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ON ZERO-SUM STOCHASTIC GAMES WITH GENERAL STATE SPACE. II

BY

ANDRZEJ S. NOWAK (WROCŁAW)

Abstract. In this paper we consider a discrete-time infinite horizon zero-sum stochastic game with a bounded non-negative reward function per stage. Under a natural integrability assumption (cf. [9], assumption (P)), we show the existence of value and optimal stationary (ε-optimal semi-stationary, $\varepsilon > 0$) strategies for a minimizer (maximizer).

This paper is a continuation of [9], so we follow all previous notation, definitions and enumeration of sections. Here we deal with positive dynamic programming (Section 6) and positive stochastic games (Section 7). The comparison and discussion of the results obtained are given in Remarks 6.1 and 7.2.

6. Positive dynamic programming. In this part of the paper we give some auxiliary results on dynamic programming for models which satisfy the condition (P) described in [9]. Throughout this section $g^{(\infty)}$ is assumed to be any stationary strategy of player II determined by $g \in \mathscr{S}_{P_B}$. Let $\{\beta_n\}$ be a sequence such that, for each $n \geqslant 1$, $0 \leqslant \beta_n \leqslant \beta_{n+1} < 1$ and $\beta_n \to 1$ as $n \to \infty$.

For each $\pi \in \Pi$ we set

$$I_n(\pi, g^{(\infty)}) = \sum_{k=1}^{\infty} \beta_n^{k-1} r_k(\pi, g^{(\infty)}),$$

where $r_k(\pi, g^{(\infty)})$ is the expected reward function at the k-th stage (see Section 2). Further, let us put

$$v_n^*(g) = \sup_{\pi \in \Pi} I_n(\pi, g^{(\infty)}) \quad \text{and} \quad v^*(g) = \sup_{\pi \in \Pi} V(\pi, g^{(\infty)}).$$

LEMMA. 6.1. *We have*

$$v^*(g) = \lim_n v_n^*(g).$$

Proof. For each $n \geq 1$ we obtain $v_n^*(g) \leq v_{n+1}^*(g) \leq v^*(g)$. Hence

$$\lim_n v_n^*(g) \leq v^*(g). \tag{6.1}$$

On the other hand, for each $\pi \in \Pi$ we get

$$V(\pi, g^{(\infty)}) = \lim_n I_n(\pi, g^{(\infty)}) \leq \lim_n v_n^*(g),$$

whence

$$v^*(g) \leq \lim_n v_n^*(g). \tag{6.2}$$

By (6.1) and (6.2) the lemma is proved.

Let $u \in M_q(S)$. For each $n \geq 1$ and $\mu \in P_A(s)$ we set

$$L_n(s, \mu, g(s))(u) = \int_X \int_Y [r(s, x, y) + \beta_n \int_S u(t) q(dt|s, x, y)] g(dy|s) d\mu$$

and

$$T_{ng}(u)(s) = \sup_{\mu \in P_A(s)} L_n(s, \mu, g(s))(u).$$

The following two lemmas may be easily deduced from Theorems 5.3 and 5.4.

LEMMA 6.2. *For each of the models* (M_1), (M_2), (M_3) *and for each* $n \geq 1$ *there is* $f_n^{(\infty)} \in \Pi_s$ *such that* $v_n^*(g) = I_n(f_n^{(\infty)}, g^{(\infty)})$.

LEMMA 6.3. *For each of the models* (M_1), (M_2), (M_3) *and for each* $n \geq 1$ *we have* $v_n^*(g) = T_{ng}(v_n^*(g))$.

The following theorem is one of the main results of this section.

THEOREM 6.1. *For each of the models* (M_1), (M_2), (M_3) *and for each* $\varepsilon > 0$ *there is a semi-stationary strategy* $\pi \in \Pi_{ss}$ *of player* I *such that* $v^*(g) < V(\pi, g^{(\infty)}) + \varepsilon$.

