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ON A CLASS OF BANACH SPACE VALUED PROCESSES WITH PROPAGATORS

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Abstract. In this paper we present a characterization of vector stochastic processes with normal propagators and apply it to study the regularity and singularity of vector stochastic processes.

0. Introduction. Let B be a Banach space, H a complex Hilbert space, and $X\colon R \to L(B, H)$ a vector stochastic process. If the formula $V_t X(s) = X(t+s)$ defines a group of continuous linear operators, we call this group the *propagator* of X (cf. [5]). In particular, a process is stationary if and only if it has a unitary propagator. Therefore, processes with propagators are a generalization of stationary processes. The purpose of this paper is to study processes with propagators and to investigate which properties of stationary processes are true for processes with propagators. Processes with propagators are not treated yet in the literature so well as stationary processes. Scalar processes with propagators were studied in [3] and [10]. Some existence conditions for propagators of vector processes were found in [5], [6], and [12]. Vector processes with hermitian propagators were described in [9]. Furthermore, some types of these processes were studied in [2] and [8].

In this paper we describe spectral representations of processes with normal propagators. We extend a theorem of Getoor [3] to the case of vector processes and give a different proof based on the dilation theorem. The final part of the paper is devoted to description of singularity and regularity of processes in terms of their propagators. For example, we prove that a process is singular if its propagator V_t satisfies the condition $\|I - V_t\| < 1$ for some $t < 0$.

1. Let B be a complex Banach space with the dual space B^* and let H be a complex Hilbert space. We denote by $L(B, H)$ the space of all continuous linear operators from B into H (by $\bar{L}(B, B^*)$ the space of all continuous antilinear operators from B into B^*). By a *stochastic process of second order* with values in B (a *vector stochastic process*) we mean a mapping $X\colon \boldsymbol{R} \to L(B, H)$ (cf. [13]). If $B = \boldsymbol{C}$, then X becomes the mapping $X\colon \boldsymbol{R} \to H$ which is said to be a *scalar stochastic process*. Clearly, if $X\colon \boldsymbol{R} \to L(B, H)$, then for each $b \in B$ the process $X(\cdot)b$ is a scalar stochastic process. We shall use the notation

$$H_t(X) = \overline{\mathrm{sp}}\,\{X(s)b\colon s \leqslant t,\ b \in B\},$$
$$H_\infty(X) = \overline{\mathrm{sp}}\,\{X(s)b\colon s \in \boldsymbol{R},\ b \in B\}, \qquad H_{-\infty}(X) = \bigcap_{t \in \boldsymbol{R}} H_t,$$

where $\mathrm{sp}\,A$ ($\overline{\mathrm{sp}}\,A$) denotes the linear space (closed linear space) spanned by $A \subset H$.

By $\mathscr{B}_{\boldsymbol{C}}$ we denote the family of Borel subsets of the complex plane $\boldsymbol{C}$. We admit after [5] the following definition:

1.1. Definition. Let $X\colon \boldsymbol{R} \to L(B, H)$. A family of linear continuous operators $\{V_u\colon u \in \boldsymbol{R}\}$, $V_u\colon H_\infty(X) \to H_\infty(X)$, $u \in \boldsymbol{R}$, is called a *propagator* of the process X if $V_s X(t) = X(t+s)$ for each $t, s \in \boldsymbol{R}$.

Note that a propagator of X is a one-parameter group of operators on $H_\infty(X)$ (cf. [5]), whence each operator V_u is invertible.

1.2. Definition. A process $X\colon \boldsymbol{R} \to L(B, H)$ is said to be of *type* $(*)$ if X has a propagator $\{V_u\colon u \in \boldsymbol{R}\}$ and the following conditions hold:

(1) for each $b \in B$ the process $X(\cdot)b$ is continuous;

(2) for each $t \in \boldsymbol{R}$ the operator V_t is normal;

(3) there exist constants $\varepsilon > 0$ and $M > 0$ such that $\|V_t\| \leqslant M$ for each $t \in (-\varepsilon, \varepsilon)$.

1.3. Example. (i) Let $B = H$, $A \in L(H, H)$, and let A be normal. The group of operators $\{e^{tA}\colon t \in \boldsymbol{R}\}$ is a vector process of type $(*)$.

