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ON THE RATE OF CONVERGENCE FOR THE WEAK LAW OF LARGE NUMBERS

BY

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Abstract. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with the common distribution F. Further, let $\{c_n\}$ be a sequence of positive numbers, and $\{b_n\}$ be a strictly increasing sequence of positive integers. The paper considers the convergence of the series

$$\sum_{n=1}^{\infty} c_n P(|X_1+\ldots+X_{b_n}| \geqslant \varepsilon b_n)$$

under the interplay of three types of conditions:

(i) convergence of this series,

(ii) an appropriate moment condition on X,

(iii) a condition imposing constraints on the behavior of the sequences $\{c_n\}$ and $\{b_n\}$.

Three theorems have been proven; in each of these two among (i)-(iii) implying the third, with one of the theorems being valid for the general case, where the random variables involved are not necessarily i.i.d.

1. Introduction. Let $X, X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(t) = P(X \leqslant t)$, and let $S_n = X_1+\ldots+X_n$ $(n \geqslant 1)$. In studying the rate of convergence in weak laws of large numbers, the convergence of the series

$$\sum_{n=1}^{\infty} P(|S_n| \geqslant n\varepsilon), \tag{1.1}$$

for some $\varepsilon > 0$, was found to be connected with the existence of second moment of X (see Hsu and Robbins [6], Erdös [3] or Révész [9]). In

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particular, Erdös [3] has shown that series (1.1) converges for some $\varepsilon > 0$ if and only if $EX^2 < \infty$ and $|EX| < \varepsilon$.

Subsequently, number of authors (notably Heyde and Rohatgi [5], Chow and Lai [2] and Lai and Lan [8]) analysed the convergence of the series of the form

$$\sum_{n=1}^{\infty} c_n P(|S_n| \geqslant a_n) \tag{1.2}$$

for various $\{c_n\}$ and $\{a_n\}$, again connecting it with the appropriate moment conditions.

Certain considerations arising in stochastic modeling for the growth of cancer tumors (see [1]), led us to the analysis of convergence of series of type (1.1) with the index of summation restricted to a subsequence.

The problem of this note may be formulated as follows. Let $\{K_n\}$ be a sequence of integers satisfying

$$1 \leqslant K_1 \leqslant K_2 \leqslant \ldots \tag{1.3}$$

and

$$\lim_{n\to\infty} K_n = \infty. \tag{1.4}$$

Consider the series

$$\sum_{n=1}^{\infty} P\{|S_{K_n}| \geqslant \varepsilon K_n\} \tag{1.5}$$

for some $\varepsilon > 0$. By grouping the terms corresponding to identical indices K_n, we may write (1.5) as

$$\sum_{n=1}^{\infty} c_n P\{|S_{b_n}| \geqslant \varepsilon b_n\} \tag{1.6}$$

where the sequences $\{b_n\}$ and $\{c_n\}$ are defined by

$$c_0 = 0, \quad c_{n+1} = \min\{r\colon K_r > K_{c_n+1}\} - 1 - c_0 - \ldots - c_n \tag{1.7}$$

and

$$b_{n+1} = K_{c_0+c_1+\ldots+c_n+1} \tag{1.8}$$

for $n = 0, 1, \ldots$

Note that, since $\lim K_n = \infty$, we have $1 \leqslant c_n < \infty$ for all $n \geqslant 1$ and $1 \leqslant b_1 < b_2 < \ldots$

We shall now drop the condition that c_n's are integers, and consider generally the problem of convergence of series (1.6), where $\{c_n\}$ is some sequence of positive real numbers and $\{b_n\}$ is a strictly increasing sequence of positive integers.

Clearly, we have here an interplay of three types of conditions:

(i) convergence of series (1.6),

(ii) and appropriate moment condition and

(iii) a condition imposing constraints on the behaviour of the sequences $\{c_n\}$ and $\{b_n\}$.

We shall prove three theorems, in each of them two among (i)-(iii) implying the third, with Theorem 1 being valid for the general case where the random variables involved are not necessarily independent and identically distributed (i.i.d.).

2. The results. We start by presenting a lemma due to von Bahr and Esséen [11], which will be needed below.

LEMMA 1. *Let $Y_1, \ldots, Y_n$ be a finite sequence of random variables. Write $S_i = Y_1 + \ldots + Y_i$ and assume that $\mathrm{E}(Y_i|S_{i-1}) = 0$, $\mathrm{E}|Y_i|^{1+\lambda} < \infty$, $i = 1, \ldots, n$, for some λ with $0 < \lambda \leqslant 1$. Then there exists a constant $C(\lambda) > 0$ such that*

(2.1) $$\mathrm{E}|S_n|^{1+\lambda} \leqslant C(\lambda) \sum_{i=1}^{n} \mathrm{E}|Y_i|^{1+\lambda}.$$

In fact, as pointed out by Rubin [10], we have

(2.2) $$C(\lambda) = \sup_x \left[\frac{|1+x|^{1+\lambda} - 1 - (1+\lambda)x}{|x|^{1+\lambda}}\right]$$

with $1 \leqslant C(\lambda) \leqslant 2$ for $0 \leqslant \lambda \leqslant 1$.