Proof. We define a sequence of sets

$$S_1 = \{s \in S: v^*(g)(s) < v_1^*(g)(s) + \varepsilon\}$$

and

$$S_n = \{s \in S: v^*(g)(s) < v_n^*(g)(s) + \varepsilon\} - \bigcup_{k=1}^{n-1} S_k.$$

It is clear that $S_n \cap S_m = \emptyset$ for $n \neq m$, and $\bigcup_{n=1}^{\infty} S_n = S$. Theorem 5.1 and Lemma 6.1 imply that $S_n \in \mathcal{S}$ for each $n \geq 1$. Without loss of generality we may assume that $S_n \neq \emptyset$ for each $n \geqslant 1$. By Lemma 6.2, there is a sequence $f_1^{(\infty)}, f_2^{(\infty)}, \ldots$ of stationary strategies of player I such that $v_n^*(g) = I_n(f_n^{(\infty)}, g^{(\infty)})$, whence $v_n^*(g) \leqslant V(f_n^{(\infty)}, g^{(\infty)})$. Let $s \in S_m$; then

$$(6.3) \qquad v^*(g)(s) < V(f_m^{(\infty)}, g^{(\infty)})(s) + \varepsilon.$$

Now, we define $\pi \in \Pi_{ss}$ as $\pi = (\tau, \tau, \tau, \ldots)$, where $\tau(s, \cdot) = f_m(\cdot)$ if $s \in S_m$. By (6.3) we obtain

$$v^*(g)(s) < V(\pi, g^{(\infty)})(s) + \varepsilon \quad \text{for each } s \in S,$$

which completes the proof.

Let $u \in M_q(S)$ and put

$$T_g(u)(s) = \sup_{\mu \in P_A(s)} L(s, \mu, g(s))(u) \quad \text{for each } s \in S.$$

We prove now the next important result of this section.

THEOREM 6.2. *Assume* (M_1), (M_2), (M_3). *Then* $v^*(g) = T_g(v^*(g))$, *and* $v^*(g)$ *is the smallest non-negative solution of this equation.*

Proof. Under (P) we have $V(\pi, \gamma) \in M_q(S)$ for all $\pi \in \Pi$, $\gamma \in \Gamma$, so Theorem 6.1 implies $v^*(g) \in M_q(S)$ for each $g \in \mathscr{S}_{P_B}$. Hence $T_g(v^*(g))$ is well defined. By Lemma 6.3, for each $n \geqslant 1$ we have

$$(6.4) \qquad v_n^*(g) = T_{ng}(v_n^*(g)).$$

Hence $v_n^*(g) \leqslant T_{ng}(v^*(g)) \leqslant T_g(v^*(g))$ for each $n \geqslant 1$ and, by Lemma 6.1,

$$(6.5) \qquad v^*(g) \leqslant T_g(v^*(g)).$$

On the other hand, (6.4) implies $v^*(g) \geqslant T_{ng}(v_n^*(g))$ for each $n \geqslant 1$. Thus, we get

$$v^*(g) \geqslant L_n(s, \mu, g(s))(v_n^*(g)) \quad \text{for each } s \in S \text{ and } \mu \in P_A(s).$$

Using Lemma 6.1 and the monotone convergence theorem we obtain

$$v^*(g) \geqslant L(s, \mu, g(s))(v^*(s)) \quad \text{for each } s \in S \text{ and } \mu \in P_A(s)$$

and, consequently,

$$(6.6) \qquad v^*(g) \geqslant T_g(v^*(g)).$$

Combining (6.5) and (6.6) we get

$$(6.7) \qquad v^*(g) = T_g(v^*(g)).$$

Let $w \geqslant 0$ be another solution of equation (6.7) and suppose that $w(s_0) < v^*(g)(s_0)$ for some $s_0 \in S$. By Theorem 6.1 we can find a strategy $f^{(\infty)} \in \Pi_s$ such that $w(s_0) < V(f^{(\infty)}, g^{(\infty)})(s_0)$. We note that

$$V(f^{(\infty)}, g^{(\infty)})(s_0) = \lim_n T_{fg}^n(u)(s_0),$$

where $u = 0$, i.e., u is the function which vanishes identically. Further, we obtain

$$w(s_0) < V(f^{(\infty)}, g^{(\infty)})(s_0) = \lim_n T^n_{fg}(u)(s_0)$$
$$\leqslant \lim_n T^n_{fg}(w)(s_0) \leqslant \lim_n T^n_g(w)(s_0) \leqslant w(s_0),$$

which is a contradiction.

Remark 6.1. Results related to Theorem 6.1 for Borel space models with some semi-continuity and/or compactness assumptions have been obtained by Himmelberg et al. ([6], Theorem 1) and by Shreve and Bertsekas ([12], Theorem 15). They have established, for any $\varepsilon > 0$, the existence of ε-optimal semi-Markov strategies. Theorem 6.2 is due to Strauch ([13], Theorem 8.2). He studied only Borel space dynamic programming models and assumed that, for some $K \geqslant 0$, $V(\pi, \gamma)(s) \leqslant K$ whenever $\pi \in \Pi$, $\gamma \in \Gamma$, and $s \in S$.

