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STOPPING GAMES FOR SYMMETRIC MARKOV PROCESSES

BY

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Abstract. Let $\mathscr{E}$ be a Dirichlet form corresponding to a symmetric Markov process $M = \{\Omega, \mathscr{M}, x_t, P^x\}$ acting on a state space X. Let g and h, $g \leqslant h$, be quasi-continuous elements of the corresponding Dirichlet space $\mathscr{F}$, and v a quasi-continuous solution of the variational inequality

$$\mathscr{E}_\alpha(v, u-v) \geqslant 0 \quad \text{for all } u \in \mathscr{F},\ g \leqslant u \leqslant h,$$

where $\alpha > 0$ and $\mathscr{E}_\alpha(u, v) = \mathscr{E}(u, v) + \alpha(u, v)$ for all $u, v \in \mathscr{F}$. It is shown in the paper that if $J_x(\tau, \sigma)$ is defined for all $x \in X$ and all stopping times τ and σ by

$$J_x(\tau, \sigma) = \mathrm{E}^x\Big(e^{-\alpha\tau\wedge\sigma}\big(I_{\tau\leqslant\alpha}h(x_\tau) + I_{\tau>\sigma}g(x_\sigma)\big)\Big),$$

then for quasi-every $x \in X$ we have

$$v(x) = \inf_\tau \sup_\sigma J_x(\tau, \sigma) = \sup_\sigma \inf_\tau J_x(\tau, \sigma).$$

Moreover, for quasi-every $x \in X$ the pair $(\hat\tau, \hat\sigma)$ such that

$$\hat\tau = \inf\{t \geqslant 0;\ h(x_t) = v(x_t)\}, \qquad \hat\sigma = \inf\{t \geqslant 0;\ g(x_t) = v(x_t)\}$$

is the saddle point of the game

$$J_x(\hat\tau, \sigma) \leqslant J_x(\hat\tau, \hat\sigma) \leqslant J_x(\tau, \hat\sigma)$$

for all stopping times τ, σ and quasi-every $x \in X$.

1. Introduction. Since early seventies it is known that the value of a stopping game associated with a diffusion process can be identified as a solution of a certain variational inequality. Papers [9], [10] and [6], [2] were the first containing general results in this direction.

In the present paper we take up a general symmetric Markov process and we assume that the obstacles, which define the cost functional, are arbitrary elements of the associated Dirichlet space $(\mathscr{F}, \mathscr{E})$ (see [7]). This assumption means, when specialized to the diffusion case, that obstacles are merely of H^1

and not of H^2 class. In the setting of Dirichlet spaces one can hope to extend known results to irregular diffusion processes with drift and diffusion coefficients being only measurable functions. Moreover, some Markov processes with non-local generators can be treated in this way. It is also of independent interest to find whether the interplay between variational inequalities and stopping games takes place if it can be reasonably formulated. The main aim of the paper is to show that this is really the case.

For one obstacle and an optimal stopping problem, analogous results were obtained by Nagai [13], and our paper can be considered as an extension of Nagai's results to the game situation. Let us notice, however, that the method used in [13] cannot be generalized to cover the present case. The main reason for this is that the value of a stopping game cannot be, in general, represented as a difference of α-potentials and, therefore, saddle stopping times cannot be defined by the corresponding additive functionals.

Main tools used in the paper are potential theory and the penalty method developed in [3].

The basic theorem is formulated in Section 2. In Section 3 we give a complete proof of the theorem under the so-called separability condition. We use here the Bismut-Nakoulim's method (see [4] and [14]) which consists in reducing the game problem to a system of quasi-variational inequalities. The proof of the first part of the theorem, in the general case, is given in Section 4. Results on penalizations, among which Theorem 3 being of independent interest, are gathered in the next section. The proof is completed in the final Section 6. As far as basic concepts and notation related to Dirichlet forms, Dirichlet spaces, and symmetric Markov processes, we refer to Nagai's paper [13] and to Fukushima's monograph [7].

The literature devoted to stopping games is quite extensive. Besides of the mentioned papers [9], [10] and [6], different aspects of such games were studied in [3]-[5], [14], [8], [16]-[18], and [11]. In particular, in some places, we follow papers [16]-[18].

The present paper is a rewritten version of the report [20].

I would like to thank my Italian colleagues I. Dolcetta and M. Matzeu for discussions on the topic of the paper we had during my stay in Istituto Guido Castelnuovo in June 1981.

2. Formulation of the main result. We assume throughout the paper that the Dirichlet space $(\mathscr{F}, \mathscr{E}_\alpha)$, $\alpha > 0$, is $C_0(X)$-regular and the Markov process $M = \{\Omega, \mathscr{M}, x_t, P^x\}$ is a Hunt process. The basic reference measure on X will be denoted by m. Thus, in particular, $\mathscr{F}$ is densely embedded in the Hilbert space $H = L^2(X, m)$ (see [7]). It is known that for an arbitrary element $g \in \mathscr{F}$ there exists its quasi-continuous version denoted by $\tilde{g}$ (see [7], p. 65).

