$k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} \to 0, \tag{4.4}$$

$$\bar{v}_n - \bar{u}_n \to \infty, \quad k_n^{(\alpha_{n,2}-1)/\alpha_{n,2}}(\bar{v}_n - \bar{u}_n) \equiv \mu_n \to \mu, \ \ \mu > 0. \tag{4.5}$$

$$k_n^{(\alpha_{n,2}-1)/\alpha_{n,2}}(2\bar{v}_n - \bar{u}_n) = a_n < a < \infty, \tag{4.6}$$

$$F_n(a2^{i(1-1/\alpha_{n,2})}) \leqslant \tilde{a}F(a2^{i(1-1/\alpha_2)}), \quad \tilde{a} < \infty. \tag{4.7}$$

Furthermore, *let*

$$V_n \Rightarrow V, \quad U_n \Rightarrow U \tag{4.8}$$

in $D[0,\infty)$ with Skorokhod J_1 topology, *where V and U are α_1- and α_2-stable Lévy processes. Then*

$$\frac{1}{k^{1/\alpha_{n,2}}}\omega_n \Rightarrow \sup_{0 \leqslant t < \infty} \big(\mu t - U(t)\big). \tag{4.9}$$

P r o o f. By Theorem 2.1 and the assumptions (4.1) and (4.2) we get

$$\omega_n = \sup_{0 \leqslant k < \infty} S_{n,k} = \sup_{0 \leqslant t < \infty} S_{n,[k_n t]}, \quad n \geqslant 1.$$

But

$$\begin{aligned}
&\frac{1}{k_n^{1/\alpha_{n,2}}} S_{n,[k_n t]} = \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} v_{n,j} - \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} u_{n,j} \\
&= \frac{k_n^{1/\alpha_{n,1}}}{k_n^{1/\alpha_{n,2}}} \frac{1}{k_n^{1/\alpha_{n,1}}} \sum_{j=1}^{[k_n t]} (v_{n,j} - \bar{v}_n) - \frac{1}{k_n^{1/\alpha_{n,2}}} \sum_{j=1}^{[k_n t]} (u_{n,j} - \bar{u}_n) + (\bar{v}_n - \bar{u}_n) \frac{[k_n t]}{k^{1/\alpha_{n,2}}} \\
&= k_n^{(\alpha_{n,2} - \alpha_{n,1})/\alpha_{n,1}\alpha_{n,2}} V_n(t) - U_n(t) + (\bar{v}_n - \bar{u}_n) \frac{[k_n t]}{k_n} k_n^{1 - 1/\alpha_{n,2}}.
\end{aligned}$$

Hence and by the mutual independence of processes V_n and U_n, together with the convergence (4.8) and assumption (4.4), we get the convergence

$$\frac{1}{k_n^{1/\alpha_{n,2}}} S_{n,[k_n \cdot]} \Rightarrow \mu e - U$$

in $D[0,\infty)$ with Skorokhod J_1 topology.

To complete the proof of the theorem, i.e. to show the convergence

$$\frac{1}{k_n^{1/\alpha_{n,2}}} \omega_n \equiv \sup_{0 \leqslant t < \infty} \frac{1}{k_n^{1/\alpha_{n,2}}} S_{n,[k_n t]} \Rightarrow \sup_{0 \leqslant t < \infty} \big(\mu t - U(t)\big),$$

we need to prove the tightness of the sequence $\{\omega_n / c_n\}$, where $c_n = k_n^{1/\alpha_{n,2}}$ (see the Heavy Traffic Invariance Principle in [8]).

Let $t_i = 2^i, i = 0, 1, 2, \ldots$ Then for any $c > 0$ we have

$$\begin{aligned}
&P\left(\frac{1}{c_n} \omega_n > c\right) = P\left(\frac{1}{c_n} \sup_{0 \leqslant t < \infty} S_{n,[k_n t]} > c\right) \\
&= P\left(\bigcup_{i=0}^{\infty} \left\{\frac{1}{c_n} \sup_{t_i \leqslant t < t_{i+1}} S_{n,[k_n t]} > c\right\}\right) \leqslant \sum_{i=0}^{\infty} P\left(\frac{1}{c_n} \sup_{t_i \leqslant t < t_{i+1}} S_{n,[k_n t]} > c\right).
\end{aligned}$$

