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ON THE CONVERGENCE OF STABLE MEASURES IN A BANACH SPACE

BY

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Abstract. In this paper a space of functions generating stable measures on a Banach space is introduced. Some properties of this space are investigated. Then as applications, necessary and sufficient conditions for the weak convergence of stable measures on a Banach space are given. The convergence of sums of independent stable random variables with Banach values is investigated as well.

1. NOTATIONS AND DEFINITIONS

Let (U, Σ, m) be a fixed finite measure space and let X be a separable Banach space with norm $\|\cdot\|$ and topological dual X^*. Denote by $L_p(X; U, \Sigma, m)$ ($L_p(X)$ for short) the set of all X-valued functions f such that

$$\|f\|_p = \{\int_U \|f(u)\|^p \, dm(u)\}^{1/p} < +\infty .$$

In the case X is the set of real numbers we write L_p instead of $L_p(X)$. $sL_p(X)$ denotes the set of all X-valued functions f such that $\langle f, x^* \rangle \in L_p$ for each $x^* \in X^*$.

In the sequel, if not otherwise stated, p is a real number with $0 < p < 2$. A Radon probability measure μ on X is called *p-stable* iff for given $a > 0$, $b > 0$ we have

$$\hat{\mu}(ax^*)\,\hat{\mu}(bx^*) = \hat{\mu}\big((a^p+b^p)^{1/p}x^*\big) \quad \text{for all } x^* \in X^*,$$

where $\hat{\mu}$ is the characteristic functional (ch.f.) of μ, i.e.

$$\hat{\mu}(x^*) = \int_X \exp\{i\langle x, x^*\rangle\}\, d\mu(x).$$

Thus, in this paper we consider only symmetric stable measures. Let $\{\theta_n^{(p)}\}$ be a sequence of independent identically distributed real random variables with ch.f. $\exp(-|t|^p)$. A Banach space X is said to be of *stable type p* if for some (each) $r \in (0, p)$ there exists a constant $c > 0$ such that

$$(\mathrm{E}\|\sum_{k=1}^{n} x_k \theta_k^{(p)}\|^r)^{1/r} \leqslant c(\sum_{k=1}^{n} \|x_k\|^p)^{1/p}$$

for all $x_1, \ldots, x_n$ in X. For properties of Banach spaces of stable type we refer the reader to [5], [10].

Let now (Ω, A, P) be a probability space. By an X-valued random variable (r.v.) we mean a strongly measurable mapping $\xi: \Omega \to X$. $\mathscr{L}(\xi)$ denotes the distribution of ξ,

$$\mathscr{L}(\xi)(B) = P\{\xi \in B\} \quad \text{for any } B \in \mathscr{B}(X),$$

where $\mathscr{B}(X)$ is the set of all Borel subsets of X. If $\mathscr{L}(\xi)$ is p-stable, then we say that ξ is *p-stable r.v.*

2. THE SPACE $S_p(X)$

Let f be an element of $sL_p(X)$. Put

$$\chi_f(x^*) = \exp\{-\int_U |\langle f(u), x^*\rangle|^p \, dm(u)\}, \qquad x^* \in X^*.$$

Denote by $S_p(X)$ the set of all $f \in sL_p(X)$ such that $\chi_f(x^*)$ is the ch.f. of a Radon measure on X. For every $f \in S_p(X)$ let μ_f denote the corresponding measure, i.e.

$$\hat{\mu}_f(x^*) = \chi_f(x^*). \tag{2.1}$$

Obviously, μ_f is p-stable.

Conversely, each p-stable measure μ on X can be written in the form (2.1). Indeed, it is well known that the ch.f. of a p-stable measure μ on X is of the form

$$\hat{\mu}(x^*) = \exp\{-\int_S |\langle x, x^*\rangle|^p \, d\sigma(x)\},$$

where σ is a finite measure on the unit sphere S of X (see e.g. [13], [17]). Taking $U = S$, $m = \sigma$, $f(x) = x$ we get (2.1). σ is called the *spectral measure* of μ.