(ii) Consider the Lebesgue measure m on $D = \{\lambda \in \boldsymbol{C}\colon |\lambda| \leqslant 1\}$ and let $H = L_2(D, m)$ be the space of all square-integrable functions on D. If we put $x(t) = e^{t\lambda}$ for $t \in \boldsymbol{R}$ and $\lambda \in D$, then the process $\{x(t)\colon t \in \boldsymbol{R}\}$ is a scalar process of type $(*)$ with the propagator given by $(V_t f)(\lambda) = e^{t\lambda} f(\lambda)$, $f \in H$.

For the proof of the characterization theorem of the process of type $(*)$ we shall need the following lemma:

1.4. Lemma. *Let $\{Q(t)\colon t \in \boldsymbol{R}\} \subset L(H, H)$ be a one-parameter group of normal operators satisfying the following continuity condition*:

(a) $\lim_{t \to 0} \|Q(t)h - h\| = 0$ *for each* $h \in H$.

Then there exists a closed normal operator A densely defined in H and such that for some $a, b \in \boldsymbol{R}$ the spectrum $\sigma(A)$ is contained in $\{\lambda \in \boldsymbol{C}\colon a \leqslant \operatorname{Re}\lambda \leqslant b\}$ and $Q(t) = e^{tA}$ for each $t \in \boldsymbol{R}$.

This lemma is an immediate consequence of Theorem 13.37 in [11].

The propagator $\{V_t\colon t \in \boldsymbol{R}\}$ of any process X of type $(*)$ satisfies condition (a). In fact, if $t_n \to 0$, then

$$V_{t_n} X(s)\, b = X(t_n + s)\, b \to X(s)\, b$$

since $X(\cdot)b$ is continuous and, therefore, $V_{t_n} h \to h$ for $h \in \operatorname{sp}\{X(t)b\colon t \in \boldsymbol{R},\, b \in B\}$. If $h \in H_\infty(X)$, then

$$h = \lim_{m \to \infty} h_m \quad \text{and} \quad h_m \in \operatorname{sp}\{X(t)b\colon t \in \boldsymbol{R},\, b \in B\}$$

($\{h_m\}$ is a sequence approximating h). Thus we have

$$\begin{aligned} \|V_{t_n} h - h\| &\leqslant \|V_{t_n} h - V_{t_n} h_m\| + \|V_{t_n} h_m - h_m\| + \|h_m - h\| \\ &\leqslant (\|V_{t_n}\| + 1)\|h_m - h\| + \|V_{t_n} h_m - h_m\| \\ &\leqslant (M+1)\|h_m - h\| + \|V_{t_n} h_m - h_m\| < \delta \end{aligned}$$

for sufficiently large n.

1.5. Remark. Consider now a σ-additive measure Φ on $\mathscr{B}_{\boldsymbol{C}}$ with values in $L(B, H)$ (with strong topology) satisfying the following orthogonal condition:

$$(\alpha) \qquad [\Phi(s_1)]^* \Phi(s_2) = 0 \quad \text{if } S_1 \cap S_2 = \emptyset.$$

Let $f\colon \boldsymbol{C} \to \boldsymbol{C}$ be a Borel bounded function. We claim that the operator $Z_f\colon B \to H$,

$$Z_f b = \int_{\boldsymbol{C}} f(\lambda)[\Phi(d\lambda)\, b],$$

belongs to $L(B, H)$. For $b \in B$, $\Phi(\cdot)b$ is an orthogonal vector measure with values in H, $\varphi_b(\cdot) = \|\Phi(\cdot)b\|^2$ is a nonnegative finite Borel measure on $\boldsymbol{C}$, and

$$\begin{aligned} \Big\|\int_{\boldsymbol{C}} f(\lambda)[\Phi(d\lambda)\, b]\Big\|^2 &= \int_{\boldsymbol{C}} |f|^2\, d\varphi_b \leqslant \big(\sup_{\boldsymbol{C}} |f(\lambda)|^2\big)\, \varphi_b(\boldsymbol{C}) \\ &= \big(\sup_{\boldsymbol{C}} |f(\lambda)|^2\big)\|\Phi(\boldsymbol{C})\, b\|^2 \leqslant \big(\sup_{\boldsymbol{C}} |f(\lambda)|^2\big)\|\Phi(\boldsymbol{C})\|^2\, \|b\|^2. \end{aligned}$$

From now on we put

$$Z_f = \int_{\boldsymbol{C}} f(\lambda)\, \Phi(d\lambda)$$

(cf. also [7]).

