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ON SOME REPRESENTATIONS CONCERNING THE STOCHASTIC INTEGRALS

BY

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Abstract. Some additive functionals concerning the Cauchy's principal value or the Hadamard's finite part are investigated. A decomposition of additive functionals is proposed by using the Hilbert transform theory.

0. Introduction. Let T be a distribution which belongs to the class $\mathscr{D}'^{1}_{L^2,\mathrm{loc}}$. Let B_t be a one-dimensional Brownian motion. Fukushima has proposed a definition of the integral $\int_0^t T(B_s)\,ds$ (cf. [1] and [2]). The definition is given so that Ito's formula

$$F(B_t) = F(B_0)+\int_0^t \frac{dF}{dx}(B_s)\,dB_s+\frac{1}{2}\int_0^t T(B_s)\,ds$$

remains still valid, where F belongs to $\mathscr{D}^{1}_{L^2,\mathrm{loc}}$ and $d^2F/dx^2 = T$ in the sense of Schwartz's distribution. Fukushima showed that the integral is a continuous additive functional of zero energy.

The additive functional defined by the Cauchy's principal value is an important example which belongs to the class of the integrals introduced in the above. In this paper, we are concerned with some properties of this functional.

In Section 1, Fukushima's definition will be introduced. In Section 2, some representations for integrals such as

$$\int_0^t \left(\mathrm{v.p.}\frac{1}{x_+}\right)(B_s)\,ds, \quad \int_0^t \left(\mathrm{v.p.}\frac{1}{x}\right)(B_s)\,ds,$$

etc, will be investigated. In Section 3, a relation between the continuous additive functional corresponding to an L^2-function and the functional defined by the Cauchy's principal value will be discussed. In the section, the Hilbert transform theory of L^2-functions will play an important role.

1. Preliminaries. Let $(\Omega, \mathscr{F}, P: \mathscr{F}_t)$ be a complete probability space with right continuous increasing family $(\mathscr{F}_t)_{t\geqslant 0}$ of sub-σ-fields of $\mathscr{F}$. Let B_t be a continuous $\mathscr{F}_t$-martingale such that

(i) $\mathrm{E}[(B_t-B_s)^2|\mathscr{F}_s]=t-s$ for $t\geqslant s\geqslant 0$;

(ii) the initial distribution μ has a compact support; that is to say, B_t is a one-dimensional $\mathscr{F}_t$-Brownian motion with a compact initial distribution μ.

Definition 1.1. A distribution T *belongs to* $\mathscr{D}'^{,1}_{L^2,\mathrm{loc}}$ if there exists a function $f\in L^2_{\mathrm{loc}}(R^1)$ such that $df/dx = T$, where the derivative is considered in the sense of Schwartz's distribution.

Definition 1.2 (M. Fukushima). We define $\int_0^t T(B_s)\,ds$ $(T\in\mathscr{D}'^{,1}_{L^2,\mathrm{loc}})$ as

$$\int_0^t T(B_s)\,ds = 2F(B_t)-2F(B_0)-2\int_0^t f(B_s)\,dB_s \ (^1),$$

where $F(x)$ is an absolutely continuous function such that

$$\frac{dF(x)}{dx}=f(x),\quad f(x)\in L^2_{\mathrm{loc}}(R^1),\quad \text{and}\quad \frac{d^2F}{dx^2}=\frac{df}{dx}=T.$$

We give some examples of distributions which belong to $\mathscr{D}'^{,1}_{L^2,\mathrm{loc}}$. In the following we denote by $\mathscr{D}$ the set of all C^∞-functions with compact support.

Example 1.1 (Dirac measure). $\delta(\varphi)=\varphi(0)$, $\varphi\in\mathscr{D}$. Let F be defined by

$$F(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ x & \text{for } x>0.\end{cases}$$

Then

$$\frac{dF}{dx}=f(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ 1 & \text{for } x>0\end{cases}$$

and $f(x)$ belongs to $L^2_{\mathrm{loc}}(R^1)$.

Example 1.2. We have

$$\text{v.p.}\frac{1}{x_+}(\varphi)=\lim_{\varepsilon\downarrow 0}\left\{(\log\varepsilon)\,\varphi(0)+\int_\varepsilon^\infty\frac{\varphi(x)}{x}\,dx\right\},\qquad \varphi\in\mathscr{D}.$$

Let F be defined by

$$F(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ x\log x-x & \text{for } x>0.\end{cases}$$

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(1) The stochastic integral $\int_0^t f(B_s)\,dB_s$ is understood in the sense of Ito.