For $f \in S_p(X)$ we put

$$\lambda_r(f) = (\int_X \|x\|^r \, d\mu_f(x))^{1/r}, \qquad r \in (0, p).$$

In view of results of [8], [9], [11] it is not hard to prove the following

THEOREM 2.1. (1) *$S_p(X)$ is a linear subspace of $L_p(X)$.*

(2) *There exist positive constants C_p and $C_{r,p}$ such that for all $f \in S_p(X)$ we have*

$$\|f\|_p = \sigma_f^{1/p}(S) = C_p \lim_{t \to +\infty} t\mu_f\{\|x\| > t\}^{1/p} \leqslant C_{r,p} \lambda_r(f)$$

where σ_f is the spectral measure of μ_f.

(3) *For $1 \leqslant r < p < 2$ $(S_p(X), \lambda_r(\cdot))$ is a Banach space; for $0 < r < p \leqslant 1$ $(S_p(X), \lambda_r(\cdot))$ is a quasi-Banach space.*

We now study some interesting properties of the space $S_p(X)$. First of all it should be noted that in general case $S_p(X)$ is not a closed subspace of $L_p(X)$ and, consequently, $S_p(X)$ is not complete with respect to $\|\cdot\|_p$. The following example shows it.

Example. Let L_p be infinite-dimensional and assume that X is not of stable type p. Then there are disjoint sets $A_n \in \Sigma$ and a sequence $x_n \in X$ having the following properties:

$$a_n = m(A_n) > 0 \quad \text{for all } n = 1, 2, \ldots,$$

$$\sum_{n=1}^{\infty} \|x_n\|^p < +\infty,$$

$$\sum_{n=1}^{\infty} x_n \theta_n^{(p)} \quad \text{diverges a.s.}$$

Consider the functions

$$f_n = \sum_{j=1}^{n} a_j^{-1/p} x_j \mathbf{1}_{A_j}, \qquad n = 1, 2, \ldots,$$

$$f = \sum_{j=1}^{\infty} a_j^{-1/p} x_j \mathbf{1}_{A_j}.$$

It is easy to see that $f_n \in S_p(X)$, $f \in L_p(X)$ and

$$\|f - f_n\|_p = \sum_{j=n+1}^{\infty} \|x_j\|^p \to 0 \quad \text{as } n \to +\infty.$$

It remains to show that $f \notin S_p(X)$. Assume to contrary that $f \in S_p(X)$. Then there exists a Radon measure μ_f with

$$\hat{\mu}_f(x^*) = \exp\{-\int_U |\langle f(u), x^*\rangle|^p \, dm(u)\} = \exp\{-\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|^p\}.$$

Consequently, by Ito-Nisio's Theorem [6] we conclude that the series

$$\sum_{n=1}^{\infty} x_n \theta_n^{(p)}$$

converges a.s. Thus we have got a contradiction.

Denote by $cL_\infty(X)$ the set of all X-valued functions f with

$$f = \sum_{n=1}^{\infty} x_n \mathbf{1}_{A_n}, \qquad \|f\|_\infty = \sup_n \|x_n\| < +\infty,$$

where $A_n \in \Sigma$ are disjoint sets and $x_n \in X$ for all $n = 1, 2, \ldots$

THEOREM 2.2[1]. (1) *In any case if X is of stable type p, then $cL_\infty(X) \subset S_p(X)$ and there exists a positive constant c depending only on X and p, r such that $\lambda_r(f) \leqslant c\|f\|_p$ for all $f \in cL_\infty(X)$.*

(2) *Conversely, if U is $[0, 1]$ with the Lebesgue measure and $cL_\infty(X) \subset S_p(X)$, then X is of stable type p.*

Proof. (1) Let $f \in cL_\infty(X)$ with

$$f = \sum_{n=1}^{\infty} x_n \mathbf{1}_{A_n}.$$

Then we have

$$\|f\|_p^p = \sum_{n=1}^{\infty} \|x_n\|^p m(A_n) < +\infty.$$

Consequently, if X is of stable type p, then the series

$$\sum_{n=1}^{\infty} \big(m(A_n)\big)^{1/p} x_n \theta_n^{(p)}$$

converges a.s. to some r.v. ξ. Evidently,

$$\begin{aligned} \mathrm{E}\exp\left(i\langle \xi, x^*\rangle\right) &= \exp\Big\{-\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|^p m(A_n)\Big\} \\ &= \exp\Big\{-\int_U |\langle f(u), x^*\rangle|^p\, dm(u)\Big\}. \end{aligned}$$

This shows that $f \in S_p(X)$. Moreover (see [10]), we have

$$\lambda_r(f) = (\mathrm{E}\|\xi\|^r)^{1/r} \leqslant c\Big(\sum_{n=1}^{\infty} \|x_n\|^p m(A_n)\Big)^{1/p} = c\|f\|_p,$$

where c is a positive constant depending only on X and p, r.