1.6. Theorem. *The process $X\colon \boldsymbol{R} \to L(B, H)$ is of type $(*)$ if and only if there exists an operator measure Φ on $\mathscr{B}_{\boldsymbol{C}}$ with values in $L(B, H_\infty(X))$, satisfying condition* (α), *supported in $\{\lambda \in \boldsymbol{C}\colon a \leqslant \operatorname{Re}\lambda \leqslant b\}$ (for some $a, b \in \boldsymbol{R}$), and such that*

$$(\beta) \qquad X(t) = \int_{\boldsymbol{C}} e^{t\lambda}\, \Phi(d\lambda).$$

Proof. We suppose that $\{V_t\colon t\in \boldsymbol{R}\}$ is the propagator of the process X of type $(*)$. Let A be a closed normal operator densely defined on $H_\infty(X)$, such that $V_t = e^{tA}$ for $t\in\boldsymbol{R}$ and $\sigma(A)\subset\{\lambda\in\boldsymbol{C}\colon a\leqslant \operatorname{Re}\lambda\leqslant b\}$ (Lemma 1.4). The spectral measure E of the operator A is concentrated on $\sigma(A)$. If we define the measure Φ for a Borel subset $S\subset\boldsymbol{C}$ by $\Phi(S) = E(S)X(0)$, then $\Phi(S)\in L(B, H_\infty(X))$ and Φ satisfies condition (α). Now the equalities

$$X(t) = V_t X(0) = e^{tA}X(0) = \Big[\int_{\boldsymbol{C}} e^{t\lambda}E(d\lambda)\Big]X(0) = \int_{\boldsymbol{C}} e^{t\lambda}\big(E(d\lambda)X(0)\big)$$
$$= \int_{\boldsymbol{C}} e^{t\lambda}\Phi(d\lambda)$$

yield (β).

Suppose now that X is of the form (β) and Φ satisfies the conditions of the theorem. For a Borel subset $S\subset\boldsymbol{C}$ we put

$$\Delta(S) = [X(0)]^*\Phi(S) = [\Phi(\boldsymbol{C})]^*\Phi(S).$$

Then $\Delta(S)\in\bar{L}(B, B^*)$ and $\big(\Delta(S)b\big)b = \big(\Phi(S)b, \Phi(\boldsymbol{C})b\big) = \|\Phi(S)b\|^2$ is a non-negative measure on $\boldsymbol{C}$. By the dilation theorem (cf. [14], (2.1)), there exist a Hilbert space $\tilde{H}$, a spectral measure $\tilde{E}(\cdot)\colon \tilde{H}\to\tilde{H}$, and an operator $Z\in L(B, \tilde{H})$ such that

$$\Delta(S) = Z^*\tilde{E}(S)Z.$$

We may assume that $\tilde{H}$ is minimal, i.e.

$$\tilde{H} = \bigvee_{S\in\mathscr{B}_{\boldsymbol{C}}} \tilde{E}(S)ZB = \overline{\operatorname{sp}}\{\tilde{E}(S)Zb\colon S\in\mathscr{B}_{\boldsymbol{C}},\ b\in B\}.$$

Then the Hilbert spaces

$$H_\infty(X) = \bigvee_{S\in\mathscr{B}_{\boldsymbol{C}}}\Phi(S)B \quad\text{and}\quad \tilde{H} = \bigvee_{S\in\mathscr{B}_{\boldsymbol{C}}}\tilde{E}(S)ZB$$

are isometric. In fact,

$$\Big\|\sum_{i=1}^{n}\alpha_i\tilde{E}(S_i)Z(b_i)\Big\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\bar{\alpha}_j\big(\Phi(S_i\cap S_j)b_i, \Phi(S_i\cap S_j)b_j\big)$$
$$= \Big\|\sum_{i=1}^{n}\alpha_i\Phi(S_i)b_i\Big\|^2.$$

Hence $\Psi\colon \Phi(S)b\to\tilde{E}(S)Zb$ has an extension to a unitary operator from $H_\infty(X)$ onto $\tilde{H}$.