But

$$\begin{aligned}
&\sup_{t_i \leqslant t < t_{i+1}} S_{n,[k_n t]} = \sup_{t_i \leqslant t < t_{i+1}} (S_{n,[k_n t]} - S_{n,[k_n t_i]} + S_{n,[k_n t_i]}) \\
&= S_{n,[k_n t_i]} + \sup_{t_i \leqslant t < t_{i+1}} (S_{n,[k_n t]} - S_{n,[k_n t_i]}) = S_{n,k_n t_i} + \sup_{0 \leqslant t \leqslant t_{i+1} - t_i} \tilde{S}_{n,[k_n t]} \\
&\qquad = S_{n,k_n 2^i} + \sup_{0 \leqslant t \leqslant 2^i} \tilde{S}_{n,[k_n t]},
\end{aligned}$$

where

$$\tilde{S}_{n,[k_n t]} = \sum_{j=1}^{[k_n t]} (\tilde{v}_{n,j} - \tilde{u}_{n,j}),$$

and the sequences $\{(\tilde{v}_{n,j}, \tilde{u}_{n,j}), j \geqslant 1\}$ and $\{(v_{n,j}, u_{n,j}), j \geqslant 1\}$ are mutually independent, $\{(\tilde{v}_{n,j}, \tilde{u}_{n,j}), j \geqslant 1\}$ being a probabilistic copy of $\{(v_{n,j}, u_{n,j}), j \geqslant 1\}$. Hence

$$P\biggl(\frac{1}{c_n}\omega_n > c\biggr) \leqslant \sum_{i=0}^{\infty} P\biggl(\frac{1}{c_n} S_{n,2^i k_n} + \frac{1}{c_n} \sup_{0\leqslant t\leqslant 2^i} \tilde{S}_{n,[k_n t]} > c\biggr). \tag{4.10}$$

But

$$\begin{aligned} \sup_{0\leqslant t\leqslant 2^i} \tilde{S}_{n,[k_n t]} &\leqslant \sum_{j=1}^{2^i k_n} \tilde{v}_{n,j} = \sum_{j=1}^{2^i k_n} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n \\ &= (2^i k_n)^{1/\alpha_{n,1}} (2^i k_n)^{-1/\alpha_{n,1}} \sum_{j=1}^{k_n} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n \\ &\overset{D}{=} (2^i k_n)^{1/\alpha_{n,1}} (\tilde{v}_{n,j} - \bar{v}_n) + 2^i k_n \bar{v}_n \\ &\overset{D}{=} (2^i k_n)^{1/\alpha_{n,1}} \tilde{v}_n + 2^i k_n \bar{v}_n, \end{aligned} \tag{4.11}$$

where $\tilde{v}_n \overset{D}{=} (\tilde{v}_{n,j} - \bar{v}_n)$ and $\tilde{v}_n$ is independent of $\{(v_{n,j}, u_{n,j}), j \geqslant 1\}$.

Notice that

$$\begin{aligned} S_{n,2^i k_n} &= \sum_{j=1}^{2^i k_n} (v_{n,j} - \bar{v}_n) - \sum_{j=1}^{2^i k_n} (u_{n,j} - \bar{u}_n) + 2^i k_n (\bar{v}_n - \bar{u}_n) \\ &= (2^i k_n)^{1/\alpha_{n,1}} (2^i k_n)^{-1/\alpha_{n,1}} \sum_{j=1}^{2^i k_n} (v_{n,j} - \bar{v}_n) \\ &\quad - (2^i k_n)^{1/\alpha_{n,2}} (2^i k_n)^{-1/\alpha_{n,2}} \sum_{j=1}^{2^i k_n} (u_{n,j} - \bar{u}_n) + 2^i k_n (\bar{v}_n - \bar{u}_n) \\ &\overset{D}{=} (2^i k_n)^{1/\alpha_{n,1}} v_n - (2^i k_n)^{1/\alpha_{n,2}} u_n + 2^i k_n (\bar{v}_n - \bar{u}_n), \end{aligned} \tag{4.12}$$

