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CONTINUITY OF NON-COMMUTATIVE STOCHASTIC PROCESSES

BY

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Abstract. The paper contains the proof of a theorem on the continuity of a "stochastic process" taking its values in an algebra of operators measurable in Nelson's sense. If the algebra considered is abelian, the theorem becomes the classical Kolmogoroff theorem on the continuity of trajectories of a stochastic process.

Introduction. We present a generalization of the classical theorem of Kolmogoroff on the continuity of stochastic processes. It may be thought of as an improved version of paper [3].

Let $\mathscr{A}$ be a von Neumann algebra with a normal faithful semifinite trace τ and $\tilde{\mathscr{A}}$ the algebra of measurable operators in Nelson's sense. Recall that $\tilde{\mathscr{A}}$ consists of closed densely defined operators x affiliated with $\mathscr{A}$ such that $\tau\left(e_{|x|}^{\perp}([0, \lambda])\right) < +\infty$ for some $\lambda > 0$, where $e_z(\cdot)$ is the spectral measure of z (see [5]).

As an analogue of a stochastic process we consider a mapping $X\colon [a, b] \to \tilde{\mathscr{A}}$. There are various notions of convergence in $\tilde{\mathscr{A}}$ and, correspondingly, various notions of the "continuity" of X. It would be advantageous to choose those which guarantee the continuity of trajectories in the commutative case. The most appropriate seems to be the so-called *Segal convergence* as defined in [4].

We say that $x_n \to x$ in *Segal's sense* if, for each $\varepsilon > 0$, there is a projection p in $\mathscr{A}$ with $\tau(p^{\perp}) < \varepsilon$, such that $(x_n - x)p \in \mathscr{A}$ and $\|(x_n - x)p\| \to 0$.

This mode of convergence is stronger than other types of convergence: almost uniform, nearly everywhere or metrically nearly everywhere, all of them being equivalent when $\mathscr{A}$ is finite and coincide with almost everywhere convergence in the classical case, i.e. for $\mathscr{A} = L^\infty(\Omega, F, P)$ and $\tau(x) = \int x dP$ (see [1] and [6]).

Let us now introduce the announced notion of continuity. We say that X is *uniformly continuous in Segal's sense* if, for each $\varepsilon > 0$, there are a projection p in $\mathscr{A}$ with $\tau(p^\perp) < \varepsilon$ and some $\delta > 0$, such that

$$\big(X(s)-X(t)\big)p \in \mathscr{A} \quad \text{and} \quad \big\|\big(X(s)-X(t)\big)p\big\| \leqslant \varepsilon$$

for $|s-t| < \delta$.

In a similar way we say that X *satisfies the Hölder condition in Segal's sense with exponent* $\gamma > 0$ *and constant* $C > 0$ if, for each $\varepsilon > 0$, there are a projection p in $\mathscr{A}$ with $\tau(p^\perp) < \varepsilon$ and some $\delta > 0$, such that

$$\big(X(s)-X(t)\big)p \in \mathscr{A} \quad \text{and} \quad \big\|\big(X(s)-X(t)\big)p\big\| \leqslant C|s-t|^\gamma$$

for $|s-t| < \delta$.

An important role in our consideration, as well as in the construction of the algebra $\tilde{\mathscr{A}}$, is played by the notion of the *measure topology*, i.e. the topology given by the fundamental system of neighbourhoods of 0 of the form

$$N(\varepsilon, \delta) = \{x \in \tilde{\mathscr{A}}\colon \text{ there exists } p \in \operatorname{Proj}\mathscr{A} \text{ such that } xp \in \mathscr{A},$$
$$\|xp\| \leqslant \varepsilon \text{ and } \tau(p^\perp) \leqslant \delta\}.$$

Accordingly, $x_n \to x$ *in measure* if, for each $\varepsilon > 0$, there is a sequence $\{p_n\}$ of projections in $\mathscr{A}$ such that

$$\tau(p_n^\perp) \to 0 \quad \text{and} \quad \|(x_n - x)p_n\| < \varepsilon.$$

Let us note that $\tilde{\mathscr{A}}$ endowed with the measure topology is a topological $*$-algebra (see [5], Th. 1).

THEOREM. *Let $\mathscr{A}$ be a semifinite von Neumann algebra with a normal faithful semifinite trace τ and let X be a mapping of an interval $[a, b]$ into the algebra $\tilde{\mathscr{A}}$. Assume that there exist strictly positive constants M, α, β such that*

$$(*) \qquad \tau\big(|X(s)-X(t)|^\beta\big) \leqslant M|s-t|^{1+\alpha}, \qquad s, t \in [a, b].$$

Then, for each $\gamma < \alpha/\beta$, X satisfies the Hölder condition with exponent γ.

Proof. Obviously, we can take $[a, b] = [0, 1]$. Fix an arbitrary γ, $0 < \gamma < \alpha/\beta$. Let η be such that $0 < \eta < 1$ and $(1-\eta)(\alpha+1-\beta\gamma) > 1+\eta$. Put

$$p_n = \bigwedge_{k=n}^{\infty} \bigwedge_{\substack{0 \leqslant i \leqslant j < 2^k \\ j-i \leqslant 2^{k\eta}}} e_{ijk},$$

where

$$e_{ijk} = e_{|X(j\cdot 2^{-k}) - X(i\cdot 2^{-k})|}\big([0, ((j-i)2^{-k})^\gamma]\big).$$

By $(*)$,

$$\tau(e_{ijk}^{\perp}) \leqslant \big((j-i)\,2^{-k}\big)^{-\beta\gamma}\,\tau\big(|X(j2^{-k})-X(i2^{-k})|\big) \leqslant M\big((j-i)\,2^{-k}\big)^{1+\alpha-\beta\gamma}.$$