Consider now the following operator group on $\tilde{H}$:

$$\tilde{V}_t = \int_{\boldsymbol{C}} e^{t\lambda}\tilde{E}(d\lambda), \qquad t\in\boldsymbol{R}.$$

Since $\operatorname{supp}\tilde{E} = \operatorname{supp}\Delta = \operatorname{supp}\Phi$, the function $e^{t\lambda}$ is bounded on $\operatorname{supp}\tilde{E}$ and the operators $\tilde{V}_t$ are bounded and normal. Let

$$\tilde{X}(t) = \tilde{V}_t Z = \int_C e^{t\lambda}(\tilde{E}(d\lambda)Z).$$

Obviously, $\Psi(X(t)) = \tilde{X}(t)$, $\tilde{H}$ is spanned by $\{\tilde{X}(t)b\colon t\in \mathbf{R}, b\in B\}$, and $\{\tilde{V}_t\colon t\in \mathbf{R}\}$ is the propagator of $\tilde{X}$. We claim that this process is of type (*). Condition (3) of Definition 1.2 follows from the equality

$$\|\tilde{V}_t\| = \sup\{e^{t\lambda}\colon \lambda\in \operatorname{supp}\tilde{E}\} = \exp\{t\max(a, b)\}.$$

If $t_n \to t_0$, then

$$\|\tilde{X}(t_n)b - \tilde{X}(t_0)b\|^2 = \Big\|\int_C \exp\{t_n\lambda\}(\tilde{E}(d\lambda)Zb) - \int_C \exp\{t_0\lambda\}(\tilde{E}(d\lambda)Zb\Big\|^2$$
$$= \int_C |\exp\{t_n\lambda\} - \exp\{t_0\lambda\}|^2 \|\tilde{E}(d\lambda)Zb\|^2$$

and the Lebesgue theorem implies that this sequence tends to zero, which proves the continuity of $\tilde{X}(\cdot)b$, $b\in B$. Clearly, $X(t) = \Psi^{-1}\tilde{X}(t)$ is also the process of type (*) with the propagator $V_t = \Psi^{-1}\tilde{V}_t\Psi$. The proof is complete.

The measure Φ will be called the *spectral measure* of the process X.

1.7. Remark. Theorem 1.6 implies that $x\colon \mathbf{R}\to H$ is a process of type (*) if and only if there exists an orthogonal vector measure Φ on $\mathscr{B}_C$ with values in $H_\infty(x)$, supported in $\{\lambda\in \mathbf{C}\colon a \leqslant \operatorname{Re}\lambda \leqslant b\}$, and such that

$$x(t) = \int_C e^{t\lambda}\Phi(d\lambda).$$

The measure $\varphi(S) = \|\Phi(S)\|^2$ is a nonnegative Borel measure on $\mathbf{C}$. Consider the Hilbert space $K = \overline{\operatorname{sp}}\{e^{t\lambda}\colon t\in \mathbf{R}\} \subset L_2(\mathbf{C}, \varphi)$ and the operator group $T_t f(\lambda) = e^{t\lambda}f(\lambda)$. The operator

$$\Psi\colon K\to H_\infty(x)\colon f\to \int_C f(\lambda)\Phi(d\lambda)$$

is unitary. We have $\Psi(e^{t\lambda}) = x(t)$ and $V_t = \Psi T_t \Psi^{-1}$, where $\{V_t\}$ is the propagator of x. Therefore, if $x\colon \mathbf{R}\to H$ is of type (*) and Φ is its spectral measure, then there exists a unitary isomorphism between $H_\infty(x)$ and K.

Given a process $X\colon \mathbf{R}\to L(B, H)$, one may consider, after Weron [13], associated processes of the form $Y_t = P_t X_t$, where $P_t\in L(H, H)$. We can prove now that a process of type (*) is associated with a stationary process. Let us recall that a process $X\colon \mathbf{R}\to L(B, H)$ is *stationary* if X has a unitary propagator (cf. [14], [1.6]).

1.8. THEOREM. *A process $X\colon \mathbf{R}\to L(B, H)$ with the propagator $\{V_t\colon t\in \mathbf{R}\}$ is of type (*) if and only if for each $t\in \mathbf{R}$ we have $V_t = P_t U_t$, where*

(i) $P_t\colon H_\infty(X)\to H_\infty(X)$ *is hermitian,*
(ii) $U_t\colon H_\infty(X)\to H_\infty(X)$ *is unitary,*

(iii) $\{P_t\colon t \in \boldsymbol{R}\}$ *and* $\{U_t\colon t \in \boldsymbol{R}\}$ *are one-parameter groups of linear continuous operators,*

(iv) $P_t U_s = U_s P_t$ *for each* $t, s \in \boldsymbol{R}$,

(v) *the processes* $Y(t) = U_t X(0)$ *and* $Z(t) = P_t X(0)$ *are of type* $(*)$.