(2) Let $U = [0, 1]$ with the Lebesgue measure m and (x_n) be a sequence in X with

$$a = \sum_{n=1}^{\infty} \|x_n\|^p < +\infty.$$

[1] See also Theorems V.10.1 and V.10.2 in [18].

We can find a partition (A_n) of $[0, 1]$ such that

$$[0, 1] = \sum_{n=1}^{\infty} A_n, \quad m(A_n) = a^{-1} \|x_n\|^p.$$

Without loss of generality we assume that $\|x_n\| \neq 0$ for all $n = 1, 2, \ldots$ Consider the following function:

$$f = a^{1/p} \sum_{n=1}^{\infty} \frac{x_n}{\|x_n\|} 1_{A_n}.$$

Then we have $\|f\|_\infty = a^{1/p}$, i.e. $f \in cL_\infty(X)$.

By assumption it follows that $f \in S_p(X)$. So there exists a Radon measure μ_f on X such that

$$\hat{\mu}_f(x^*) = \exp\{-\int_U |\langle f(u), x^*\rangle|^p dm(u)\}$$

$$= \exp\{-\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|^p\}.$$

By Ito-Nisio's Theorem it implies that the series

$$\sum_{n=1}^{\infty} x_n \theta_n^{(p)}$$

converges a.s. in X.

From Theorems 2.1, 2.2 and the example above we get

COROLLARY 2.3 [2]. *The following conditions are equivalent*:

(1) *X is of stable type p.*

(2) *$S_p(X) = L_p(X)$ for any (U, Σ, m).*

(3) *There exists a positive constant C depending only on X and p, r such that for any (U, Σ, m) and for all $f \in S_p(X)$ we have*

$$\lambda_r(f) \leqslant C\|f\|_p.$$

(4) *$S_p(X)$ is complete with respect to $\|\cdot\|_p$.*

One can ask whether $\varphi f \in S_p(X)$, provided $f \in S_p(X)$ and $\varphi \in L_\infty$, where L_∞ denotes the set of all measurable real-valued functions φ with

$$\|\varphi\|_\infty = \sup_U |\varphi(u)| < +\infty.$$

As is shown below, it is true. For this purpose we need the two following lemmas.

LEMMA 2.4. *Let $f_n \in S_p(X)$ and $f \in L_p(X)$ such that $\|f_n - f\|_p \to 0$ as $n \to$*

[2] See also Theorem V.10.2 in [18] and Theorem 5 in [8].

$+\infty$ *and* $\{\mu_{f_n}\}$ *is uniformly tight. Then* $f \in S_p(X)$ *and the sequence* $\{\mu_{f_n}\}$ *converges weakly to* μ_f.

Proof. Since $\{\mu_{f_n}\}$ is uniformly tight, there exists a subsequence $\{\mu_{f_{n_k}}\}$ converging weakly to some Radon measure μ on X. In particular we have

$$\hat{\mu}_{f_{n_k}}(x^*) = \exp\{-\int_U |\langle f_{n_k}(u), x^*\rangle|^p \, dm(u)\} \to \hat{\mu}(x^*) \quad \text{as } k \to +\infty.$$

On the other hand, because of $\|f_n - f\|_p \to 0$ we get

$$\lim_{k \to +\infty} \int_U |\langle f_{n_k}, x^*\rangle|^p \, dm(u) = \int_U |\langle f(u), x^*\rangle|^p \, dm(u),$$

which shows that

$$\hat{\mu}(x^*) = \exp\{-\int_U |\langle f(u), x^*\rangle|^p \, dm(u)\},$$

i.e. $f \in S_p(X)$. Moreover, as is shown above, the sequence $\{\mu_{f_n}\}$ has only one limit point. Hence it must converge weakly to μ_f.

LEMMA 2.5. *If* $f \in S_p(X)$ *and* $A \in \Sigma$, *then* $\mathbf{1}_A f \in S_p(X)$.