We first prove

THEOREM 1. *Let $Y_1, Y_2, \ldots$ be a sequence of random variables with $\mathrm{E}(Y_i|S_{i-1}) = 0$, $i = 1, 2, \ldots$, where $S_0 = 0$, $S_i = Y_1 + \ldots + Y_i$, $i = 1, 2, \ldots$*

Assume that, for some sequence $\{\lambda_n\}$ with $0 < \lambda_n \leqslant 1$, we have $\mathrm{E}|Y_i|^{1+\lambda} < \infty$, $i = 1, 2, \ldots$, where $\lambda = \sup_n \lambda_n$, and the sequences $\{c_n\}$ and $\{b_n\}$ satisfy the condition

(2.3) $$\sum_{n=1}^{\infty} c_n \bar{\theta}_n b_n^{-\lambda_n} < \infty$$

where

(2.4) $$\bar{\theta}_n = \frac{1}{b_n} \sum_{j=1}^{b_n} \mathrm{E}|Y_j|^{1+\lambda_n}.$$

Then for every $\varepsilon > 0$ we have

(2.5) $$\sum_{n=1}^{\infty} c_n P\{|Y_1 + \ldots + Y_{b_n}| \geqslant \varepsilon b_n\} < \infty.$$

Proof. We may estimate the terms of the series in (2.5), using Markov inequality and Lemma 1, as follows:

$$(2.6)\qquad c_n P\{|Y_1+\ldots+Y_{b_n}|\geqslant \varepsilon b_n\} = c_n P\{|S_{b_n}|^{1+\lambda_n}\geqslant (\varepsilon b_n)^{1+\lambda_n}\} \leqslant c_n \frac{\mathrm{E}|S_{b_n}|^{1+\lambda_n}}{(\varepsilon b_n)^{1+\lambda_n}} \leqslant c_n C(\lambda_n)\bar{\theta}_n b_n^{-\lambda_n}\varepsilon^{-(1+\lambda_n)}.$$

The theorem now follows from (2.3), since $\sup_n C(\lambda_n)\leqslant 2$, and $\varepsilon^{-(1+\lambda_n)}\leqslant \varepsilon^{-1}$ or ε^{-2}, depending on whether $\varepsilon\geqslant 1$ or $\varepsilon<1$.

In particular, in the case of i.i.d. random variables $X, X_1, X_2, \ldots$ we obtain

COROLLARY 1. *Assume that* $\mathrm{E}|X|^{1+\lambda}<\infty$ *for some* λ *with* $0<\lambda\leqslant 1$. *Moreover, let* $\mathrm{E}X = 0$ *and assume that the sequences* $\{b_n\}$ *and* $\{c_n\}$ *satisfy the condition*

$$(2.7)\qquad \sum_{n=1}^{\infty} c_n b_n^{-\lambda}<\infty.$$

Then series (1.6) *converges for every* $\varepsilon>0$.

Observe that for $\tau>0$, if we put $c_n=n^{\tau}$, $b_n=n$ and $\lambda>1+\tau$, we obtain the sufficiency part of Theorem 1 of Katz [7].

We prove

THEOREM 2. *Assume that* $\liminf_{n\to\infty} c_n>0$. *If, for some* $\lambda>0$,

$$(2.8)\qquad \limsup_{n\to\infty}\frac{b_{n+1}^{\lambda}(b_{n+1}-b_n)}{c_n b_n}<\infty$$

and series (1.6) *converges for some* $\varepsilon>0$, *then* $\mathrm{E}|X|^{1+\lambda}<\infty$ *and* $|\mathrm{E}X|<\varepsilon$.

Proof. Using the inequality (see Feller [4], p. 149)

$$(2.9)\qquad P\{|X_1+\ldots+X_n|\geqslant t\}\geqslant \tfrac{1}{2}(1-e^{-n[1-F(t)+F(-t)]})$$

we infer from the convergence of series (1.6) that

$$(2.10)\qquad \sum_{n=1}^{\infty} c_n(1-e^{-b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]})<\infty.$$

Since $b_n\uparrow\infty$ and c_n's are bounded away from 0 for n large enough, we have

$$(2.11)\qquad \lim_{n\to\infty} b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]=0$$

and hence

$$(2.12)\qquad \sum_{n=1}^{\infty} c_n b_n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]<\infty.$$

Again (see Feller [4], p. 151), we have $\mathrm{E}|X|^{1+\lambda} < \infty$ iff

$$\int_0^\infty x^\lambda [1-F(x)+F(-x)]\,dx < \infty. \tag{2.13}$$

Also, from (2.8), it follows that, for some constant M, we have

$$b_{n+1}^\lambda (b_{n+1}-b_n) \leqslant M c_n b_n, \quad n = 1, 2, \ldots \tag{2.14}$$

Since the sequence $\{b_n\}$ is strictly increasing, while $1-F(t)+F(-t)$ is nonincreasing, we bound the integral in (2.13) as follows:

$$\begin{aligned} &\int_0^\infty x^\lambda [1-F(x)+F(-x)]\,dx \\ &\leqslant \sum_{n=1}^\infty (\varepsilon b_{n+1})^\lambda [1-F(\varepsilon b_n)+F(-\varepsilon b_n)](b_{n+1}-b_n)+(\varepsilon b_1)^{1+\lambda} \\ &\leqslant M\varepsilon^\lambda \sum_{n=1}^\infty c_n b_n [1-F(\varepsilon b_n)+F(-\varepsilon b_n)]+(\varepsilon b_1)^{1+\lambda} \end{aligned} \tag{2.15}$$

The fact that the last sum is finite in view of (2.12), implies that $\mathrm{E}|X|^{1+\lambda} < \infty$.