7. The infinite horizon stochastic games satisfying the condition (P). Before proving the main results of this section we give some notation and auxiliary lemmas. Let $\{\beta_n\}$ be a sequence such that $0 \leqslant \beta_n \leqslant \beta_{n+1} < 1$ for each $n \geqslant 1$ and $\lim_n \beta_n = 1$. Let us put

$$I_n(\pi, \gamma) = \sum_{k=1}^{\infty} \beta_n^{k-1} r_k(\pi, \gamma).$$

It is clear that I_n $(n \geqslant 1)$ satisfies the condition (D) from [9]. The value function w_n^* for the modified positive stochastic game with the reward function I_n, $n \geqslant 1$, exists by Theorem 5.1. Moreover, $w_n^* \in B(S)$ for each $n \geqslant 1$ since r is bounded and $0 \leqslant \beta_n < 1$.

For any $u \in M_q(S)$ and $n \geqslant 1$, we put

$$L_n(s, \mu, \lambda)(u) = \int_X \int_Y [r(s, x, y) + \beta_n \int_S u(t) q(dt|s, x, y)]\, d\mu\, d\lambda,$$

where $s \in S$, $\mu \in P_A(s)$, $\lambda \in P_B(s)$, and

$$T_n(u)(s) = \sup_{\mu \in P_A(s)} \inf_{\lambda \in P_B(s)} L_n(s, \mu, \lambda)(u) = \inf_{\lambda \in P_B(s)} \sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(u)$$

if the last equality holds.

By Theorem 5.3 we obtain

LEMMA 7.1. *For all models* (M_1), (M_2), (M_3) *and for each* $n \geqslant 1$, $T_n(w_n^*)$ *is well defined and* $w_n^* = T_n(w_n^*)$.

We note that $0 \leqslant w_n^* \leqslant w_{n+1}^* \leqslant \underline{v}^* \leqslant v^*(g)$ for each $g \in \mathcal{S}_{P_B}$ and each $n \geqslant 1$. Let $w^* = \lim_n w_n^*$. Then

$$0 \leqslant w^* \leqslant \underline{v}^*. \tag{7.1}$$

Moreover, Theorem 5.1 and the condition (P) imply that $w^* \in M_q(S)$.

LEMMA 7.2. *For each $s \in S$ and $\lambda \in P_B(s)$ we have*

(i) $$\lim_n \sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*) = \sup_{\mu \in P_A(s)} L(s, \mu, \lambda)(w^*),$$

(ii) $$\lim_n \sup_{\mu \in P_A(s)} \inf_{\lambda \in P_B(s)} L_n(s, \mu, \lambda)(w_n^*) = \sup_{\mu \in P_A(s)} \lim_n \inf_{\lambda \in P_B(s)} L_n(s, \mu, \lambda)(w_n^*).$$

Proof. Since, for each $n \geqslant 1$,

$$\sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*) \leqslant \sup_{\mu \in P_A(s)} L(s, \mu, \lambda)(w^*),$$

we have

$$\lim_n \sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*) \leqslant \sup_{\mu \in P_A(s)} L(s, \mu, \lambda)(w^*). \tag{7.2}$$

On the other hand, the monotone convergence theorem implies

$$L(s, \mu, \lambda)(w^*) = \lim_n L_n(s, \mu, \lambda)(w_n^*) \leqslant \lim_n \sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*)$$

for each $s \in S$, $\mu \in P_A(s)$, and $\lambda \in P_B(s)$. Thus, we obtain

$$\sup_{\mu \in P_A(s)} L(s, \mu, \lambda)(w^*) \leqslant \lim_n \sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*). \tag{7.3}$$

Combining (7.2) and (7.3) we complete the proof of (i). The proof of part (ii) is similar.

LEMMA 7.3. *Assume* (M_1). *Then for each $n \geqslant 1$ we have*

(i) $$\sup_{\mu \in P_A(s)} L_n(s, \mu, \cdot)(w_n^*) \in C(P_B(s))$$

for each $s \in S$, and

(ii) $$\sup_{\mu \in P_A(\cdot)} L_n(\cdot, \mu, \cdot)(w_n^*) \in B(P(Y) \times S).$$

Proof. Since $w_n^* \in B(S)$, $n \geqslant 1$, we infer that $L_n(s, \cdot, \cdot)(w_n^*)$ is continuous on $P_A(s) \times P_B(s)$, being a compact metric space. Hence (i) follows immediately. In order to prove (ii) we refer to (4.2).