Proof. Consider cylindrical measures μ_1, μ_2 defined as follows:

$$\hat{\mu}_1(x^*) = \exp\{-\int_A |\langle f, x^*\rangle|^p \, dm\},$$

$$\hat{\mu}_2(x^*) = \exp\{-\int_{U \setminus A} |\langle f, x^*\rangle|^p \, dm\}.$$

Evidently, we have

$$\hat{\mu}_f(x^*) = \hat{\mu}_1(x^*) \cdot \hat{\mu}_2(x^*),$$

i.e. $\mu_f = \mu_1 * \mu_2$. Since μ_f is a Radon measure on X and μ_1, μ_2 are symmetric, μ_1 and μ_2 are Radon measures on X as well (see [14]), which shows that $\mathbf{1}_A f \in S_p(X)$.

THEOREM 2.6. *If* $\varphi \in L_\infty$ *and* $f \in S_p(X)$, *then* $\varphi f \in S_p(X)$ *and*

$$\lambda_r(\varphi f) \leqslant 2\|\varphi\|_\infty \lambda_r(f). \tag{2.2}$$

Proof. At first we assume that φ is a simple function with

$$\varphi = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}.$$

Then we can write

$$\varphi f = \sum_{k=1}^{n} \alpha_k (\mathbf{1}_{A_k} f).$$

Noting that $S_p(X)$ is a linear subspace, we get $\varphi f \in S_p(X)$ (by Lemma 2.5). Furthermore, we can choose independent r.v.'s $\xi_1, \ldots, \xi_n$ with values in X such that

$$\mathscr{L}(\xi_k) = \mu_{1_{A_k}f}, \quad k = 1, \ldots, n,$$

$$\mathscr{L}\Big(\sum_{k=1}^{n} \xi_k\Big) = \mu_f,$$

$$\mathscr{L}\Big(\sum_{k=1}^{n} \alpha_k \xi_k\Big) = \mu_{\varphi f}.$$

By this way and noting that (see, e.g., [16])

$$P\{\|\sum_{k=1}^{n} \alpha_k \xi_k\| > t\} \leqslant 2P\{\max_{\alpha} |\alpha_k| \,\|\sum_{k=1}^{n} \xi_k\| > t\},$$

we obtain inequality (2.2) immediately.

The general case follows from the special case by choosing suitable simple functions and using Lemma 2.4 and Theorem 3.3 below.

Remark. The assertion of Theorem 2.6 for the case $p = 2$ was pointed out in [5] by using random integral. Recently, Rosiński [15] has constructed a general random integral. In paricular, he has proved that $f \in S_p(X)$ iff f is M-integrable, where M is a random measure defined on Σ and taking values in $L_0(\Omega, A, P)$ with

$$\mathrm{E} \exp itM(A) = \exp\big(-m(A)|t|^p\big), \quad A \in \Sigma.$$

We refer the reader to [15] and [18] for details.

To end this section we should like to raise two following problems.

Problem 1. Is it true that $S_p(X) \subset S_q(X)$ for $1 \leqslant q < p < 2$?

Clearly, the answer is "yes" if X is of stable type q. Dang Hung Thang [2] gave a positive answer to the problem for the case $X = l_s(B)$ with $1 \leqslant s < q$ and B is a Banach space of stable type q (although $l_s(B)$ is not of stable type q).

Problem 2. Is the set of all simple functions dense in $S_p(X)$ with respect to norm $\lambda_r(\cdot)$?

If the answer is "yes", then by Theorem 3.3 we could prove that:

For any p-stable measure μ on a Banach space there exists a sequence of p-stable measures $\{\mu_n\}$ such that the support of σ_n is a finite set for all n and μ_n converges weakly to μ, where σ_n is the spectral measure of μ_n.

3. THE CONVERGENCE IN $S_p(X)$

Our aim in this section is to investigate the convergence relation between $f \in S_p(X)$ and μ_f.