Proof. The "if" part is rather obvious: if P_t is hermitian and U_t is unitary, then $V_t = U_t P_t$ is normal. Furthermore, $\|V_t\| = \|U_t P_t\| \leqslant \|P_t\|$. Hence condition (3) of Definition 1.2 holds. Since $\{P_t\}$ and $\{U_t\}$ are the strong continuous operator groups (this follows from (v); cf. Lemma 1.4 and the remarks after this lemma), the operator group $\{V_t\}$ is also strong continuous, which implies continuity of the process $X(t)b$, $b \in B$.

Let now $X\colon \boldsymbol{R} \to L(B, H)$ be of type $(*)$ with the propagator $\{V_t\colon t \in \boldsymbol{R}\}$ and let E be the spectral measure of A such that $V_t = e^{tA}$, $t \in \boldsymbol{R}$. Then

$$V_t = \int_{\boldsymbol{C}} e^{t\lambda} E(d\lambda) = \int_{\boldsymbol{C}} \exp\{t\lambda_1 + it\lambda_2\} E(d\lambda) = \int_{\boldsymbol{C}} \exp\{t\lambda_1\} E(d\lambda) \int_{\boldsymbol{C}} \exp\{it\lambda_1\} E(d\lambda)$$

$$= \int_{\boldsymbol{R}} \exp\{t\lambda_1\} E_{\boldsymbol{R}}(d\lambda_1) \int_{\boldsymbol{R}} \exp\{it\lambda_2\} E_I(d\lambda_2) = P_t U_t,$$

where $\lambda_1 = \operatorname{Re}\lambda$, $\lambda_2 = \operatorname{Im}\lambda$, and $E_{\boldsymbol{R}}(S) = E(S \times \boldsymbol{R})$, $E_I(S) = E(\boldsymbol{R} \times S)$ for a Borel subset $S \subset \boldsymbol{R}$.

Clearly, for any $t, s \in \boldsymbol{R}$ the operator P_t is selfadjoint, the operator U_s is unitary, and P_t and U_s commute. Putting

$$A_{\boldsymbol{R}} = \int_{\boldsymbol{R}} v E_{\boldsymbol{R}}(dv) \quad \text{and} \quad A_I = \int_{\boldsymbol{R}} v E_I(dv)$$

we have $P_t = \exp\{tA_{\boldsymbol{R}}\}$ and $U_t = \exp\{tA_I\}$. Thus $\{P_t\colon t \in \boldsymbol{R}\}$ and $\{U_t\colon t \in \boldsymbol{R}\}$ are one-parameter groups of bounded operators. Moreover, since $\operatorname{supp} E_{\boldsymbol{R}}$ is bounded, $A_{\boldsymbol{R}}$ is continuous, which in turn implies that the representation $\{P_t\colon t \in \boldsymbol{R}\}$ is continuous in the operator norm. Therefore, the process $Z_t = P_t X(0)$ is of type $(*)$. Since $U_t = P_t^{-1} V_t$, the process $Y_t = U_t X(0)$ is also of type $(*)$.

1.9. Corollary. *Let* $X\colon \boldsymbol{R} \to L(B, H)$ *be a process of type* $(*)$ *with the propagator* $\{V_t\colon t \in \boldsymbol{R}\}$ *and the spectral measure* Φ. *Then*

$$X(t) = P_t Y(t) = U_t Z(t),$$

where $Y(t) = U_t X(0)$ *is stationary,* $Z(t) = P_t X(0)$ *is of type* $(*)$ *with the hermitian propagator. Moreover,*

$$Y(t) = \int_{\boldsymbol{R}} e^{it\lambda} \Phi_I(d\lambda) \quad \text{and} \quad Z(t) = \int_{\boldsymbol{R}} e^{t\lambda} \Phi_{\boldsymbol{R}}(d\lambda),$$

where $\Phi_I(S) = \Phi(\boldsymbol{R} \times S)$ *and* $\Phi_{\boldsymbol{R}}(S) = \Phi(S \times \boldsymbol{R})$ *for any Borel subset* $S \subset \boldsymbol{R}$.