Using (4.2) and Lemma 1.5 we can prove the following

LEMMA 7.4. *Assume* (M_2) *or* (M_3). *Then for each $n \geqslant 1$ we have*

(i) $$\sup_{\mu \in P_A(s)} L_n(s, \mu, \cdot)(w_n^*) \in \underset{\sim}{C}(P_B(s))$$

for each $s \in S$ and, moreover, (ii) *of Lemma* 7.3 *holds.*

LEMMA 7.5. *For each of the models* (M_1), (M_2), (M_3) *we have $w^* = T(w^*)$.*

Proof. It is clear that, for each $s \in S$ and $\lambda \in P_B(s)$, the sequence $\sup_{\mu \in P_A(s)} L_n(s, \mu, \lambda)(w_n^*)$, $n \geqslant 1$, is non-decreasing and bounded. Using Lemma 1.1 and Lemmas 7.2, 7.3, and 7.4 for the models (M_1), (M_2), and (M_3), respectively, we obtain

$$\lim_n T_n(w_n^*)(s) = \lim_n \inf_{\lambda\in P_B(s)} \sup_{\mu\in P_A(s)} L_n(s, \mu, \lambda)(w_n^*)$$
$$= \inf_{\lambda\in P_B(s)} \lim_n \sup_{\mu\in P_A(s)} L_n(s, \mu, \lambda)(w_n^*) = \inf_{\lambda\in P_B(s)} \sup_{\mu\in P_A(s)} L(s, \mu, \lambda)(w^*)$$

and, similarly,

$$\lim_n T_n(w_n^*)(s) = \lim_n \sup_{\mu\in P_A(s)} \inf_{\lambda\in P_B(s)} L_n(s, \mu, \lambda)(w_n^*) = \sup_{\mu\in P_A(s)} \inf_{\lambda\in P_B(s)} L(s, \mu, \lambda)(w^*).$$

Consequently, by Lemma 7.1, we get $w^*(s) = T(w^*)(s)$ for each $s\in S$, which completes the proof.

LEMMA 7.6. *For each of the models* (M_1), (M_2), (M_3) *there is* $g\in\mathcal{S}_{P_B}$ *such that, for each* $s\in S$,

$$w^*(s) = T_g(w^*)(s) = \sup_{\mu\in P_A(s)} L(s, \mu, g(s))(w^*).$$

Proof. By Lemmas 7.2, 7.3, and 7.4 we have

$$\sup_{\mu\in P_A(\cdot)} L(\cdot, \mu, \cdot)(w^*) \in M(P(Y)\times S) \tag{7.4}$$

in all the cases (M_1), (M_2), (M_3). Moreover, since the sequence $\sup_{\mu\in P_A(s)} L_n(s, \mu, \cdot)(w_n^*)$ is non-decreasing, we get

$$\sup_{\mu\in P_A(s)} L(s, \mu, \cdot)(w^*) \in \underset{\sim}{C}(P_B(s)) \tag{7.5}$$

for each $s\in S$.

Now, for each $s\in S$, we define

$$D(s) = \{\lambda\in P_B(s)\colon \sup_{\mu\in P_A(s)} L(s, \mu, \lambda)(w^*) \leqslant T(w^*)(s)\}.$$

It follows from (7.4) and (7.5), Lemma 1.11 and Theorem 3.5 of [4] that in all the cases (M_1), (M_2), (M_3), the set $D(s)$ is compact and non-empty, and D has a measurable graph. In the cases (M_2) and (M_3), D is also measurable by Lemmas 1.8 and 1.7, respectively. Now we prove the measurability of D in the case (M_1).

For each $n\geqslant 1$ and $s\in S$ we define

$$D_n(s) = \{\lambda\in P_B(s)\colon \sup_{\mu\in P_A(s)} L_n(s, \mu, \lambda)(w_n^*) \leqslant T(w^*)(s)\}.$$

Clearly, $D_n(s)$ is non-empty and compact for each $n\geqslant 1$ and $s\in S$. From Lemmas 1.10, 1.11, and 7.3 we infer that D_n, $n\geqslant 1$, is measurable. From Lemma 7.2 it follows that

$$D(s) = \bigcap_{n=1}^{\infty} D_n(s) \quad \text{for each } s\in S.$$

Moreover, by Theorem 4.1 of [4], D is a measurable multifunction. Now, the lemma follows (in all the cases (M_1), (M_2), (M_3)) from the Kuratowski and Ryll-Nardzewski theorem on selectors ([8], Theorem 1).