PROPOSITION 3.1. *Let $\{f_\alpha\}$ be a family of elements in $S_p(X)$ and $\{\mu_\alpha = \mu_{f_\alpha}\}$ be the corresponding family of p-stable measures on X. If $\{\mu_\alpha\}$ is uniformly tight, then $\{f_\alpha\}$ is bounded in $L_p(X)$.*

Proof. Recall that a family of Radon measures $\{\mu_\alpha\}$ on X is called *uniformly tight* iff for any $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset X$ such that $\mu_\alpha(X \setminus K_\varepsilon) < \varepsilon$ for all α. In [3] we have proved that if $\{\mu_\alpha\}$ is a family of p-stable measures on X and it is uniformly tight, then

$$\sup_\alpha \int_X \|x\|^r d\mu_\alpha(x) < +\infty \qquad \text{with } 0 < r < p.$$

Hence, by Theorem 2.1, it follows that

$$\sup_\alpha \|f_\alpha\|_p \leqslant C_{r,p} \sup_\alpha \lambda_r(f_\alpha) < +\infty.$$

Remark. In general, it is possible that $\mu_f = \mu_g$ for $f, g \in S_p(X)$ and $f \neq g$. So the assertion that the weak convergence of $\{\mu_n\} = \{\mu_{f_n}\}$ implies the converge of $\{f_n\}$ in $L_p(X)$ is false. However, in any case we have the following

PROPOSITION 3.2. *If $f_n \in S_p(X)$, $f \in S_p(X)$ and μ_n weakly converges to μ_f, then $\|f_n\|_p \to \|f\|_p$ and $\lambda_r(f_n) \to \lambda_r(f)$ as $n \to +\infty$.*

Proof. The first assertion is an immediate consequence of Theorem 2.1 and Theorem 4.3, the second one follows from Corollary 3.4 of [3].

The next theorem is the main result of this section.

THEOREM 3.3. *If $f_n \in S_p(X)$, $f \in S_p(X)$ and, for some $r \in (0, p)$, $\lambda_r(f_n - f) \to 0$ as $n \to +\infty$, then μ_n weakly converges to $\mu = \mu_f$.*

Proof. Let Φ be an embedding of L_p into L_r (see [1]) and T_n, T the linear operators defined as follows:

$$T_n\colon \left.\begin{matrix} X^* \to L_p \\ x^* \to \langle f_n, x^* \rangle \end{matrix}\right\} \qquad T\colon \left.\begin{matrix} X^* \to L_p \\ x^* \to \langle f, x^* \rangle \end{matrix}\right\}.$$

Put $L_n = \Phi \circ T_n$, $L = \Phi \circ T$. Because of $f_n \in S_p(X)$, $f \in S_p(X)$ there exist X-valued r.v.'s φ_n, φ such that

$$\left.\begin{matrix} L_n x^* = \langle \varphi_n, x^* \rangle \\ \mu_n = \mathscr{L}(\varphi_n) \end{matrix}\right\} \qquad \left.\begin{matrix} L x^* = \langle \varphi, x^* \rangle \\ \mu = \mathscr{L}(\varphi) \end{matrix}\right\}$$

(see Lemma 2 in [8]). Now it is easy to check that

$$(L_n - L)(x^*) = \langle \varphi_n - \varphi, x^* \rangle,$$
$$\mu_{f_n - f} = \mathscr{L}(\varphi_n - \varphi),$$
$$\lambda_r(f_n - f) = (\mathrm{E}\,\|\varphi_n - \varphi\|^r)^{1/r} \to 0 \quad \text{as } n \to +\infty,$$

where E stands for the mathematical expectation. In particular, this implies that μ_n converges weakly to μ.

As a corollary of Theorem 3.3 and Corollary 2.3 we obtain

COROLLARY 3.4. *Let X be a Banach space of stable type p and $f_n \in L_p(X)$, $f \in L_p(X)$. If $f_n \to f$ in $L_p(X)$, then μ_n weakly converges to μ.*

Remark. The assertion of Corollary 3.4 for the case $p = 2$ was pointed out in [19].

From the example in Section 2 and Corollary 3.4 we get

COROLLARY 3.5. *The following conditions are equivalent:*

(1) *X is of stable type p.*

(2) *For any (U, Σ, m) the convergence of a sequence $\{f_n\}$ in $L_p(X)$ implies the weak convergence of the corresponding sequence $\{\mu_n\}$.*

4. THE UNIFORM TIGHTNESS OF SPECTRAL MEASURES

Let $\{\mu_\alpha\}$ be a family of p-stable measures on a separable Banach space X with ch.f.

$$\hat{\mu}_\alpha(x^*) = \exp\{-\int_S |\langle x, x^* \rangle|^p\, d\sigma_\alpha(x)\}, \quad x^* \in X^*,\ 0 < p < 2,$$

where S is the unit sphere of X and σ_α is a finite measure on S for any α.