2. In this section we study regularity and singularity of stochastic processes with propagators. Recall that a process $X\colon \boldsymbol{R} \to L(B, H)$ is *regular* if $H_{-\infty}(X) = \{0\}$ and it is *singular* if $H_{\infty}(X) = H_{-\infty}(X)$.

2.1. LEMMA. *The process X: $\mathbf{R} \to L(B, H)$ is singular if and only if there exist $t > 0$ and $s \in \mathbf{R}$ such that $H_s(X)$ is the invariant subspace of the operator V_t.*

Proof. The "only if" part follows from the fact that the singularity of X implies the equality $H_s(X) = H_\infty(X)$ for each $s \in \mathbf{R}$.

If $t > 0$, $s \in \mathbf{R}$, and $V_t H_s(X) \subset H_s(X)$, then $V_t H_s(X) = H_s(X)$ (because $V_t H_s(X) = H_{t+s}(X)$ for all $t, s \in \mathbf{R}$ and $H_s(X) \subset H_{t+s}(X)$ for $t > 0$). Hence $V_{-t} H_s(X) = V_t^{-1} H_s(X) = H_s(X)$. For any natural number k we have

$$H_{kt+s}(X) = V_t^k H_s(X) = H_s(X), \qquad H_{-kt+s}(X) = V_t^{-k} H_s(X) = H_s(X).$$

Consequently, $H_u(X) = H_s(X)$ for each $u \in \mathbf{R}$, which is equivalent to singularity of X.

2.2. THEOREM. *Suppose that $\{V_t: t \in \mathbf{R}\}$ is the propagator of X: $\mathbf{R} \to L(B, H)$. If $\|I - V_{s_0}\| < 1$ for some $s_0 < 0$, then X is singular.*

Proof. Notice that for any operator $T \in L(H, H)$ such that $\|I - T\| < 1$ and $TH_0 \subset H_0$ for some closed subspace $H_0 \subset H$ we have $T^{-1} H_0 \subset H_0$. In fact,

$$T^{-1} = \sum_{n=0}^{\infty} (I-T)^n$$

(the series convergent in the operator norm). Hence

$$T^{-1} h = \Big(\sum_{n=0}^{\infty} (I-T)^n\Big) h = \sum_{n=0}^{\infty} (I-T)^n h \in H_0 \quad \text{for any } h \in H_0.$$

Suppose now that there exists $s_0 < 0$ such that $\|I - V_{s_0}\| < 1$. For each $s < 0$ and $t \in \mathbf{R}$ we have $V_s H_t(X) = H_{t+s}(X) \subset H_t(X)$. Thus

$$V_{s_0} H_t(X) \subset H_t(X) \quad \text{and} \quad V_{-s_0} H_t(X) = V_{s_0}^{-1} H_t(X) \subset H_t(X)$$

for any $t \in \mathbf{R}$. Consequently, for $u = -s_0 > 0$ we get $V_u H_t(X) \subset H_t(X)$ for each t, and Lemma 2.1 implies singularity of X.

2.3. COROLLARY. *If the propagator of a process X considered as a representation is continuous in the operator norm, then X is singular. In particular, X is singular if $V_t = e^{tA}$ for each $t \in \mathbf{R}$ and $A \in L(H_\infty(X), H_\infty(X))$.*

2.4. COROLLARY. *Let X: $\mathbf{R} \to L(B, H)$ be a process of type $(*)$. If the spectral measure Φ of X has a bounded support, then X is singular.*

Proof. We have $\operatorname{supp} \Phi = \operatorname{supp} E$, where E is the spectral measure of the operator A such that $V_t = e^{tA}$ is the propagator of X. Hence $\sigma(A)$ is a bounded subset of $\mathbf{C}$ and the operator A is bounded. It follows from Corollary 2.3 that X is singular.