Then

$$\frac{dF}{dx} = f(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \log x & \text{for } x > 0, \end{cases}$$

where $f(x)$ belongs to $L^2_{\mathrm{loc}}(R^1)$ and

$$\frac{d}{dx} f(x) = \text{v.p.} \frac{1}{x_+}.$$

Example 1.3 (Cauchy's principal value). We have

$$\text{v.p.} \frac{1}{x}(\varphi) = \lim_{\varepsilon \downarrow 0} \left\{ \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx \right\}, \qquad \varphi \in \mathscr{D}.$$

Let F be defined by $F(x) = x \log|x| - x$. Then $dF/dx = f(x) = \log|x|$, where $f(x)$ belongs to $L^2_{\mathrm{loc}}(R^1)$ and

$$\frac{d}{dx} \log|x| = \text{v.p.} \frac{1}{x}.$$

2. Some representations. H. Tanaka has pointed out that

$$(B_t)^+ = (B_0)^+ + \int_0^t H(B_s) dB_s + \tfrac{1}{2} L_t^0 \tag{2.1}$$

(see, e.g., [3], [5], and [6]), where

$$(x)^+ = \begin{cases} 0 & \text{for } x \leqslant 0, \\ x & \text{for } x > 0, \end{cases} \qquad H(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ 1 & \text{for } x > 0, \end{cases}$$

and L_t^0 is the local time of the Brownian motion B_t at 0. Combining (2.1) and Definition 1.2, we obtain

$$\int_0^t \delta(B_s) ds = L_t^0. \tag{2.2}$$

In the following theorem we give two representations concerning the stochastic integrals:

THEOREM 2.1. *The following equalities hold in the L^2-sense:*

$$\int_0^t \left(\text{v.p.} \frac{1}{x_+} \right)(B_s) ds = \lim_{\varepsilon \downarrow 0} \left\{ (\log \varepsilon) L_t^0 + \int_0^t I_{[\varepsilon, \infty)}(B_s) \frac{ds}{B_s} \right\}, \tag{2.3}$$

$$\int_0^t \left(\text{v.p.} \frac{1}{x} \right)(B_s) ds = \lim_{\varepsilon \downarrow 0} \left\{ \int_0^t I_{(-\infty, -\varepsilon]}(B_s) \frac{ds}{B_s} + \int_0^t I_{[\varepsilon, \infty)}(B_s) \frac{ds}{B_s} \right\} \tag{2.4}$$

(cf. [4], p. 72).

Proof of (2.3). Set

$$F_\varepsilon(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ (x^2\log\varepsilon)/2\varepsilon & \text{for } 0<x<\varepsilon,\\ \varepsilon-\frac{1}{2}(\log\varepsilon)\,\varepsilon+x\log x-x & \text{for } \varepsilon\leqslant x.\end{cases}$$

Then

$$F'_\varepsilon(x)=f_\varepsilon(x)=\begin{cases}0 & \text{for } x<0,\\ [(\log\varepsilon)/\varepsilon]\,x & \text{for } 0<x<\varepsilon,\\ \log x & \text{for } \varepsilon\leqslant x,\end{cases}$$

and

$$F''_\varepsilon(x)=\begin{cases}0 & \text{for } x<0,\\ (\log\varepsilon)/\varepsilon & \text{for } 0<x<\varepsilon,\\ 1/x & \text{for } \varepsilon\leqslant x.\end{cases}$$

By Ito's formula, we have

$$(2.5)\qquad \frac{1}{2}\left\{\int_0^t I_{[0,\varepsilon)}(B_s)\frac{\log\varepsilon}{\varepsilon}\,ds+\int_0^t I_{[\varepsilon,\infty)}(B_s)\frac{ds}{B_s}\right\}$$

$$=F_\varepsilon(B_t)-F_\varepsilon(B_0)-\int_0^t f_\varepsilon(B_s)\,dB_s \quad \text{a.s.}$$

We will show that

$$(2.6)\qquad \lim_{\varepsilon\downarrow 0}\left\{F_\varepsilon(B_t)-F_\varepsilon(B_0)-\int_0^t f_\varepsilon(B_s)\,dB_s\right\}=\frac{1}{2}\int_0^t\left(\text{v.p.}\frac{1}{x_+}\right)(B_s)\,ds$$

holds in the L^2-sense.