In this section we investigate the relation between the uniform tightness of $\{\mu_\alpha\}$ and the uniform tightness of the corresponding family of spectral measures $\{\sigma_\alpha\}$.

THEOREM 4.1. *If $\{\mu_\alpha\}$ is uniformly tight, then so is $\{\sigma_\alpha\}$.*

Proof. Without loss of generality we can suppose that X has a Schauder basis (since every separable Banach space can be embedded into a Banach space having a Schauder basis, e.g. $C_{[0,1]}$).

Let (e_n) be a Schauder basis in X. By Theorem 3.5 of [3] the uniform tightness of $\{\mu_\alpha\}$ is equivalent to the conditions

$$\sup_\alpha \int_X \|x\|^r\, d\mu_\alpha(x) < +\infty,$$

$$\lim_{N\to\infty} \sup_\alpha \int_X \|V_N x\|^r\, d\mu_\alpha(x) = 0$$

for any $r \in (0, p)$, where

$$V_N x = \sum_{k=N}^{\infty} \langle x, e_k^* \rangle e_k$$

and (e_n^*) is the dual basis of (e_n).

Note that $\mu_\alpha \circ V_N^{-1}$ is the p-stable measure on X with ch.f.

$$(\mu_\alpha \circ V_N^{-1})\hat{}(x^*) = \exp\{-\int_S |\langle V_N x, x^* \rangle|^p d\sigma_\alpha(x)\}.$$

Consequently, by Theorem 2.1,

$$\sup_\alpha \sigma_\alpha(S) = \sup_\alpha \int_S \|x\|^p d\sigma_\alpha(x) \leqslant c_{r,p} \sup_\alpha \{\int_X \|x\|^r d\mu_\alpha(x)\}^{p/r} < +\infty.$$

$$\sup_\alpha \int_S \|V_N x\|^p d\sigma_\alpha(x) \leqslant c_{r,p} \sup_\alpha \{\int_X \|V_N x\|^r d\mu_\alpha(x)\}^{p/r} \to 0 \quad \text{as } N \to +\infty.$$

In view of Theorem 3 in [12] this permits us to conclude that the family $\{\sigma_\alpha\}$ is uniformly tight.

The next theorem characterizes those Banach spaces for which the converse to Theorem 4.1 is true.

THEOREM 4.2. *For any separable Banach space X the following conditions are equivalent:*

(1) *X is of stable type p.*

(2) *$\{\mu_\alpha\}$ is uniformly tight iff so is $\{\sigma_\alpha\}$, where $\{\mu_\alpha\}$ is an arbitrary family of p-stable measures on X.*

Proof. By the argument as in the proof of Theorem 4.1 we can assume that X has a Schauder basis (e_n).

(1) → (2). In view of Theorem 4.1 it is sufficient to show that the uniform tightness of $\{\sigma_\alpha\}$ implies the uniform tightness of $\{\mu_\alpha\}$.

As was shown in [12], the uniform tightness of $\{\sigma_\alpha\}$ is equivalent to the following conditions:

$$\sup_\alpha \sigma_\alpha(S) < +\infty, \tag{4.1}$$

$$\lim_{N \to +\infty} \sup_\alpha \sigma_\alpha\{x: \|V_N x\| > \varepsilon\} = 0 \quad \text{for any } \varepsilon > 0. \tag{4.2}$$

Now note that

$$\int_S \|V_N x\|^p d\sigma_\alpha = \int_{\{x \in S: \|V_n x\| > \varepsilon\}} \|V_N x\|^p d\sigma_\alpha + \int_{\{x \in S: \|V_n x\| \leqslant \varepsilon\}} \|V_N x\|^p d\sigma_\alpha$$
$$\leqslant K\sigma_\alpha\{x: \|V_N x\| > \varepsilon\} + \varepsilon^p \sigma_\alpha(S)$$