Let g and h be quasi-continuous versions of arbitrary elements from $\mathscr{F}$

satisfying the inequality

(1) $$g \leqslant h \text{ quasi-everywhere (q.e.).}$$

For any pair of stopping times (τ, σ) the cost functional J_x is defined by

(2) $$J_x(\tau, \sigma) = \mathrm{E}^x\big(e^{-\alpha\tau\wedge\sigma}\big(I_{\tau\leqslant\sigma} h(x_\tau) + I_{\tau>\sigma} g(x_\sigma)\big)\big),$$

and K stands for the following closed and convex subset of $\mathscr{F}$:

(3) $$K = \{u \in \mathscr{F};\ g \leqslant u \leqslant h \ m\text{-a.e.}\}.$$

The following theorem is the main result of the paper:

THEOREM 1. *There exist a quasi-continuous function $v \in K$ which solves the variational inequality*

(4) $$\mathscr{E}_\alpha(v, u-v) \geqslant 0 \quad \textit{for all } u \in K$$

and a properly exceptional set N such that, for all $x \in X \setminus N$,

$$v(x) = \sup_\sigma \inf_\tau J_x(\tau, \sigma) = \inf_\tau \sup_\sigma J_x(\tau, \sigma).$$

Moreover, the pair $(\hat{\tau}, \hat{\sigma})$ such that

(5) $$\hat{\tau} = \inf\{t \geqslant 0;\ v(x_t) = h(x_t)\},$$

(6) $$\hat{\sigma} = \inf\{t \geqslant 0;\ v(x_t) = g(x_t)\}$$

is the saddle point of the game

$$J_x(\hat{\tau}, \sigma) \leqslant J_x(\hat{\tau}, \hat{\sigma}) \leqslant J_x(\tau, \hat{\sigma})$$

for all stopping times τ and σ and $x \in X \setminus N$.

For completeness and later references we formulate a slightly generalized version of Nagai's result [13].

THEOREM 2 (Nagai [13]). *There exist a quasi-continuous function $w \in \mathscr{F}$ which solves the variational inequality*

(7) $$w \geqslant g, \quad \mathscr{E}_\alpha(w, u-w) \geqslant 0 \quad \textit{for } u \in \mathscr{F},\ u \geqslant g,$$

and a properly exceptional set N such that, for all $x \in X \setminus N$,

$$w(x) = \sup_\sigma E^x\big(e^{-\alpha\sigma} g(x_\sigma)\big) = \mathrm{E}^x\big(e^{-\hat{\sigma}} g(x_{\hat{\sigma}})\big),$$

where $\hat{\sigma}$ is defined by (6) with v replaced by w. Moreover, w is the smallest α-potential majorizing the function g m-a.e.

3. The case of separable obstacles. In this section we prove Theorem 1 under the following separability condition (see [4] and [14]):

There exist α-potentials $w_1, w_2 \in \mathscr{F}$ such that

(8) $$g \leqslant w_1 - w_2 \leqslant h \ m\text{-a.e.}$$

PROPOSITION 1. *Let g and h be arbitrary elements from $\mathcal{F}$ such that $g \leqslant h$ m-a.e. There exists a pair $(\bar{v}, \underline{v}) \in \mathcal{F} \times \mathcal{F}$ satisfying the quasi-variational problem*

$$\text{(9)} \qquad \bar{v} \geqslant \underline{v}+g \quad \text{and} \quad \mathcal{E}_\alpha(\bar{v}, u-\bar{v}) \geqslant 0 \quad \text{for } u \geqslant \underline{v}+g,\ u \in \mathcal{F},$$

$$\text{(10)} \qquad \underline{v} \geqslant \bar{v}-h \quad \text{and} \quad \mathcal{E}_\alpha(\underline{v}, u-v) \geqslant 0 \quad \text{for } u \geqslant \bar{v}-h,\ u \in \mathcal{F},$$

if and only if the separability condition (8) *holds. Moreover, if there exists a solution to the problem* (9)-(10), *then it is unique if and only if the contact set*

$$\text{(11)} \qquad \{x;\ \tilde{h}(x) = \tilde{g}(x)\},$$

defined in terms of some quasi-continuous modifications $\tilde{h}$, $\tilde{g}$ of elements h, g, is of capacity zero.