Let

$$F(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ x\log x-x & \text{for } x>0.\end{cases}$$

Then

$$F'(x)=f(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ \log x & \text{for } x>0.\end{cases}$$

Therefore, by the definition of the integral

$$\int_0^t\left(\text{v.p.}\frac{1}{x_+}\right)(B_s)\,ds,$$

we have

$$\mathrm{E}\left[\left|\frac{1}{2}\int_0^t\left(\text{v.p.}\frac{1}{x_+}\right)(B_s)\,ds-F_\varepsilon(B_t)+F_\varepsilon(B_0)+\int_0^t f_\varepsilon(B_s)\,dB_s\right|^2\right]$$
$$\leqslant 3\mathrm{E}\,[|F(B_t)-F_\varepsilon(B_t)|^2]+3\mathrm{E}\,[|F(B_0)-F_\varepsilon(B_0)|^2]+$$
$$+3\mathrm{E}\Big[\int_0^t |f(B_s)-f_\varepsilon(B_s)|^2\,ds\Big]=3I_\varepsilon^{(1)}+3I_\varepsilon^{(2)}+3I_\varepsilon^{(3)}.$$

Since $|F(x)-F_\varepsilon(x)|\leqslant\varepsilon-\frac{1}{2}\varepsilon\log\varepsilon$ $(0<\varepsilon<1)$, we see that $I_\varepsilon^{(1)}\to 0$ and $I_\varepsilon^{(2)}\to 0$ as $\varepsilon\downarrow 0$.

For $I_\varepsilon^{(3)}$ we have

$$I_\varepsilon^{(3)}\leqslant\mathrm{E}\left[\int_0^t I_{[0,\varepsilon)}(B_s)\left|\frac{\log\varepsilon}{\varepsilon}B_s-\log(B_s)\right|^2 ds\right]$$
$$\leqslant 4\int_{R^1}\mu(dx)\int_0^t\int_0^\varepsilon\frac{1}{\sqrt{2\pi s}}\exp\left\{-\frac{|x-y|^2}{2s}\right\}\log^2(y)\,dyds$$
$$\leqslant 4\int_0^t\frac{ds}{\sqrt{2\pi s}}\int_0^\varepsilon\log^2(y)\,dx.$$

These inequalities imply that $I_\varepsilon^{(3)}\to 0$ as $\varepsilon\downarrow 0$. Thus we have proved (2.6).

Combining (2.5) and (2.6) we obtain

$$\text{(2.7)}\qquad \int_0^t\left(\text{v.p.}\frac{1}{x_+}\right)(B_s)\,ds=\lim_{\varepsilon\downarrow 0}\left\{\int_0^t I_{[0,\varepsilon)}(B_s)\frac{\log\varepsilon}{\varepsilon}\,ds+\int_0^t I_{[\varepsilon,\infty)}(B_s)\frac{ds}{B_s}\right\}$$
$$=\lim_{\varepsilon\downarrow 0}J_\varepsilon.$$

Reduce J_ε to the form

$$\text{(2.8)}\qquad J_\varepsilon=\left\{(\log\varepsilon)\,L_t^0+\int_0^t I_{[\varepsilon,\infty)}(B_s)\frac{ds}{B_s}\right\}+$$
$$+\left\{\int_0^t I_{[0,\varepsilon)}(B_s)\frac{\log\varepsilon}{\varepsilon}\,ds-(\log\varepsilon)\,L_t^0\right\}=J_\varepsilon^{(1)}+J_\varepsilon^{(2)},$$

say. Then, for the proof of (2.3), it remains to show that $J_\varepsilon^{(2)}\to 0$ in L^2 as $\varepsilon\downarrow 0$.

By Tanaka's formula, we observe that

$$\text{(2.9)}\qquad J_\varepsilon^{(2)}=(\log\varepsilon)\left\{\int_0^t I_{[0,\varepsilon)}(B_s)\frac{ds}{\varepsilon}-2(B_t)^+ +2(B_0)^+ +2\int_0^t H(B_s)\,dB_s\right\}.$$

Set

$$S_\varepsilon(x)=\begin{cases}0 & \text{for } x\leqslant 0,\\ \frac{1}{2}\varepsilon^{-1}x^2 & \text{for } 0<x<\varepsilon,\\ \frac{1}{2}\varepsilon+(x-\varepsilon) & \text{for } \varepsilon\leqslant x.\end{cases}$$