(because of $\|V_N x\| \leqslant K$ for all $x \in S$, where K is the basis constant of (e_n)). Consequently, using conditions (4.1) and (4.2), we obtain

$$\sup_{\alpha} \int_{S} \|x\|^p d\sigma_\alpha(x) < +\infty, \tag{4.3}$$

$$\lim_{N\to\infty} \sup_{\alpha} \int_{S} \|V_N x\|^p d\sigma_\alpha(x) = 0. \tag{4.4}$$

If X is of stable type p, then (4.3), (4.4) and Corollary 2.3 imply, for any $r \in (0, p)$, the following conditions:

$$\sup_{\alpha} \int_{X} \|x\|^r d\mu_\alpha(x) < +\infty, \qquad \lim_{N\to+\infty} \sup_{\alpha} \int_{X} \|V_N x\|^r d\mu_\alpha(x) = 0,$$

which shows that $\{\mu_\alpha\}$ is uniformly tight (see [3]).

(2) $\to$ (1). Suppose that (x_n) is a sequence of elements in X with

$$\sum_{n=1}^{\infty} \|x_n\|^p < +\infty.$$

We show that the random series

$$\sum_{n=1}^{\infty} x_n \theta_n^{(p)} \tag{4.5}$$

converges a.s. We take for σ_n the measure on S defined as follows:

$$\sigma_n\left(\frac{x_k}{\|x_k\|}\right) = \|x_k\|^p, \qquad k = 1, 2, \ldots, n.$$

It is easy to see that σ_n is the spectral measure of the following p-stable measure:

$$\mu_n = \mathscr{L}\Big(\sum_{k=1}^{n} x_k \theta_k^{(p)}\Big).$$

Obviously, this family $\{\sigma_n\}$ is uniformly tight, so, by the assumption, $\{\mu_n\}$ is also uniformly tight. Hence, by Ito-Nisio's Theorem (see [6]) it follows that series (4.5) converges a.s.

Remark. Theorem 4.2 is a simple corollary of Theorem V.10.3 in [18]. However, it should be noted that we have proved this theorem without using stochastic integral.

In general case we can not assert that the weak convergence of a sequence of p-stable measures implies the weak convergence of the corresponding sequence of spectral measures. But we always have the following

THEOREM 4.3. *Let X be a separable Banach space and $\{\mu_n\}$ a sequence of p-stable measures on X. If μ_n converges weakly to μ, then $\sigma_n(s)$ converges to $\sigma(S)$, where σ_n and σ are spectral measures of μ_n and μ, respectively.*

Proof. According to Theorem 4.1 we have $\sup \sigma_n(S) < +\infty$ (since $\{\sigma_n\}$ is uniformly tight). Let s be a limit point of the sequence $\{\sigma_n(S)\}$. So there exists a subsequence $\{\sigma_{n_k}(S)\}$ with

$$\lim_{k\to+\infty} \sigma_{n_k}(S) = s.$$

It remains to show that $s = \sigma(S)$. Since $\{\mu_{n_k}\}$ is uniformly tight, so is $\{\sigma_{n_k}\}$ by Theorem 4.1. Hence, by Prokhorov's Theorem (see [14]), the sequence $\{\sigma_{n_k}\}$ contains a subsequence $\{\sigma_{n_k'}\}$ converging weakly to a finite measure σ_∞ on S. This implies, for all $x^* \in X^*$, that

$$\lim_{k\to\infty} \sigma_{n_k'}(S) = \sigma_\infty(S) = s,$$

$$\lim_{k\to\infty} \int_S |\langle x, x^*\rangle|^p \, d\sigma_{n_k'} = \int_S |\langle x, x^*\rangle|^p \, d\sigma_\infty.$$

On the other hand, since μ_n converges weakly to μ, we get

$$\lim_{n\to\infty} \int_S |\langle x, x^*\rangle|^p \, d\sigma_n = \int_S |\langle x, x^*\rangle|^p \, d\sigma.$$

Consequently,

$$\int_S |\langle x, x^*\rangle|^p \, d\sigma = \int_S |\langle x, x^*\rangle|^p \, d\sigma_\infty \quad \text{for all } x^* \in X^*.$$

By Theorem 2.1 it follows that $\sigma(S) = \sigma_\infty(S)$, i.e. $s = \sigma(S)$, as desired.

5. THE CONVERGENCE OF SUMS OF INDEPENDENT p-STABLE BANACH VALUED RANDOM VARIABLES

We now apply results of previous sections to study the convergence of sums of independent p-stable Banach valued random variables.