Proof. If there exists a solution $(\bar{v}, \underline{v})$ of the problem (9)-(10), then the functions $\bar{v}$ and $\underline{v}$ are α-potentials (see Theorem 2 or Theorem 3.2.1 in [7]) and they clearly separate obstacles h and g. On the other hand, if the separability condition (8) holds, then for each n we have $w_1 \geqslant \bar{v}_n$ and $w_2 \geqslant \underline{v}_n$ m-a.e., where the functions $\bar{v}_n$ and $\underline{v}_n$ $(n = 1, 2, \ldots)$ are defined by induction as follows. If $n = 0$, then $\bar{v}_0 = \underline{v}_0 = 0$ and, for general n, $\bar{v}_{n+1}$ and $\underline{v}_{n+1}$ are unique solutions of the following variational inequalities, respectively:

$$\text{(12)} \qquad \bar{v}_{n+1} \in \mathcal{F}, \quad \bar{v}_{n+1} \geqslant \underline{v}_n+g,$$

$$\mathcal{E}_\alpha(\bar{v}_{n+1}, u-\bar{v}_{n+1}) \geqslant 0 \quad \text{for } u \in \mathcal{F},\ u \geqslant \underline{v}_n+g,$$

$$\text{(13)} \qquad \underline{v}_{n+1} \in \mathcal{F}, \quad \underline{v}_{n+1} \geqslant \bar{v}_n-h,$$

$$\mathcal{E}_\alpha(\underline{v}_{n+1}, u-\underline{v}_{n+1}) \geqslant 0 \quad \text{for } u \in \mathcal{F},\ u \geqslant \bar{v}_n-h.$$

Consequently, $\bar{v}_n$ and $\underline{v}_n$ are α-potentials. Moreover, a simple induction shows that both sequences $(\bar{v}_n)$ and $(\underline{v}_n)$ are increasing and bounded from above by α-potentials w_1 and w_2, respectively. Therefore, $(\bar{v}_n)$ and $(\underline{v}_n)$ converge in $(\mathcal{F}, \mathcal{E}_\alpha)$ to some α-potentials $\bar{v}$ and $\underline{v}$ which solve the initial problem (9)-(10).

The uniqueness can be proved in a similar way as in [14].

COROLLARY. *If the pair $(\bar{v}, \underline{v})$ is a solution of the problem* (9)-(10), *then the difference $v = \bar{v}-\underline{v}$ is the unique solution of the problem* (4).

The proof is immediate.

Now we are in a position to prove the following partial result:

PROPOSITION 2. *Under the separability condition* (8), *Theorem* 1 *holds true.*

Proof. Without loss of generality we can assume that the functions $\bar{v}$, $\underline{v}$, and v are quasi-continuous. Consequently, the functions $\bar{v}$ and $\underline{v}$ are solutions, in the sense specified in Theorem 2, of stopping time problems with gain functions $\underline{v}+g$ and $\bar{v}-h$, respectively. Using the same argument as in [13] one can find a properly exceptional set N such that, for all $x \in X \setminus N$

and moments $\hat{\sigma}$, $\hat{\tau}$,

$$\begin{aligned}\hat{\sigma} &= \inf\{t \geqslant 0;\ \bar{v}(x_t) = \underline{v}(x_t)+g\} \\ &= \inf\{t \geqslant 0;\ v(x_t) = g(x_t)\}, \\ \hat{\tau} &= \inf\{t \geqslant 0;\ \underline{v}(x_t) = \bar{v}(x_t)-h\} \\ &= \inf\{t \geqslant 0;\ v(x_t) = h(x_t)\},\end{aligned}$$

we have

$$\bar{v}(x) = \mathrm{E}^x\big(e^{-\alpha\sigma}\bar{v}(x_\sigma)\big) \quad \text{for all } \sigma \leqslant \hat{\sigma},$$

$$\underline{v}(x) = \mathrm{E}^x\big(e^{-\alpha\tau}\underline{v}(x_\tau)\big) \quad \text{for all } \tau \leqslant \hat{\tau}.$$

Moreover, one can assume that for all initial states $x \in X\setminus N$ the processes $\big(e^{-\alpha t}\bar{v}(x_t)\big)$ and $\big(e^{-\alpha t}\underline{v}(x_t)\big)$ are non-negative P^x-supermartingales. Thus, for arbitrary stopping times τ, σ we obtain

$$\bar{v}(x) \geqslant \mathrm{E}^x\big(e^{-\alpha\sigma}\bar{v}(x_\sigma)\big), \qquad \underline{v}(x) \geqslant \mathrm{E}^x\big(e^{-\alpha\tau}\underline{v}(x_\tau)\big), \qquad x \in X\setminus N.$$

Consequently,

$$\begin{aligned}v(x) = \bar{v}(x)-\underline{v}(x) &\leqslant \mathrm{E}^x\big(e^{-\alpha\hat{\sigma}\wedge\tau}(\bar{v}-\underline{v})(x_{\hat{\sigma}\wedge\tau})\big) \leqslant \mathrm{E}^x\big(e^{-\alpha\hat{\sigma}\wedge\tau}v(x_{\hat{\sigma}\wedge\tau})\big) \\ &\leqslant \mathrm{E}^x\big(e^{-\alpha\hat{\sigma}}I_{\hat{\sigma}<\tau}\,g(x_{\hat{\sigma}})+e^{-\alpha\tau}I_{\tau\leqslant\hat{\sigma}}\,h(x_\tau)\big) \leqslant J_x(\tau, \hat{\sigma}).\end{aligned} \tag{14}$$

In (14) we used the property that the processes $\big(v(x_t)\big)$, $\big(g(x_t)\big)$, $\big(h(x_t)\big)$ are P^x-a.e. right-continuous and that $P_x\big(h(x_t) \geqslant v(x_t),\ t \geqslant 0\big) = 1$ for all $x \in X\setminus N$. In the same way one proves that $v(x) \geqslant J_x(\hat{\tau}, \sigma)$ q.e.