Then

$$S'_\varepsilon(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ \varepsilon^{-1} x & \text{for } 0 < x < \varepsilon \\ 1 & \text{for } \varepsilon \leqslant x \end{cases}$$

and

$$S''_\varepsilon(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1/\varepsilon & \text{for } 0 < x < \varepsilon, \\ 0 & \text{for } \varepsilon \leqslant x. \end{cases}$$

By Ito's formula, we have

$$\int_0^t I_{[0,\varepsilon)}(B_s)\frac{ds}{\varepsilon} = 2S_\varepsilon(B_t) - 2S_\varepsilon(B_0) - 2\int_0^t S'_\varepsilon(B_s)\,dB_s. \tag{2.10}$$

By (2.9) and (2.10) we obtain

$$\begin{aligned} J_\varepsilon^{(2)} = (\log\varepsilon)\big[&2\{S_\varepsilon(B_t) - (B_t)^+\} + \\ &+ 2\{(B_0)^+ - S_\varepsilon(B_0)\} + 2\{\int_0^t (H(B_s) - S'_\varepsilon(B_s))\,dB_s\}\big]. \end{aligned} \tag{2.11}$$

Since

$$|S_\varepsilon(x) - (x)^+| \leqslant \varepsilon/2 \quad \text{and} \quad |H(B_s) - S'_\varepsilon(B_s)| \leqslant I_{[0,\varepsilon)}(B_s),$$

we infer from (2.11) that

$$\begin{aligned} \mathrm{E}[(J_\varepsilon^{(2)})^2] &\leqslant 12(\log\varepsilon)^2\,\mathrm{E}[\{S_\varepsilon(B_t) - (B_t)^+\}^2] + \\ &\quad + 12(\log\varepsilon)^2\,\mathrm{E}[\{S_\varepsilon(B_0) - (B_0)^+\}^2] + \\ &\quad + 12(\log\varepsilon)^2\,\mathrm{E}\Big[\int_0^t \{S'_\varepsilon(B_s) - H(B_s)\}^2\,ds\Big] \\ &\leqslant 3(\log\varepsilon)^2\varepsilon^2 + 3(\log\varepsilon)^2\varepsilon^2 + \\ &\quad + 12(\log\varepsilon)^2 \int_{R^1}\mu(dx)\int_0^t\int_0^\varepsilon \frac{1}{\sqrt{2\pi s}}\exp\Big\{-\frac{|x-y|^2}{2s}\Big\}\,ds \\ &\leqslant 6(\log\varepsilon)^2\varepsilon^2 + 12(\log\varepsilon)^2\varepsilon\Big(\int_0^t \frac{ds}{\sqrt{2\pi s}}\Big). \end{aligned} \tag{2.12}$$

Thus, (2.12) implies

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}[(J_\varepsilon^{(2)})^2] = 0. \tag{2.13}$$

By (2.7), (2.8), and (2.13), we conclude that (2.3) holds.

Proof of (2.4). Define the distribution v.p.(x_-) by

$$\left(\text{v.p.}\frac{1}{x_-}\right)(\varphi) = \lim_{\varepsilon\downarrow 0}\left\{-(\log\varepsilon)\,\varphi(0) + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}\,dx\right\}, \qquad \varphi\in\mathscr{D}.$$

Then, we observe that

$$\text{v.p.}\frac{1}{x} = \text{v.p.}\frac{1}{x_+} + \text{v.p.}\frac{1}{x_-}.$$

In the same way as in the proof of (2.3), we can see that

$$\int_0^t\left(\text{v.p.}\frac{1}{x_-}\right)(B_s)\,ds = \lim_{\varepsilon\downarrow 0}\left\{-(\log\varepsilon)\,L_t^0 + \int_0^t I_{(-\infty,-\varepsilon]}(B_s)\frac{ds}{B_s}\right\} \quad \text{in } L^2.$$

Thus, we conclude that (2.4) holds.

Remark 2.1. Let p.f.x_+^λ be the Hadamard's finite part:

$$\text{p.f.}\,x_+^\lambda(\varphi) = \lim_{\varepsilon\downarrow 0}\left\{\frac{\varepsilon^{\lambda+1}}{\lambda+1}\varphi(0) + \int_\varepsilon^\infty x^\lambda\,\varphi(x)\,dx\right\}, \qquad \varphi\in\mathscr{D}.$$

Then, in the case where the index λ satisfies the condition $-3/2 < \lambda < -1$, the distribution p.f.x_+^λ belongs to the class $\mathscr{D}'^{,1}_{L^2,\mathrm{loc}}$.