2.5. Remark. Note that the process $Z(t) = P_t X(0)$ (Z, X, P_t, U_s as in Theorem 1.8) is singular since its spectral measure $\Phi_{\mathbf{R}}$ has a bounded support. Therefore, the conditions $U_s^{-1} H_s(X) \subset H_s(X)$ and $U_s H_s(Z) \subset H_s(Z)$ for $s \in \boldsymbol{R}$ are sufficient for singularity of X because $U_s^{-1} H_s(X) \subset H_s(X)$ implies $H_s(Z) \subset H_s(X)$, while $U_s H_s(Z) \subset H_s(Z)$ gives $H_s(Z) \subset H_s(X)$, and hence

$$H_\infty(X) = H_\infty(Z) = H_{-\infty}(Z) = H_{-\infty}(X).$$

2.6. Example. We show that the condition given in Theorem 2.2 is not necessary. Let $x\colon \boldsymbol{R} \to H$ be a stationary (scalar) process. Then x has a unitary propagator $\{U_t\colon t \in \boldsymbol{R}\}$ and

$$x(t) = \int_{\boldsymbol{R}} e^{it\lambda} \Phi(d\lambda),$$

where Φ is an orthogonal vector measure with values in $H_\infty(x)$. It follows from Remark 1.7 that

$$\|U_t - U_s\| = \sup\{|e^{it\lambda} - e^{is\lambda}|\colon \lambda \in \operatorname{supp}\Phi\} = \sup_{\operatorname{supp}\Phi} [2 - 2\cos((t-s)\lambda)]^{1/2}.$$

If $s = 0$, then

$$\|U_t - I\| = \sup_{\operatorname{supp}\Phi} (2 - 2\cos(t\lambda))^{1/2}. \tag{1}$$

Let now Φ be an orthogonal vector measure on $\boldsymbol{R}$ supported in $\boldsymbol{R} \setminus [a, b]$, $-\infty < a < b < \infty$, $\varphi(\cdot) = \|\Phi(\cdot)\|^2$, and let φ_a be the absolutely continuous (with respect to the Lebesgue measure on $\boldsymbol{R}$) part of the measure φ. We have then

$$\int_{\boldsymbol{R}} \left(\log \frac{d\varphi_a}{dt}(t)\right) \frac{1}{1+t^2}\, dt = -\infty$$

and it is well known (cf. [15]) that in this case the process x is singular. On the other hand, from (1) we get $\|U_t - I\| = 2$ for each $t \in \boldsymbol{R}$.

2.7. Remark. By Theorem 1.8, a process X of type $(*)$ takes the form $X(t) = P_t Y(t)$, where the process Y is stationary. Singularity and regularity of vector stationary processes are studied in [1]. The relations between regularity (singularity) properties of processes X and Y such that $X(t) = P_t Y(t)$ are found by Weron [13], and for scalar processes by Mandrekar [4].

2.8. Proposition. *Let $x\colon \boldsymbol{R} \to H$ be a scalar process of type $(*)$ with the propagator $\{V_t\colon t \in \boldsymbol{R}\}$, $V_t = U_t P_t$ for $\{U_t\colon t \in \boldsymbol{R}\}$ and $\{P_t\colon t \in \boldsymbol{R}\}$ given by Theorem 1.8, and $y(t) = P_t x(0)$. Suppose that $P_t H_t(y) \subset H_t(y)$ for $t \in \boldsymbol{R}$. If the spectral measure Φ is absolutely continuous with respect to the Lebesgue measure m on the complex plaine $\boldsymbol{C}$ and*

$$\int_{\boldsymbol{R}} \frac{d\|\Phi\|^2}{dm}(u+iv)\, dv > 0$$

for $\tilde{m}$-almost every $v \in \mathbf{R}$ ($\tilde{m}$ denotes the Lebesgue measure on $\mathbf{R}$), then the process x is regular.

Proof. In this case Φ is a vector measure with values in the Hilbert space $H_{\infty}(x)$. If Φ is absolutely continuous with respect to m, then Φ_I, the projection Φ of an imaginary axis ($\Phi_I(S) = \Phi(\mathbf{R} \times S)$), is absolutely continuous with respect to m. The nonnegative measure $\varphi_I(S) = \|\Phi_I(S)\|^2$ has density of the form

$$\frac{d\varphi_I}{dm}(v) = \int_{\mathbf{R}} \frac{d\|\Phi\|^2}{dm}(u+iv)\,du$$

whose right-hand side is positive $\tilde{m}$-a.e. by assumption. Since Φ_I is the spectral measure of the stationary process y (cf. Corollary 1.9), it follows from the theory of stationary processes that y is regular (cf. [15]). The relation $P_t H_t(y) \subset H_t(y)$ implies $H_t(x) \subset H_t(y)$ for each t and $H_{-\infty}(x) \subset H_{-\infty}(y) = \{0\}$, which completes the proof.

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