where $v_n \overset{D}{=} (v_{n,j} - \bar{v}_n) \overset{D}{=} \tilde{v}_n$, $u_n \overset{D}{=} (u_{n,j} - \bar{u}_n)$ and $v_n, \tilde{v}_n, u_n$ are mutually independent. Hence, by (4.10)–(4.12), we get

$$\begin{aligned} P\biggl(\frac{1}{c_n}\omega_n > c\biggr) &\leqslant \sum_{i=0}^{\infty} P\bigl(k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} 2^{i/\alpha_{n,1}} v_n - 2^{i/\alpha_{n,2}} u_n \\ &+ k_n^{1/\alpha_{n,1} - 1/\alpha_{n,2}} 2^{i/\alpha_{n,1}} \tilde{v}_n + 2^i k_n^{1-1/\alpha_{n,2}} (\bar{v}_n - \bar{u}_n) + 2^i k_n^{1-1/\alpha_{n,2}} \bar{v}_n > c\bigr). \end{aligned} \tag{4.13}$$

But

$$v_n + \tilde{v}_n = 2^{1/\alpha_{n,1}} 2^{-1/\alpha_{n,1}} (v_n + \tilde{v}_n) \overset{D}{=} 2^{1/\alpha_{n,1}} v_n.$$

Hence, since $\beta_n = 1/\alpha_{n,2} - 1/\alpha_{n,1}$, we get

$$
\begin{aligned}
(4.14)\quad & P\Big(\frac{1}{c_n}\omega_n > c\Big) \leqslant \\
&\leqslant \sum_{i=0}^{\infty} P\big(k_n^{-\beta_n}(2^{1/\alpha_{n,1}})^{i+1}v_n - 2^{i/\alpha_{n,2}}u_n + 2^i k_n^{1-1/\alpha_{n,2}}(2\bar{v}_n - \bar{u}_n) > c\big) \\
&\leqslant \sum_{i=0}^{\infty} P\big(k_n^{-\beta_n}(2^{1/\alpha_{n,1}})^{i+1}v_n - 2^{i/\alpha_{n,2}}u_n + 2^i a > c\big) \\
&= \sum_{i=0}^{\infty} P\big(v_n - k_n^{\beta_n}(2^{i\beta_n}2^{-1/\alpha_{n,1}}u_n - 2^{i(1-1/\alpha_{n,1})}2^{-1/\alpha_{n,1}}a) > k_n^{\beta_n}2^{-(i+1)/\alpha_{n,1}}c\big) \\
&\leqslant \sum_{i=0}^{\infty} P\big(v_n - k_n^{\beta_n}2^{-1/\alpha_{n,1}}(2^{i\beta_n}u_n - 2^{i(1-1/\alpha_{n,1})}a) > 0\big).
\end{aligned}
$$

Let f_n be the density of the distribution function of u_n, and assume that $A_{n,i}$ and $B_{n,i}$ are the following sets in R:

$$
\begin{aligned}
A_{n,i} &= \{x : 2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a < 0\} = \{x : x < 2^{i(1-1/\alpha_{n,2})}a\}, \\
B_{n,i} &= R - A_{n,i}.
\end{aligned}
$$

Furthermore, let

$$C_{n,i} = \int\limits_{A_{n,i}} P\big(v_n - k_n^{\beta_n}2^{-1/\alpha_{n,1}}(2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a) > 0\big) f_n(x)dx,$$

$$D_{n,i} = \int\limits_{B_{n,i}} P\big(v_n - k_n^{\beta_n}2^{-1/\alpha_{n,1}}(2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a) > 0\big) f_n(x)dx.$$

Then, using (4.14), we get

$$P\Big(\frac{1}{c_n}\omega_n > c\Big) \leqslant \sum_{i=0}^{\infty} C_{n,i} + \sum_{i=0}^{\infty} D_{n,i}. \tag{4.15}$$

Since $v_n \geqslant 0$, we have

$$
\begin{aligned}
\sum_{i=0}^{\infty} C_{n,i} &= \sum_{i=0}^{\infty} \int\limits_{A_{n,i}} f_n(x)dx = \sum_{i=0}^{\infty} P(u_n \leqslant a2^{i(1-1/\alpha_{n,2})}) \\
&= \sum_{i=0}^{\infty} F_n(a2^{i(1-1/\alpha_{n,2})}) \leqslant \sum_{i=0}^{\infty} F_n(z_{n,i}).
\end{aligned}
$$