By the similar argument as in the proof of Theorem 2.1, one obtains the following equality in the L^2-sense:

$$\int_0^t(\text{p.f.}\,x_+^\lambda)(B_s)\,ds = \lim_{\varepsilon\downarrow 0}\left\{\frac{\varepsilon^{\lambda+1}}{\lambda+1}L_t^0 + \int_0^t I_{[\varepsilon,\infty)}(B_s)\,B_s^\lambda\,ds\right\} \quad \left(-\frac{3}{2} < \lambda < -1\right).$$

Remark 2.2. It is known that the additive functional defined by $\int_0^t T(B_s)\,ds$ is of bounded variation if and only if T is a signed Radon measure (cf., e.g., [2]). The additive functionals treated in Theorem 2.1 as well as in Remark 2.1 are not of bounded variation.

3. Hilbert transform of L^2-functions and the additive functionals defined by the Cauchy's principal value. Let C_t^a be defined as

$$C_t^a = \int_0^t\left(\text{v.p.}\frac{1}{x-a}\right)(B_s)\,ds.$$

C_t^a is a continuous additive functional but it is not of bounded variation with respect to t. First, we shall investigate the regularity of C_t^a.

PROPOSITION 3.1. *The following relation holds*:

$$\lim_{b\to a}\mathrm{E}\left[|C_t^a - C_t^b|^2\right] = 0. \tag{3.1}$$

To prove the proposition, we need the following lemma:

LEMMA 3.1. (A) *The collection of random variables defined by*

$$[\{(B_t-a)\log(|B_t-a|)-(B_t-b)\log(|B_t-b|)\}^2;\ a,\ b\in[c,\ d]]$$

is uniformly integrable, where $-\infty<c<d<\infty$ *and* $0<t<\infty$.

(B) *The collection of random variables defined by*

$$[\{(B_0-a)\log(|B_0-a|)-(B_0-b)\log(|B_0-b|)\}^2;\ a,\ b\in[c,\ d]]$$

is uniformly integrable, where $-\infty<c<d<\infty$.

(C) *The collection of random variables defined by*

$$[\{\log(|B_s-a|)-\log(|B_s-b|)\}^2;\ a,\ b\in[c,\ d],\ s\in[0,\ T]]$$

is uniformly integrable with respect to $(\Omega\times[0,\ T],\ \mathscr{F}\otimes\mathscr{B}[0,\ T],\ P\otimes ds)$, *where* $-\infty<c<d<\infty$ *and* $0<T<\infty$.

Proof of (A). A simple calculus shows that

$$\begin{aligned}
&\mathrm{E}[\{(B_t-a)\log(|B_t-a|)-(B_t-b)\log(|B_t-b|)\}^4]\\
&\leqslant 8\mathrm{E}[\{(B_t-a)\log(|B_t-a|)\}^4]+8\mathrm{E}[\{(B_t-b)\log(|B_t-b|)\}^4]\\
&\leqslant 8\sup_{a\in[c,d]}\int_{R^1}\mu(dx)\int_{R^1}\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(x-y)^2}{2t}\right\}(y-a)^4\log^4(|y-a|)\,dy+\\
&\quad+8\sup_{b\in[c,d]}\int_{R^1}\mu(dx)\int_{R^1}\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(x-y)^2}{2t}\right\}(y-b)^4\log^4(|y-b|)\,dy<\infty.
\end{aligned}$$

These inequalities imply that the collection is uniformly integrable.

Proof of (B). We have

$$\begin{aligned}
&\mathrm{E}[\{(B_0-a)\log(|B_0-a|)-(B_0-b)\log(|B_0-b|)\}^4]\\
&\leqslant 8\sup_{a\in[c,d]}\int_{R^1}\mu(dx)(x-a)^4\log(|x-a|)+8\sup_{b\in[c,d]}\int_{R^1}\mu(dx)(x-b)^4\log^4(|x-b|)\\
&<\infty.
\end{aligned}$$

These inequalities imply the desired result.

Proof of (C). We observe that

$$\begin{aligned}
&\int_0^T\mathrm{E}[\{\log(|B_s-a|)-\log(|B_s-b|)\}^4]\,ds\\
&\qquad\leqslant 8\sup_{a\in[c,d]}\int_0^T\int_{R^1}\mu(dx)\int_{R^1}\frac{1}{\sqrt{2\pi s}}\exp\left\{-\frac{(x-y)^2}{2s}\right\}\log^4(|y-a|)\,dyds+\\
&\qquad+8\sup_{b\in[c,d]}\int_0^T\int_{R^1}\mu(dx)\int_{R^1}\frac{1}{\sqrt{2\pi s}}\exp\left\{-\frac{(x-y)^2}{2s}\right\}\log^4(|y-b|)\,dyds\\
&\qquad<\infty.
\end{aligned}$$

Thus, we conclude that the collection is uniformly integrable.