4. Proof of the first part of Theorem 1. We need the following purely analytical results.

PROPOSITION 3. *Let (g_n) and (h_n) converge in $(\mathscr{F}, \mathscr{E}_\alpha)$ to g and h, respectively, and let $g_n \leqslant h_n$ m-a.e. $(n = 1, 2, \ldots)$. If $v_n, v \in \mathscr{F}$ $(n = 1, 2, \ldots)$ and*

$$\mathscr{E}_\alpha(v_n, w-v_n) \geqslant 0 \quad \textit{for all } u \in \mathscr{F},\ g_n \leqslant u \leqslant h_n, \tag{15}$$

$$\mathscr{E}_\alpha(v, w-v) \geqslant 0 \quad \textit{for all } u \in \mathscr{F},\ g \leqslant u \leqslant h, \tag{16}$$

then $v_n \to v$ in $(\mathscr{F}, \mathscr{E}_\alpha)$.

The main tool in proving this proposition is the following lemma due to Ancona [1]:

LEMMA 1. *If $v_n \to v$ in $(\mathscr{F}, \mathscr{E}_\alpha)$, then also $v_n^+ \to v^+$ in $(\mathscr{F}, \mathscr{E}_\alpha)$.*

Proof of Proposition 3. Since $\mathscr{E}_\alpha(v_n, v_n) \leqslant \mathscr{E}_\alpha(g_n, g_n)$ for $n = 1, 2, \ldots$, the sequence (v_n) is $(\mathscr{F}, \mathscr{E}_\alpha)$-bounded. Let $\hat{v}$ be a weak limit in $(\mathscr{F}, \mathscr{E}_\alpha)$ of a subsequence (v_{n_k}). If $u \in \mathscr{F}$ and $g \leqslant u \leqslant h$, then by Lemma 1 we have $w_n = (g_n \vee u) \wedge h_n \to u$ in $(\mathscr{F}, \mathscr{E}_\alpha)$, and $g_n \leqslant w_n \leqslant h_n$ for $n = 1, 2, \ldots$ Since $\mathscr{E}_\alpha(v_{n_k}, w_{n_k}-v_{n_k}) \geqslant 0$, we get

$$\mathscr{E}_\alpha(v_{n_k}, w_{n_k}) \geqslant \mathscr{E}_\alpha(v_{n_k}, v_{n_k}). \tag{17}$$

Putting $k \to \infty$ in (17), we obtain

$$\mathcal{E}_\alpha(\hat{v}, u) \geqslant \varliminf_k \mathcal{E}_\alpha(v_{n_k}, v_{n_k}) \geqslant \mathcal{E}_\alpha(\hat{v}, \hat{v}).$$

Therefore, $\hat{v}$ is a solution of (16) and the uniqueness gives $\hat{v} = v$. Since

$$\varlimsup_k \mathcal{E}_\alpha(v_{n_k}, v_{n_k}) \leqslant \mathcal{E}_\alpha(\hat{v}, \hat{v}),$$

$v_n \to v$ in $(\mathcal{F}, \mathcal{E}_\alpha)$.

PROPOSITION 4. *For arbitrary obstacles $g, h \in \mathcal{F}$, $g \leqslant h$ m-a.e., there exists a sequence of separable obstacles (g_n, h_n) convergent in $(\mathcal{F}, \mathcal{E}_\alpha)$ to (g, h).*

Proof. Let R_γ, $\gamma > 0$, be the resolvent operator associated with $(\mathcal{F}, \mathcal{E}_\alpha)$. Then for any $\gamma > 0$ we have $\gamma R_\gamma g \leqslant \gamma R_\gamma h$ and $\gamma R_\gamma g \to g$, $\gamma R_\gamma h \to h$ in $(\mathcal{F}, \mathcal{E}_\alpha)$ (see [7], p. 20). From the resolvent identity we obtain

$$\gamma R_\gamma g = R_\alpha\big(\gamma(\alpha-\gamma) R_\gamma g - \gamma g\big) = R_\alpha(f_1) - R_\alpha(f_2)$$

for some non-negative functions $f_1, f_2 \in L^2(X, m)$. Therefore, for every $\gamma > 0$, obstacles $\gamma R_\gamma g$ and $\gamma R_\gamma h$ are separable.