THEOREM 5.1. *Let $\{\xi_n\}$ be a sequence of independent X-valued p-stable r.v.'s. If the random series*

$$\sum_{n=1}^{\infty} \xi_n \tag{5.1}$$

converges a.s., then

$$\sum_{n=1}^{\infty} \sigma_n(S) < +\infty, \tag{5.2}$$

where σ_n is the spectral measure of the p-stable measure $\mathscr{L}(\xi_n)$.

Proof. Put

$$S_n = \sum_{k=1}^{n} \xi_k, \quad \mu_n = \mathscr{L}(S_n), \quad \Delta_n = \sum_{k=1}^{n} \sigma_k.$$

It is easy to see that

$$\hat{\mu}_n(x^*) = \exp\{-\int_S |\langle x, x^*\rangle|^p d\Delta_n(x)\},$$

i.e. Δ_n is the spectral measure of the p-stable measure μ_n on X. If the random series (5.1) converges a.s., then, evidently, the sequence $\{\mu_n\}$ converges weakly to a measure μ on X. So, by Theorem 4.3, we have

$$\lim_{n\to+\infty} \Delta_n(S) = \Delta(S),$$

where Δ is the spectral measure of the p-stable measure μ, which implies condition (5.2).

THEOREM 5.2. *Let X be a separable Banach space. The following conditions are equivalent*:

(1) *X is of stable type p with $0 < p < 2$.*

(2) *For any sequence $\{\xi_n\}$ of independent X-valued p-stable r.v.'s series* (5.1) *converges a.s. iff condition* (5.2) *is satisfied.*

Proof. (1)→(2). In view of Theorem 5.1 it remains to show that if condition (5.2) holds, then random series (5.1) converges a.s. For this purpose we put

$$\sigma(B) = \sum_{n=1}^{\infty} \sigma_n(B), \quad B \in \mathscr{B}(X).$$

From condition (5.2) it follows that σ is a finite measure on s. If X is of stable type p, then, by Corollary 2.3, the functional defined by the formula

$$\hat{\mu}(x^*) = \exp\{-\int_S |\langle x, x^*\rangle|^p d\sigma(x)\}$$

is the ch.f. of a p-stable measure μ on X.

On the other hand, we have

$$\lim_{n\to\infty} \hat{\mu}_n(x^*) = \lim_{n\to+\infty} \exp\{-\int_S |\langle x, x^*\rangle|^p d\Delta_n(x)\}$$
$$= \exp\{-\int_S |\langle x, x^*\rangle|^p d\sigma\} = \hat{\mu}(x^*)$$

for all $x^* \in X^*$. Consequently, by Ito-Nisio's Theorem we conclude that random series (5.1) converges a.s.

(2)→(1). Let (x_n) be a sequence of elements in X with

$$\sum_{n=1}^{\infty} \|x_n\|^p < +\infty.$$

Put $\xi_n = x_n \theta_n^{(p)}$. Then we get

$$\sigma_n(S) = \|x_n\|^p, \quad \sum_{n=1}^{\infty} \sigma_n(S) = \sum_{n=1}^{\infty} \|x_n\|^p < +\infty.$$

Hence (2) implies that the random series

$$\sum_{n=1}^{\infty} x_n \theta_n^{(p)}$$

converges a.s. The proof is completed.

Corollary 5.3. *Let X be a separable Banach space. The following conditions are equivalent*:

(1) *X is of stable type p with $0 < p < 2$.*

(2) *For any sequence $\{\xi_n\}$ of independent X-valued p-stable r.v.'s series* (5.1) *converges a.s. iff the series*

$$\sum_{n=1}^{\infty} (\mathrm{E}\,\|\xi_n\|^r)^{p/r}$$

converges for some (*each*) $r \in (0, p)$.

Remark. Corollary 5.3 (and, consequently, Theorem 5.2) is not valid for $p = 2$. A necessary and sufficient condition for (2) to be true for $p = 2$ is: X is isomorphic to a Hilbert space (see [7]).

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After finishing the paper the author has learned that recently Problems 1 and 2 in this paper have been solved by J. Rosiński in *Random Integrals of Banach Space Valued Functions*, preprint, 1982.

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