Proof of Proposition 3.1. By the definition of C_t^a, we obtain

$$C_t^a = 2(B_t-a)\log(|B_t-a|)-2(B_t-a)-2(B_0-a)\log(|B_0-a|)+$$
$$+2(B_0-a)-2\int_0^t \log(|B_s-a|)\,dB_s.$$

Then

$$\text{(3.2)}\qquad \mathrm{E}[|C_t^a-C_t^b|^2]$$
$$\leqslant 12\,\mathrm{E}[\{(B_t-a)\log(|B_t-a|)-(B_t-b)\log(|B_t-b|)\}^2]+$$
$$+12\,\mathrm{E}[\{(B_0-a)\log(|B_0-a|)-(B_0-b)\log(|B_0-b|)\}^2]+$$
$$+12\,\mathrm{E}\Big[\int_0^t \{\log(|B_s-a|)-\log(|B_s-b|)\}^2\,ds\Big].$$

Observe that each integrant of the right-hand side of (3.2) converges to zero as b tends to a. Then, by Lemma 3.1, relation (3.1) holds.

Corollary 3.1. *There exists a version of*

$$\int_0^t \left(\text{v.p.}\frac{1}{x-a}\right)(B_s)\,ds$$

which is measurable with respect to $(t, a, \omega)\in[0, \infty)\times R^1\times\Omega$.

In the following, we denote by C_t^a a measurable version of

$$(t, a, \omega)\mapsto\int_0^t \left(\text{v.p.}\frac{1}{x-a}\right)(B_s)\,ds.$$

Let g be a function which belongs to $L^2(R^1)$. Consider the Hilbert transform of g:

$$(\mathscr{H}g)(x) = \frac{1}{\pi}\left(\text{v.p.}\frac{1}{x}*g\right)(x),$$

where $*$ means the convolution operator in the theory of distributions. Then it is well known that $\mathscr{H}g$ also belongs to $L^2(R^1)$ and

$$\int_{-\infty}^{\infty} g^2(x)\,dx = \int_{-\infty}^{\infty} (\mathscr{H}g(x))^2\,dx$$

holds (cf., e.g., [7]-[9]).

The main result of this section is the following

Theorem 3.1. (i) *Let g belong to $L^2(R^1)$ and let its support be compact. Then*

$$\text{(3.3)}\qquad \int_{R^1} C_t^a g(a)\,da = \int_0^t\left(\text{v.p.}\frac{1}{x}*g\right)(B_s)\,ds \quad a.s.$$

(ii) *Let g and g_n belong to $L^2(R^1)$ and let their support be compact. Suppose that*

$$\lim_{n\to\infty} \int_{R^1} (g_n(x)-g(x))^2\,dx = 0.$$

Then

$$\lim_{n\to\infty} \int_{R^1} C_t^a g_n(a)\,da = \int_{R^1} C_t^a g(a)\,da \tag{3.4}$$

in the L^2-sense.

Proof. (i) Set $F(x) = x\log|x| - x$. Since g is an L^2-function with compact support and $F \in L^2_{\mathrm{loc}}(R^1)$, we obtain

$$(F*g)' = F'(x)*g = \log|x|*g \tag{3.5}$$

and

$$(F*g)'' = \mathrm{v.p.}\frac{1}{x}*g. \tag{3.6}$$

On the other hand, by the definition of C_t^a, we have

$$\int_{R^1} C_t^a g(a)\,da = 2\int_{R^1} \{F(B_t-a)-F(B_0-a)\}\,g(a)\,da -$$

$$-2\int_{R^1} \{\int_0^t F'(B_s-a)\,dB_s\}\,g(a)\,da.$$

Applying the next lemma to the second term of the right-hand side of this equality, we have

$$\int_{R^1} C_t^a g(a)\,da = 2(F*g)(B_t) - 2(F*g)(B_0) - 2\int_0^t (F'*g)(B_s)\,dB_s.$$

By (3.5), the right-hand side of the above equality is equal to

$$2(F*g)(B_t) - 2(F*g)(B_0) - 2\int_0^t (F*g)'(B_s)\,dB_s,$$

which by the definition of $\int_0^t (F*g)''(B_s)\,ds$ is reduced to

$$\int_0^t (F*g)''(B_s)\,ds.$$

Thus, by (3.6), we obtain (3.3).