Now, by condition (4.7) and the inequality $z_{n,i} = a2^{i(1-1/\alpha_{n,2})} \leqslant a2^{i(1-1/\alpha_2)} \equiv z_i$, we get

$$\sum_{i=0}^{\infty} C_{n,i} \leqslant \tilde{a}\sum_{i=0}^{\infty} F(z_i) = \tilde{a}\sum_{i=0}^{\infty} F(z_i)\exp(z_i^{-\alpha_2})\exp(-z_i^{-\alpha_2}). \tag{4.16}$$

Theorem 1 in Feller [4], p. 448, says that if G_α is the distribution function of the strictly stable distribution with $0 < \alpha < 1$, then the following convergence holds:

$$G_\alpha(x)\exp(x^{-\alpha}) \to 0 \quad \text{as } x \to 0. \tag{4.17}$$

Hence and by the convergence, $z_i \to 0$, the sequence $F(z_i)\exp(z_i^{-\alpha_2})$ is bounded, say by $b < 0$. Therefore,

$$\sum_{i=0}^{\infty} C_{n,i} \leqslant b\tilde{a} \sum_{i=0}^{\infty} \exp(-z_i^{-\alpha_2}). \tag{4.18}$$

This and the relations

$$z_i^{-\alpha_2} = a^{-\alpha_2}(2^{i(1-1/\alpha_2)})^{-\alpha_2} = a^{-\alpha_2}2^{i(1-\alpha_2)} \geqslant a^{-\alpha_2} i 2^{1-\alpha_2}$$

give

$$\begin{aligned}\sum_{i=0}^{\infty} \exp(-z_i^{-\alpha_2}) &\leqslant \sum_{i=0}^{\infty} \big(\exp(-a^{-\alpha_2}2^{1-\alpha_2})\big)^i \\ &= \big(1 - \exp(-a^{-\alpha_2}2^{1-\alpha_2})\big)^{-1} < \infty,\end{aligned}$$

which implies that the series $\sum_{i=0}^{\infty} C_{n,i}$ is uniformly bounded with respect to n by the finite series $b\tilde{a}\sum_{i=0}^{\infty}\exp(-z_i^{-\alpha_2})$.

Now, using the relation $P(v_n > x) \approx x^{-\alpha_{n,1}}$ for large x to the second component of (4.14), we get

$$\begin{aligned}\sum_{i=0}^{\infty} D_{n,i} &= \sum_{i=0}^{\infty} \int_{B_{n,i}} P\big(v_n > k_n^{\beta_n} 2^{-1/\alpha_{n,1}}(2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a)\big) f_n(x)dx \\ &\approx \sum_{i=0}^{\infty} \int_{B_{n,i}} \big(k_n^{\beta_n} 2^{-1/\alpha_{n,1}}(2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a)\big)^{-\alpha_{n,1}} f_n(x)dx \\ &= 2k_n^{-\beta_n\alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{n,i}} (2^{i\beta_n}x - 2^{i(1-1/\alpha_{n,1})}a)^{-\alpha_{n,1}} f_n(x)dx \\ &\leqslant 2k_n^{-\beta_n\alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{n,i}} |2^{i\beta_n}x - a|^{-\alpha_{n,1}} f_n(x)dx \\ &\leqslant 2k_n^{-\beta_n\alpha_{n,1}} \sum_{i=0}^{\infty} \int_{B_{n,i}} (2^{-\beta_n\alpha_{n,1}})^i |x - 2^{-i\beta_n}a|^{-\alpha_{n,1}} f_n(x)dx \\ &\leqslant 2k_n^{-\beta_n\alpha_{n,1}} \sum_{i=0}^{\infty} (2^{-\beta_n\alpha_{n,1}})^i E|u_n - a|^{-\alpha_{n,1}} \\ &\leqslant 2k_n^{-\beta_n\alpha_{n,1}} E|u_n - a|^{-\alpha_{n,1}}(1 - 2^{-\beta_n\alpha_{n,1}})^{-1}.\end{aligned}$$