We are now ready to state the main result of this section:

THEOREM 7.1. *For each of the models* (M_1), (M_2), (M_3) *the value function* v^* *exists and* $v^* = T(v^*) = w^*$. *Moreover, for any* $\varepsilon > 0$, *player* I *has an* ε*-optimal semi-stationary strategy and player* II *has an optimal stationary strategy.*

Proof. By Lemmas 7.5, 7.6, and Theorem 6.2, we have

$$w^* = T(w^*) = T_g(w^*) \geqslant \sup_{\pi\in\Pi} V(\pi, g^{(\infty)}) \tag{7.6}$$

for some $g \in \mathcal{S}_{P_B}$. Hence $w^* \geqslant \bar{v}^*$, which together with (7.1) and (7.6) implies

$$w^* = v^* = T(v^*) = \sup_{\pi\in\Pi} V(\pi, g^{(\infty)}),$$

i.e., the value function v^* exists and player II has the optimal stationary strategy $g^{(\infty)}$. By Theorem 5.4 there is a sequence $\{f_n^{(\infty)}\} \subset \Pi_s$, $n \geqslant 1$, such that

$$w_n^* = \inf_{\gamma\in\Gamma} I_n(f_n^{(\infty)}, \gamma), \quad n \geqslant 1.$$

Hence

$$w_n^* \leqslant \inf_{\gamma\in\Gamma} V(f_n^{(\infty)}, \gamma), \quad n \geqslant 1.$$

Since $v^* = \lim_n w_n^*$, the construction of ε-optimal (for any $\varepsilon > 0$) semi-stationary strategies for player I is similar to that from the proof of Theorem 6.1.

Remark 7.1. Van der Wal has given an example ([14], Example 2.26) satisfying the assumptions of Theorem 7.1, where a stationary (even Markov) ε-optimal strategy for player I does not exist. The optimal strategies for player I need not exist as the same example shows (cf. also [1] and [10]).

Finally, we prove the following

THEOREM 7.2. *Suppose one of the assumptions* (M_1), (M_2) *or* (M_3) *is satisfied and assume that* $v^* \in B(S)$ *and the set*

$$\{q(\cdot|s, x, y)\colon s\in S,\ x\in A(s),\ y\in B(s)\} \tag{7.7}$$

is relatively compact in the s-topology in $P(S)$. *Then* $w_n^* \to v^*$ *uniformly on* S *as* $n \to \infty$.

Proof. By Lemma 7.1, for each $n \geqslant 1$ we have $w_n^* = T_n(w_n^*)$ and, by Theorem 7.1, $v^* = T(v^*)$ and $w_n^* \to v^*$ pointwise as $n \to \infty$.

It is easy to check that

$$\begin{aligned}\sup_{s\in S}|v^*(s)-w_n^*(s)| &= \sup_{s\in S}\left|T(v^*)(s)-T_n(w_n^*(s))\right| \\ &\leqslant \sup_{s\in S}\sup_{x\in A(s)}\sup_{y\in B(s)}\Big|\int_S v^*(t)\,q(dt|s,x,y) \\ &\quad -\beta_n\int_S w_n^*(t)\,q(dt|s,x,y)\Big| \\ &\leqslant (1-\beta_n)\|v^*\|+\sup_{s\in S}\sup_{x\in A(s)}\sup_{y\in B(s)}\int_S |v^*(t)-w_n^*(t)|\,q(dt|s,x,y).\end{aligned}$$

Since $v^*-w_n^*$ monotonically decreases to 0, by Lemma 1.2, we obtain

$$\begin{aligned}&\lim_n \sup_{s\in S}|v^*(s)-w_n^*(s)| \\ &\qquad\leqslant \lim_n \sup_{s\in S}\sup_{x\in A(s)}\sup_{y\in B(s)}\int_S |v^*(t)-w_n^*(t)|\,q(dt|s,x,y) = 0,\end{aligned}$$

which completes the proof.

COROLLARY 7.1. *Under the assumptions of Theorem* 7.2 *player* I *has an* ε-*optimal stationary strategy for each* $\varepsilon > 0$.