(ii) By (i), we see that

$$\int_{R^1} C_t^a g_n(a)\,da = \pi\int_0^t (\mathscr{H}g_n)(B_s)\,ds \tag{3.7}$$

and

$$\int_{R^1} C_t^a g(a)\,da = \pi \int_0^t (\mathscr{H} g)(B_s)\,ds. \tag{3.8}$$

Since

$$\int_{R^1} (g_n(a)-g(a))^2\,da = \int_{R^1} (\mathscr{H} g_n(a)-\mathscr{H} g(a))^2\,da,$$

the assumption on $\{g_n, g\}$ implies that $\mathscr{H} g_n$ converges to $\mathscr{H} g$ in $L^2(R^1)$. Then, by (3.7) and (3.8), we observe that

$$\lim_{n\to\infty} \int_{R^1} C_t^a g_n(a)\,da = \lim_{n\to\infty} \pi \int_0^t (\mathscr{H} g_n)(B_s)\,ds = \pi \int_0^t (\mathscr{H} g)(B_s)\,ds = \int_{R^1} C_t^a g(a)\,da$$

in the L^2-sense.

LEMMA 3.2. *The Fubini type relation*

$$\int_0^t \Big\{ \int_{R^1} F'(B_s-a)\,g(a)\,da \Big\}\,dB_s = \int_{R^1} \Big\{ \int_0^t F'(B_s-a)\,dB_s \Big\}\,g(a)\,da \tag{3.9}$$

holds a.s., where $F(x) = x\log|x|-x$ *and* $g(x)$ *is an* L^2*-function with compact support.*

Proof. To prove this lemma, we follow essentially the same way as in the proof of Lemma 4.1 in [3], Chapter 3.

Without loss of generality we suppose that g is non-negative. Denote the collection of all square integrable (continuous) $\mathscr{F}_t$-martingales by M_2 (M_2^c). Since $F' * g$ belongs to $L^2(R^1)$, the left-hand side of (3.9) is well defined as an element of M_2^c.

On the other hand, $a \mapsto \int_0^t F'(B_s-a)\,dB_s$ is Borel measurable, and for every $T>0$ we have

$$\begin{aligned}
&E\Big[\int_{R^1} g(a)\,da \max_{0\leq t\leq T} \Big|\int_0^t F'(B_s-a)\,dB_s\Big|\Big] \\
&\leq \int_{R^1} g(a)\,da \Big\{E\Big[\max_{0\leq t\leq T} \Big|\int_0^t F'(B_s-a)\,dB_s\Big|^2\Big]\Big\}^{1/2} \\
&\leq 2\int_{R^1} g(a)\,da \int_0^T ds \int_{R^1} \mu(dx) \int_{R^1} \log^2(|y-a|)\frac{1}{\sqrt{2\pi s}}\exp\Big\{-\frac{(x-y)^2}{2s}\Big\}\,dy \\
&< \infty.
\end{aligned}$$

Hence

$$\int_{R^1} g(a)\,da \max_{0\leq t\leq T} \Big|\int_0^t F'(B_s-a)\,dB_s\Big| < \infty \ \text{ a.s.},$$

which implies that

$$t \mapsto \int_{R^1} g(a)\,da \int_0^t F'(B_s-a)\,dB_s$$

is continuous a.s. Thus the right-hand side of (3.9) is well defined as an $\mathscr{F}_t$-adapted continuous process. Now, we show that it is an element of M_2^c. It is square integrable because

$$\begin{aligned}
&\mathrm{E}\Big[\Big\{\int_{R^1} g(a)\,da\Big(\int_0^t F'(B_s-a)\,dB_s\Big)\Big\}^2\Big]\\
&\qquad \leqslant \mathrm{E}\Big[\Big(\int_{R^1} g(a)\,da\Big)\Big(\int_{R^1} g(a)\,da\Big(\int_0^t F'(B_s-a)\,dB_s\Big)^2\Big)\Big]\\
&\qquad = \Big(\int_{R^1} g(a)\,da\Big)\int_{R^1} g(a)\,da\,\mathrm{E}\Big[\Big(\int_0^t F'(B_s-a)\,dB_s\Big)^2\Big]\\
&\qquad \leqslant \Big(\int_{R^1} g(a)\,da\Big)\int_{R^1} g(a)\,da\int_0^t ds\int_{R^1}\mu(dx)\times\\
&\qquad\quad \times\int_{R^1}\log^2(|y-a|)\frac{1}{\sqrt{2\pi s}}\exp\Big\{-\frac{(x-y)^2}{2s}\Big\}dy\\
&\qquad < \infty.
\end{aligned}$$