We proceed now to proving the first part of Theorem 1. Let g_n and h_n be separable, quasi-continuous obstacles such that $g_n \leqslant h_n$ q.e. for $n = 1, 2, \ldots$ and g_n and h_n are convergent in $(\mathcal{F}, \mathcal{E}_\alpha)$ to g and h, respectively. Moreover, let $J_x^n(\tau, \sigma)$ be the corresponding cost functionals:

$$J_x^n(\tau, \sigma) = \mathrm{E}^x\big(e^{-\alpha\tau\wedge\sigma}\big(I_{\tau\leqslant\sigma} h_n(x_\tau) + I_{\sigma<\tau} g_n(x_\sigma)\big)\big), \qquad x \in X.$$

Then

$$\text{(18)} \quad |J_x^n(\tau, \sigma) - J_x(\tau, \sigma)| \leqslant \mathrm{E}^x\big(e^{-\alpha\tau}|(h_n-h)|(x_\tau)\big) + \mathrm{E}^x\big(e^{-\alpha\sigma}|(g_n-g)|(x_\sigma)\big) \leqslant G_n(x) + H_n(x),$$

where

$$G_n(x) = \sup_\sigma \mathrm{E}^x\big(e^{-\alpha\sigma}|(g_n-g)|(x_\sigma)\big),$$

$$H_n(x) = \sup_\tau \mathrm{E}^x\big(e^{-\alpha\tau}|(h_n-h)|(x_\tau)\big).$$

From Theorem 2 and the fact that the solution of a variational problem (7) has minimal energy it follows that

$$\mathcal{E}_\alpha(G_n, G_n) \leqslant \mathcal{E}_\alpha(|g_n-g|, |g_n-g|) \leqslant \mathcal{E}_\alpha(g_n-g, g_n-g),$$

$$\mathcal{E}_\alpha(H_n, H_n) \leqslant \mathcal{E}_\alpha(|h_n-h|, |h_n-h|) \leqslant \mathcal{E}_\alpha(h_n-h, h_n-h),$$

and therefore sequences (G_n) and (H_n) converge to 0 in the sense of $(\mathcal{F}, \mathcal{E}_\alpha)$. Consequently, we can assume that $G_n(x) \to 0$ and $H_n(x) \to 0$ as $n \to \infty$ for q.e. $x \in X$. Let

$$u_n(x) = \inf_{\tau} \sup_{\sigma} J_x^n(\tau, \sigma), \qquad u(x) = \inf_{\tau} \sup_{\sigma} J_x(\tau, \sigma).$$

By (18), $|u_n - u| \to 0$ q.e. But Proposition 2 implies that the functions u_n can be identified as solutions of variational inequalities (15) and, by Proposition 3, u_n converge to v. Therefore, $v = u$ outside of a set of capacity zero. This proves the required result.

5. A convergence theorem. After some preparatory results on penalization we prove Theorem 3 needed for the identification of the pair $(\hat{\tau}, \hat{\sigma})$ as the saddle point of the game (see formulae (5) and (6)). We omit proofs of Propositions 5 and 6 below because of their similarity to those contained in [3] and [16] (see also [20]).

PROPOSITION 5. *For arbitrary elements $g, h \in H = L^2(X, m)$ there exist unique solutions $w^\beta, v^\beta \in H$ and $(\bar{v}^\beta, \underline{v}^\beta) \in H \times H$ of the following equations:*

$$w^\beta = \beta R_\alpha (g - w^\beta)^+, \tag{19}$$

$$\underline{v}^\beta = \beta R_\alpha \big((g - v^\beta)^+ - (v^\beta - h)^+\big), \tag{20}$$

$$\underline{v}^\beta = \beta R_\alpha (\bar{v}^\beta - h - \underline{v}^\beta)^+, \tag{21}$$

$$\bar{v}^\beta = \beta R_\alpha (\underline{v}^\beta + g - \bar{v}^\beta)^+, \qquad \beta > 0. \tag{22}$$

Moreover,

$$v^\beta = \bar{v}^\beta - \underline{v}^\beta, \qquad \beta > 0. \tag{23}$$

In the next proposition, M^β stands for the set of all progressively measurable processes (u_t) satisfying $0 \leqslant u_t \leqslant \beta$ for all $t \geqslant 0$. If $u, u^1, u^2 \in M^\beta$, $x \in X$, then

$$J_x^\beta(u) = \mathrm{E}^x\Big(\int_0^\infty \exp\big[-\int_0^t (\alpha + u_s)\, ds\big] u_t\, g(x_t)\, dt\Big),$$

$$J_x^\beta(u^1, u^2) = \mathrm{E}^x\Big(\int_0^\infty \exp\big[-\int_0^t (\alpha + u_s^1 + u_s^2)\, ds\big] \big(u_t^1\, g(x_t) + u_t^2\, g(x_t)\big)\, dt\Big).$$