The above is finite because of the equation (25.5) in Sato [7], p. 162, which gives

$$Eu_n^{-\alpha_{n,1}} = \gamma_n^{-\alpha_{n,1}/\alpha_{n,2}} \frac{\Gamma(1+\alpha_{n,1}/\alpha_{n,2})}{\Gamma(1+\alpha_{n,1})} = \tilde{c} < \infty, \quad \text{where } \gamma_n = O(1).$$

Since $\alpha_{n,2} \leqslant \alpha_2 \leqslant \alpha_1 \leqslant \alpha_{n,2}$, we have

$$-\beta_n \alpha_{n,1} = 1 - \alpha_{n,1}/\alpha_{n,2} \leqslant 1 - \alpha_1/\alpha_2,$$

which gives $2^{\alpha_{n,1}/\alpha_{n,2}} \geqslant 2^{\alpha_1/\alpha_2}$. Hence for some finite $D < \infty$ we have

$$\sum_{i=0}^{\infty} D_{n,i} \leqslant D \sum_{i=0}^{\infty} (2^{-\beta_n \alpha_{n,1}})^i \leqslant D \sum_{i=0}^{\infty} 2^{1-\alpha_1/\alpha_2} < \infty, \tag{4.19}$$

which implies that the series $\sum_{i=0}^{\infty} D_{n,i}$ is uniformly bounded with respect to n by the convergent series. This and (4.18) imply that we can pass with n under the sum in (4.14), so we get

(4.20)
$$\begin{aligned}\lim_n P\Big(\frac{1}{c_n}\omega_n > c\Big) &\leqslant \sum_{i=0}^{\infty} \lim_n P\big(k_n^{-\beta_n}(2^{1/\alpha_{n,1}})^{i+1} v_n - 2^{i/\alpha_{n,2}} u_n + 2^i a_n > c\big) \\ &= \sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c),\end{aligned}$$

where the random variable u is strictly α_2-stable. But

$$\begin{aligned}\sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c) &\leqslant \sum_{i=0}^{\infty} P(u < 2^{i(1-1/\alpha_2)} a) \\ &= \sum_{i=0}^{\infty} F(x_i) \exp(x_i^{-\alpha_2}) \exp(-x_i^{-\alpha_2}) \leqslant \tilde{d} \sum_{i=0}^{\infty} \exp(-2^{i(1-\alpha_2)}) < \infty,\end{aligned}$$

where in view of (4.19) we have

$$\sup_i F(x_i) \exp(x_i^{-\alpha_2}) < \tilde{d} < \infty.$$

This implies that the series $\sum_{i=0}^{\infty} P(-2^{i/\alpha_2} u + 2^i a > c)$ is uniformly bounded with respect to c by the convergent series $\tilde{d} \sum_{i=0}^{\infty} \exp(-2^{i(1-\alpha_2)})$. Therefore, passing with $c \to \infty$ under the sum on the right-hand side of (4.20), we get

$$\lim_{c\to\infty} \lim_n P\Big(\frac{1}{c_n}\omega_n > c\Big) = 0,$$

which, in turn, implies that the sequence $\left\{\frac{1}{c_n}\omega_n\right\}$ is tight. This completes the proof of the theorem. ∎

REMARK 4.1. *Condition* (4.7) *holds true when the distribution function of* $(u_n - \bar{u}_n)\frac{\sigma}{\sigma_n}$ *is* F.

By Remark 3.2 we get the following corollary.

COROLLARY 4.1. *If* U *is an* α*-stable Lévy process with* $\alpha = 1/2$ *and such that* $U(1) \sim S_{1/2}(\sigma, 1)$, *then*

$$\sup_{0 \leqslant s < \infty} \big(\mu s - U(s)\big) \stackrel{D}{=} \frac{\mu}{\sqrt{\sigma}} \sup_{0 \leqslant s < \infty} \left(\mathcal{W}(s) - \frac{\sqrt{\sigma}}{\mu} s\right) \equiv M,$$

where $\mathcal{W}$ *is a Wiener process and*

$$P(M > x) = \exp\big(-2(\sigma/\mu^2)x\big), \qquad x \geqslant 0.$$

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