The process considered is an $\mathscr{F}_t$-martingale because if $t > s > 0$ and $A \in \mathscr{F}_s$, then

$$\mathrm{E}\Big[I_A\int_{R^1} g(a)\,da\int_s^t F'(B_u-a)\,dB_u\Big] = \int_{R^1} g(a)\,da\,\mathrm{E}\Big[I_A\int_s^t F'(B_u-a)\,dB_u\Big] = 0.$$

In a similar way, for $N \in M_2$ we have

$$\begin{aligned}
&\mathrm{E}\Big[I_A\int_{R^1} g(a)\,da\Big(\int_s^t F'(B_u-a)\,dB_u\Big)(N_t-N_s)\Big]\\
&\qquad = \int_{R^1} g(a)\,da\,\mathrm{E}\Big[I_A\Big(\int_s^t F'(B_u-a)\,dB_u\Big)(N_t-N_s)\Big]\\
&\qquad = \int_{R^1} g(a)\,da\,\mathrm{E}\Big[I_A\int_s^t F'(B_u-a)\,d\langle B, N\rangle_u\Big]\\
&\qquad = \mathrm{E}\Big[I_A\int_s^t\Big(\int_{R^1} F'(B_u-a)\,g(a)\,da\Big)d\langle B, N\rangle_u\Big].
\end{aligned}$$

Thus

$$t \mapsto \int_{R^1}\Big\{\int_0^t F'(B_s-a)\,dB_s\Big\}g(a)\,da = L_t$$

is an element of M_2^c such that, for every $N \in M_2$,

$$\langle N, L\rangle_t = \int_0^t \{\int_{R^1} F'(B_s - a) g(a) da\} d\langle B, N\rangle_s.$$

Consequently, by Proposition II-2-4 in [3] we have

$$L_t = \int_0^t \{\int_{R^1} F'(B_u - a) g(a)\} dB_s,$$

which completes the proof.

THEOREM 3.1′. *Both parts* (i) *and* (ii) *of Theorem* 3.1 *are true without the assumptions that* g *and* g_n *have compact supports.*

Proof([2]). (i) Let $L_t^a(\omega)$ be the local time of the Brownian motion B_t at a. The functions $a \mapsto L_t^a(\omega)$ belong to $L^2(R^1)$ a.s. because they are continuous and $L_t^a(\omega) = 0$ when $|a| > \max_{0 \leqslant s \leqslant t} |B_s|$. Therefore, we define A_t by

$$A_t g = \pi \int_0^t (\mathscr{H} g)(B_s) ds = \pi \int_{R^1} L_t^a (\mathscr{H} g)(a) da \quad \text{for } g \in L^2(R^1).$$

Since

$$\int_{R^1} g^2(a) da = \int_{R^1} (\mathscr{H} g)^2(a) da,$$

we see that A_t are linear functionals on $L^2(R^1)$ a.s.

On the other hand, Theorem 3.1 shows that

$$A_t h = \pi \int_0^t (\mathscr{H} h)(B_s) ds = \int_{R^1} C_t^a h(a) da$$

for any $L^2(R^1)$-function h with compact support.

Thus we conclude that

$$A_t g = \pi \int_0^t (\mathscr{H} g)(B_s) ds = \int_{R^1} C_t^a g(a) da \text{ a.s.} \quad \text{for } g \in L^2(R^1).$$

(ii) This part can be proved analogously as (i), so we omit its proof.

Define $\mathscr{H}^*$ by

$$\mathscr{H}^* g = -\frac{1}{\pi}\left(\text{v.p.}\frac{1}{x} * g\right).$$

([2]) For the proof we owe much to M. Yor (private communication). We wish to express our thanks to him.

Then it is well known that $\mathscr{H}(\mathscr{H}^* g) = g$ and $\mathscr{H}^*(\mathscr{H} g) = g$ for $g \in L^2(R^1)$.

COROLLARY 3.2. *If f belongs to $L^2(R^1)$, then*

$$\int_0^t f(B_s)\, ds = \frac{1}{\pi} \int_{R^1} C_t^a (\mathscr{H}^* f)(a)\, da \quad a.s.$$

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