PROPOSITION 6. *Let g and h be quasi-continuous functions belonging to $\mathscr{F}$. Then there exists a properly exceptional set N such that, for quasi-continuous solutions $w^\beta, v^\beta \in \mathscr{F}$ of equations (19) and (20) and for all $x \in X \setminus N$,*

$$w^\beta(x) = \sup_{u \in M^\beta} J_x^\beta(x),$$

$$v^\beta(x) = \inf_{u^1 \in M^\beta} \sup_{u^2 \in M^\beta} J_x^\beta(u^1, u^2) = \sup_{u^2 \in M^\beta} \inf_{u^1 \in M^\beta} J_x^\beta(u^1, u^2).$$

Moreover,

$$w^\beta(x) \leqslant \sup_{\tau} \mathrm{E}^x\big(e^{-\alpha\tau} g(x_\tau)\big), \qquad x \in X \setminus N.$$

THEOREM 3. *If $h, g \in \mathscr{F}$, $h \leqslant g$, and the separability condition* (8) *holds, then $v^\beta \to v$ in $(\mathscr{F}, \mathscr{E}_\alpha)$ as $\beta \uparrow +\infty$.*

For the proof we need two lemmas.

LEMMA 2. *An increasing sequence of α-potentials bounded from above by an α-potential is convergent in the norm $\mathscr{E}_\alpha$ to an α-potential.*

The lemma follows from Lemma 3.3.2 of [7].

LEMMA 3. *Assume that $g_n \uparrow g$ in $(\mathscr{F}, \mathscr{E}_\alpha)$ and that w_n $(n = 1, 2, \ldots)$ are solutions of the equations*

$$w_n = \beta_n R_\alpha (g_n - w_n)^+.$$

If $\beta_n \uparrow +\infty$, then $w_n \uparrow w$ in $(\mathscr{F}, \mathscr{E}_\alpha)$ where w is the solution of (7).

Proof. Without loss of generality we can assume that the functions w_n, g_n, and w are quasi-continuous and that outside of a set N given in Proposition 6 we have $g^n \uparrow g$. It follows from Theorem 2 and Proposition 6 that the sequence (w_n) is increasing and that for $x \in X \setminus N$

$$w(x) = \hat{w}(x) = \lim_n w_n(x) \geqslant g(x).$$

Lemma 2 implies that (w_n) converges in $(\mathscr{F}, \mathscr{E}_\alpha)$ to an α-potential $\hat{w}$. On the other hand, w is the smallest α-potential majorizing g (see Theorem 2), so $\hat{w} \geqslant w$ and, finally, $\hat{w} = w$.

Proof of Theorem 3. We show that families $(\bar{v}^\beta)$ and $(\underline{v}^\beta)$ are increasing with respect to $\beta > 0$ and bounded by α-potentials $\bar{v}$ and $\underline{v}$ introduced in Proposition 1. Then Lemma 2 will imply that $\bar{v}^\beta \uparrow \bar{v}$ and $\underline{v}^\beta \uparrow \underline{v}$ in $(\mathscr{F}, \mathscr{E}_\alpha)$ and Proposition 2 together with formula (23) will give the desired result. Let us define α-potentials $\bar{v}_n^\beta$ and $\underline{v}_n^\beta$, $\beta > 0$, $n = 0, 1, \ldots$, inductively:

$$\bar{v}_0^\beta = \underline{v}_0^\beta = 0,$$

$$\bar{v}_{n+1}^\beta = \beta R_\alpha (\underline{v}_n^\beta + g - \bar{v}_{n+1}^\beta)^+, \tag{24}$$

$$\underline{v}_{n+1}^\beta = \beta R_\alpha (\bar{v}_n^\beta - h - \underline{v}_{n+1}^\beta)^+. \tag{25}$$

One can easily show by induction (cf. the proof of Lemma 3) that

$$\bar{v}_{n+1}^\beta \geqslant \bar{v}_n^\beta, \quad \underline{v}_{n+1}^\beta \geqslant \underline{v}_n^\beta \tag{26}$$

and

$$\bar{v}_n \geqslant \bar{v}_n^\beta, \quad \underline{v}_n \geqslant v_n^\beta \tag{27}$$

m-a.e., $n = 0, 1, 2, \ldots$ Functions $\bar{v}_n$ and $\underline{v}_n$ in (27) are exactly those introduced in the proof of Proposition 1. Since $\bar{v} \geqslant \bar{v}_n \geqslant \bar{v}_n^\beta$ and $\underline{v} \geqslant \underline{v}_n \geqslant \underline{v}_n^\beta$ for all $\beta > 0$ and $n = 0, 1, \ldots$, passing in (24) and (25) to the limit with $n \uparrow \infty$, and using Proposition 5 and Lemma 2 we obtain $\bar{v}_n^\beta \uparrow \bar{v}^\beta \leqslant \bar{v}$ and $\underline{v}_n^\beta \uparrow \underline{v}^\beta \leqslant \underline{v}$ in $(\mathscr{F}, \mathscr{E}_\alpha)$. Moreover, for fixed n, both families $(\bar{v}_n^\beta)$ and $(\underline{v}_n^\beta)$ are increasing with respect to β, and so the same is true for $(\bar{v}^\beta)$ and $(\underline{v}^\beta)$.