Let $\mu = \mathrm{E}X$. In the case $|\mu| > \varepsilon$, we can find an interval of the form $(\mu-\delta, \mu+\delta) \subset (-\varepsilon, \varepsilon)^c$ for some $\delta > 0$, such that by the weak law of large numbers we have

$$1 = \lim_{n\to\infty} P\left\{\left|\frac{S_{b_n}}{b_n}-\mu\right| < \delta\right\} \leqslant \lim_{n\to\infty} P\left\{\left|\frac{S_{b_n}}{b_n}\right| \geqslant \varepsilon\right\}. \tag{2.16}$$

This means that series (1.6) cannot converge, since $\liminf_{n\to\infty} c_n > 0$, leading thereby to a contradiction. The argument in the case with $|\mu| = \varepsilon$ being similar, proves that we must have $|\mathrm{E}X| < \varepsilon$.

For the next theorem we shall use the following lemma (see Feller [4], p. 277):

LEMMA 2. *Suppose that $\lambda_{n+1}/\lambda_n \to 1$ and $a_n \to \infty$ as $n \to \infty$. If U is a monotone function such that*

$$\lim_{n\to\infty} [\lambda_n U(a_n x)] = \chi(x) \leqslant \infty \tag{2.17}$$

exists on a dense set and χ is finite and positive in some interval, then U varies regularly and $\chi(x) = cx^\varrho$ for some $-\infty < \varrho < \infty$.

We now prove

THEOREM 3. *Let $b_n/b_{n+1} \to 1$. Assume that for some $\lambda > 0$*

(2.18) $$\lim_{x\to\infty} x^{1+\lambda}[1-F(x)+F(-x)]$$

exists and is positive, say equal to c. Then the convergence of series (1.6) *for some* $\varepsilon > 0$ *implies* (2.7).

Proof. As in the proof of Theorem 2, convergence of (1.6) implies (2.12). Let us write the series in (2.12) as

(2.19)
$$\sum c_n b_n [1-F(\varepsilon b_n)+F(-\varepsilon b_n)] = \sum (c_n b_n^{-\lambda}) \{b_n^{1+\lambda}[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]\}.$$

We now apply Lemma 2 with $\lambda_n = b_n^{1+\lambda}$, $a_n = b_n$, $U(t) = 1-F(t)+F(-t)$ and $x = \varepsilon$. As a result, $\lim_{n\to\infty} \lambda_n U(a_n x)$ becomes $\lim_{n\to\infty} b_n^{1+\lambda}[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]$, which exists and is positive in view of the assumption of the theorem. Consequently, the latter limit equals $c\varepsilon^{\varrho}$ for some ϱ. In fact, replacing x by εx in (2.18) we infer that $\varrho = -(1+\lambda)$. From the convergence of (2.19) it follows now that $\sum c_n b_n^{-\lambda} < \infty$, as asserted.

As an example, consider the case when X has the central t-distribution with 2 degrees of freedom, so that $EX^2 = \infty$ and $E|X| < \infty$. Here the limit (2.18) exists with $\lambda = 1$ and $c = 1/2$, so that Theorem 2 applies.

Note that since the sequence $\{b_n\}$ is strictly increasing, condition (2.8) may be written as

(2.20) $$\liminf_{n\to\infty} \frac{c_n b_n^{-\lambda}}{(b_{n+1}/b_n)^{\lambda}\left(\frac{b_{n+1}}{b_n}-1\right)} > 0.$$

Now, if (2.7) holds, then $c_n b_n^{-\lambda} \to 0$, so that condition (2.20) (and hence (2.8)) may hold only if $b_{n+1}/b_n \to 1$.

Let us also note that the existence of the positive limit (2.18) implies $E|X|^{1+\lambda} = \infty$, although $E|X|^{1+\sigma} < \infty$, for all $0 < \sigma < \lambda$. Conversely, if

(2.21) $$\sigma_0 = \sup\{\sigma\colon \int_0^\infty x^{\sigma}[1-F(x)+F(-x)]\,dx < \infty\}$$

and

(2.22) $$\int_0^\infty x^{\sigma_0}[1-F(x)+F(-x)]\,dx = \infty,$$

then

(2.23) $$\lim_{x\to\infty} x^{1+\sigma}[1-F(x)+F(-x)] = 0$$

for all $\sigma < \sigma_0$. Here we cannot say that the limit (2.23) is positive or 0 in the case with $\sigma = \sigma_0$.

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