Remark 7.2. Positive stochastic games with finite sets of states and actions were examined by Parthasarathy [10]. Everett has given an example ([2], Example 1) of the positive stochastic game with finite state space in which player I has no optimal strategy, stationary or any other. Kamerud studied a positive (unbounded) stochastic game with a countable state space and finite action spaces and showed that such a game has a value, player II has an optimal stationary strategy, and player I has an ε-optimal semi-stationary strategy for each $\varepsilon > 0$ (cf. [7], Theorem III.6, or [11], Theorem 1.5). Theorem 7.1 generalizes that result. Moreover, Theorem 7.1 generalizes Theorem 1 of Frid [3] and Theorem 2 of Himmelberg et al. [5], where S is assumed to be an SB-space and X and Y are assumed to be finite sets.

We close this section with the following example which shows that the relative compactness of the set (7.7) in the s-topology in $P(S)$ is an essential assumption in Theorem 7.2:

Example. Let $s_1 = 1$, $0 < s_{n+1} < s_n < 1$ for each $n \geqslant 2$, and

$$\lim_n s_n = 0.$$

Let $S = (0, 1)$, $X = A(s) = [0, 1]$, and $Y = B(s) = \{1\}$ for each $s\in S$. Assume that $q(\cdot|s, x, y) = q(\cdot|s)$ for each $s\in S$, $x\in X$, $y\in Y$ and, moreover, $q(\cdot|s) \ll m$ for each $s\in S$, where m is the Lebesgue measure on (0, 1). For each $s\in S$, denote by $h(\cdot, s)$ the Radon-Nikodym derivative of $q(\cdot|s)$ with respect to m and suppose that

$$h(t,s) = \begin{cases}(s_n-s)^{-1} & \text{for each } t\in[s, s_n) \text{ and } s\in[s_{n+1}, s_n),\\ 0 & \text{for other } t\in(0,1).\end{cases}$$

By Lemma 1.2 it can be easily verified that the set $\{q(\cdot|s): s \in S\}$ is not relatively compact in the s-topology in $P(S)$.

Recall that v_1^* (see Section 4) denotes the value function in the one-stage stochastic game. Under our assumptions we get

$$v_1^*(s) = \sup_{x \in X} r(s, x, 1) \quad \text{for each } s \in S.$$

Let the reward function r be non-negative and such that

$$(7.8) \qquad v_1^*(s) = \frac{1}{n} \frac{(s_n - s)^{1/n}}{(s_n - s_{n+1})^{1/n}}$$

whenever $s \in [s_{n+1}, s_n)$.

Observe that

$$(7.9) \qquad 0 < v_1^*(s) \leqslant 1/n \quad \text{for each } s \in [s_{n+1}, s_n),\ n \geqslant 1,$$

and

$$(7.10) \qquad v_1^*(s_{n+1}) = 1/n, \quad n \geqslant 1.$$

By a simple calculation, we obtain the following value function v^*:

$$(7.11) \qquad v^*(s) = v_1^*(s)(n+1) \quad \text{if } s \in [s_{n+1}, s_n),\ n \geqslant 1.$$

Consequently, by (7.9), v^* is bounded. Taking any sequence $\{\beta_k\}$ such that $0 < \beta_k < \beta_{k+1} < 1$ for each $k \geqslant 1$ and $\lim_k \beta_k = 1$, we obtain the following value functions w_k^*, $k \geqslant 1$, for the modified positive stochastic games described above:

$$(7.12) \qquad w_k^*(s) = v_1^*(s)(n+1)\frac{1}{n(1-\beta_k)+1} \quad \text{if } s \in [s_{n+1}, s_n),\ n \geqslant 1.$$

By (7.10) and (7.11) we get

$$\lim_n v^*(s_{n+1}) = 1,$$

and by (7.10) and (7.12) we have

$$\lim_n w_k^*(s_{n+1}) = 0 \quad \text{for each } k \geqslant 1.$$

Therefore, w_k^* does not converge uniformly on S to v^* as $k \to \infty$. It is proper to add that the stochastic game presented above satisfies also the condition (D) from Section 2.

Added in proof. The models (M_1) and (M_3) studied in parts I and II of this paper are more general than the so-called *Borel model* considered by P. R. Kumar and T. H. Shiau in the paper *Existence of value and randomized strategies in zero-sum discrete-time stochastic dynamic games* (SIAM J.

Control Optimization 19 (1981), p. 617-634). Dynamic games with a standard Borel state space, which include the positive games and the absorbing or discounted stochastic games under semi-continuity assumptions similar to (M_3), have been studied by U. Rieder in the paper *On semi-continuous dynamic games* (to appear).

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Institute of Mathematics
Technical University of Wrocław
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland

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