6. Identification of the saddle point. To prove the second part of Theorem 1 we start with some lemmas.

LEMMA 4. *Let v and v^{β}, $\beta > 0$, be quasi-continuous solutions of* (4) *and* (20), *respectively. There exist an increasing sequence of closed sets (X_k) and an increasing sequence of positive numbers $\beta_n \uparrow \infty$ such that on every X_k the functions h, g, v, v^{β_n} are bounded and continuous and $\underline{v}^{\beta_n} \to v$ uniformly as $n \to \infty$. Moreover,* $\mathrm{Cap}(X \setminus X_k) \to 0$ *as $k \to \infty$.*

Proof. Let g_n, h_n, and v_n be the functions given in Proposition 3 and let v_n^{β} be the solution of the penalized problem with obstacles g_n and h_n. It follows from Theorem 3 that $v_n^{\beta} \to v_n$ as $\beta \uparrow \infty$ in $(\mathscr{F}, \mathscr{E}_{\alpha})$. Therefore, one can find a sequence $\beta_n \uparrow \infty$ such that $v^n - v_n^{\beta_n} \to 0$ also $(\mathscr{F}, \mathscr{E}_{\alpha})$. Using Proposition 6 and reasoning as in (18) but with the penalized functionals J_x^{β} instead of J_x, we obtain

$$|v_n^{\beta_n} - v^{\beta_n}| \leqslant H_{2n} + G_{2n} \text{ q.e.}$$

But

$$|v - v^{\beta_n}| \leqslant |v - v_n| + |v_n - v_n^{\beta_n}| + |v_n^{\beta_n} - v^{\beta_n}| \tag{28}$$

and the right-hand side of (28) converges also in $(\mathscr{F}, \mathscr{E}_{\alpha})$. Reasoning in the same way as in the proof of Theorem 3.1.4 of [7] we can show that there exists an increasing sequence of closed sets X_k such that the right-hand sides of (28) are bounded on each X_k and an appropriately chosen subsequence converges to 0 uniformly. Moreover, the sequence (X_k) can be selected in such a way that $\mathrm{Cap}(X \setminus X_k) \to 0$ and that the functions h, g, v, v^{β_n} have the required properties (see the proofs of Theorems 3.1.3 and 3.1.2 in [7]).

LEMMA 5. *Let a function $u \in \mathscr{F}$ be quasi-continuous and $\alpha > 0$ a positive constant. Then for q.e. $x \in X$ the family of random variables $\{e^{-\alpha T} u(x_T);\ T$ any stopping time$\}$ is P^x-uniformly integrable.*

Proof. Since there always exists a quasi-continuous α-potential majorizing the function u q.e. (see Theorem 2), we can assume that u itself is an α-potential. In the same way as in the proof of Lemma 4.4.1 in [7] one can show that there exists a properly exceptional set N such that for all $x \in X \setminus N$ the process $Z_t = e^{-\alpha t} u(x_t)$, $t \geqslant 0$, is a right-continuous non-negative P^x-supermartingale with the property

$$\lim_{t \uparrow \infty} Z_t = 0.$$

To prove the required uniform integrability it is enough to show (see [12], p. 102) that, for any increasing family of stopping times $T_n \uparrow \infty$, $\mathrm{E}^x(Z_{T_n}) \to 0$ as $n \to \infty$. From the representation Theorem 5.11 in [7] it follows that there exist a positive continuous additive functional A_t and a properly exceptional set

$N_1 \supset N$ such that

$$u(x) = \mathrm{E}^x\Big(\int_0^\infty e^{-\alpha s}\, dA_s\Big) \quad \text{for } x \in X \setminus N_1. \tag{29}$$

By the strong Markov property we have

$$u(x) = \mathrm{E}^x\Big(\int_0^{T_n} e^{-\alpha s}\, dA_s + e^{-\alpha T_n} u(x_{T_n})\Big) = \mathrm{E}^x(Z_{T_n}) + \mathrm{E}^x\Big(\int_0^{T_n} e^{-\alpha s}\, dA_s\Big). \tag{30}$$

Since $T_n \uparrow \infty$ P^x-a.e., from (29) and (30) we obtain $\mathrm{E}^x(Z_{T_n}) \to 0$.

We proceed now to the proof of the second part of Theorem 1, which is a modification of the proof of a similar property for optimal stopping given in [15] (see also [16]). Let us introduce the following stopping times:

$$\tau^\beta = \inf\{t \geqslant 0;\ v^\beta(x_t) \geqslant h(x_t)\}, \qquad \sigma^\beta = \inf\{t \geqslant 0;\ v^\beta(x_t) \leqslant g(x_t)\},$$

$$\tau_\gamma = \inf\{t \geqslant 0;\ v(x_t) + \gamma \geqslant h(x_t)\}, \qquad \sigma_\gamma = \inf\{t \geqslant 0;\ v(x_t) - \gamma \leqslant g(x_t)\},$$

$$T_k = \inf\{t \geqslant 0;\ x_t \in X \setminus X_k\},$$

where $\beta > 0$, $\gamma > 0$, and $k = 1, 2, \ldots$ One can always assume that $P^x(T_k \uparrow \infty) = 1$ for all $x \in X \setminus N$, where the set N has also properties specified in Proposition 6. The proof will be completed if we show that for arbitrary stopping times τ and σ such that $\sigma \leqslant \hat{\tau}$ and $\tau \leqslant \hat{\sigma}$ P^x-a.e. the relations

$$v(x) \leqslant \mathrm{E}^x\big(e^{-\alpha\tau} v(x_\tau)\big), \tag{31}$$

$$v(x) \geqslant \mathrm{E}^x\big(e^{-\alpha\sigma} v(x_\sigma)\big) \tag{32}$$

hold for $x \in X \setminus N$. To see this notice that (31) and (32) imply

$$v(x) = \mathrm{E}^x\big(e^{-\alpha\hat{\tau}\wedge\hat{\sigma}} v(x_{\hat{\tau}\wedge\hat{\sigma}})\big) = J_x(\hat{\tau}, \hat{\sigma}).$$

Moreover,

$$\begin{aligned} J_x(\hat{\tau}, \sigma) &= \mathrm{E}^x\big(e^{-\alpha\hat{\tau}} v(x_{\hat{\tau}}) I_{\hat{\tau}\leqslant\sigma} + e^{-\alpha\sigma} g(x_\sigma) I_{\sigma<\hat{\tau}}\big) \\ &\leqslant \mathrm{E}^x\big(e^{-\alpha\hat{\tau}} v(x_{\hat{\tau}}) I_{\hat{\tau}\leqslant\sigma} + e^{-\alpha\sigma} v(x_\sigma) I_{\sigma<\hat{\tau}}\big) \\ &\leqslant \mathrm{E}^x\big(e^{-\alpha\hat{\tau}\wedge\sigma} v(x_{\hat{\tau}\wedge\sigma})\big) \leqslant v(x). \end{aligned}$$

The last inequality follows from (32) because $\hat{\tau} \wedge \sigma \leqslant \hat{\tau}$. In the same way one can show that $J_x(\tau, \hat{\sigma}) \geqslant v(x)$.

We prove now, for instance, (32). Assume that $\sigma \leqslant \tau^\beta$ P^x-a.e. for some $\beta > 0$. Then from (20) and the strong Markov property we obtain

$$v^\beta(x) \geqslant \mathrm{E}^x\big(e^{-\alpha\sigma} v^\beta(x_\sigma)\big), \quad x \in X \setminus N. \tag{33}$$

Letting $\beta \uparrow \infty$ and applying the Lebesgue domination principle to the right-hand side of (33) (cf. the estimate in Proposition 6), we obtain

$$v(x) \geqslant \mathrm{E}^x\big(e^{-\alpha\sigma} v(x_\sigma)\big), \quad x \in X \setminus N. \tag{34}$$

Exactly as in [15] we get

$$\lim_{\gamma\downarrow 0}\tau_\gamma = \hat{\tau}, \qquad \lim_{\gamma\downarrow 0}\sigma_\gamma = \hat{\sigma} \qquad \mathrm{P}^x\text{-a.e.}$$

Let us fix now k and γ. Since $v^\beta \to v$ uniformly on X_k, we can find p such that $|v^{\beta_n}(x)-v(x)| < \gamma$ for $n > p$ and $x \in X_k$. If $t < \tau_\gamma$, then $v(x_t)+\gamma < h(x_t)$. Therefore, for $t < \tau_\gamma \wedge T_k \wedge \sigma$ we obtain

$$v^{\beta_n}(x_t) \leqslant v(x_t)+\gamma < h(x_t).$$

Consequently, $\tau_\gamma \wedge T_k \wedge \sigma \leqslant \tau^{\beta_n}$ and, by (34),

(35) $$v(x) \geqslant \mathrm{E}^x\big(\exp[-\alpha(\tau_\gamma \wedge T_k \wedge \sigma)]\, v(x_{\tau_\gamma \wedge T_k \wedge \sigma})\big), \qquad x \in X\setminus N.$$

Lemma 5 and the quasi-left continuity of the Markov process allow us to pass in (35) to the limit first with $\gamma \downarrow 0$, and then with $k \uparrow \infty$ to obtain (32) for arbitrary $\sigma \leqslant \hat{\tau}$.

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