Setting $\mu = (1-\eta)(\alpha+1-\beta\gamma)-(1+\eta) > 0$, we get

$$\tau(p_n^{\perp}) \leqslant \sum_{k=n}^{\infty} \sum_{\substack{0\leqslant i\leqslant j<2^k \\ j-i\leqslant 2^{k\eta}}} M\,2^{(k\eta-k)(1+\alpha-\beta\gamma)} = M\sum_{k=n}^{\infty} \sum_{\substack{0\leqslant i\leqslant j<2^k \\ j-i\leqslant 2^{k\eta}}} 2^{-k(1+\eta+\mu)}$$

$$\leqslant M\sum_{k=n}^{\infty} 2^k(2^{k\eta}+1)\,2^{-k(1+\eta+\mu)} = M\sum_{k=n}^{\infty} 2^{-k\mu}(1+2^{-k\eta})$$

$$\leqslant M\,2^{1+\mu-n\mu}/(2^n-1) \to 0 \quad \text{as } n\to\infty.$$

Thus, for $k \geqslant n \geqslant \eta^{-1}$, $j-i \leqslant 2^{k\eta}$ and $0 \leqslant i \leqslant j < 2^k$, we have

$$(1)\quad \big\|\big(X(j\cdot 2^{-k})-X(i\cdot 2^{-k})\big)p_n\big\| \leqslant \big\|\big(X(j\cdot 2^{-k})-X(i\cdot 2^{-k})\big)e_{ijk}\big\| \leqslant \big((j-i)\,2^{-k}\big)^{\gamma}.$$

Let D denote the set of the dyadic numbers from $[0, 1[$. For $n > \eta^{-1}$, put $\delta_n = 2^{n(\eta-1)}$. We assert that if $s, t \in D$, $s < t$ and $t-s < \delta_n$, then

$$\big\|\big(X(s)-X(t)\big)p_n\big\| \leqslant C\,|s-t|^{\gamma} \quad \text{for some } C = C(\gamma).$$

To this end, take $k \geqslant n$ satisfying

$$2^{(k+1)(\eta-1)} \leqslant t-s < 2^{k(\eta-1)}$$

and represent the numbers s and t in the form

$$s = i\cdot 2^{-k}-2^{-h_1}-\ldots-2^{-h_u}, \quad k < h_1 < \ldots < h_u,$$
$$t = j\cdot 2^{-k}-2^{-l_1}+\ldots+2^{-l_v}, \quad k < l_1 < \ldots < l_v.$$

By (1), for $w = 1, \ldots, u$,

$$\big\|\big(X(i\cdot 2^{-k}-2^{-h_1}-\ldots-2^{-h_{w-1}})-X(i\cdot 2^{-k}-2^{-h_1}-\ldots-2^{-h_w})\big)p_n\big\| \leqslant 2^{-\gamma h_w}.$$

Therefore $\big(X(i\cdot 2^{-k})-X(s)\big)p_n \in \mathscr{A}$ and

$$\big\|\big(X(i\cdot 2^{-k})-X(s)\big)p_n\big\| \leqslant \sum_{w=1}^{u} 2^{-\gamma h_w} \leqslant \sum_{w=k+1}^{\infty} 2^{-\gamma w} = C_1\,2^{-\gamma k},$$

where $C_1 = (2^{\gamma}-1)^{-1}$. Similarly,

$$\big\|\big(X(t)-X(j\cdot 2^{-k})\big)p_n\big\| \leqslant C_1\,2^{-\gamma k}.$$

Taking into account the last two inequalities and (1), we obtain

$$\big\|\big(X(s)-X(t)\big)p_n\big\| \leqslant |s-t|^{\gamma}+2C_1\,2^{-\gamma k} \leqslant C\,|s-t|^{\gamma}$$

with $C = 2C_1+1$ (observe that $2^{-k} \leqslant 2^{(k+1)(\eta-1)} \leqslant |s-t|$ for $k \geqslant \eta^{-1}$).

That proves our assertion and, together with $\tau(p_n^{\perp}) \to 0$, gives the following condition:

(2) for each $\varepsilon > 0$, there are some $p \in \operatorname{Proj} \mathscr{A}$ with $\tau(p^{\perp}) < \varepsilon$ and $\delta > 0$, such that

$$\|(X(s)-X(t))p\| \leqslant C|s-t|^{\gamma}$$

for $s, t \in D$, $|s-t| < \delta$.

Let now $D = \{t_n\}$. For $\varepsilon > 0$ we choose a projection p_0 with $\tau(p_0^{\perp}) < \varepsilon/2$ so that (2) holds. For every n we take $q_n \in \operatorname{Proj} \mathscr{A}$ satisfying $X(t_n) q_n \in \mathscr{A}$ and $\tau(q_n^{\perp}) < \varepsilon/2^{n+1}$ (see [5], Th. 2 (ii)). Put

$$p = p_0 \wedge \bigwedge_{n=1}^{\infty} q_n.$$

Then $X(t_n)p \in \mathscr{A}$, $\tau(p^{\perp}) < \varepsilon$ and

$$\|(X(s)-X(t))p\| \leqslant \|(X(s)-X(t))p_0\|, \qquad s, t \in D.$$

Therefore we may and shall require that $X(s)p$, $X(t)p \in \mathscr{A}$ in condition (2).

Fix now an arbitrary $\varepsilon > 0$ and take p and δ as in (2). Let $s, t \in [0, 1]$, $s < t$, $t-s < \delta$. Choose sequences $s_n \downarrow s$, $t_n \uparrow t$, $s_n, t_n \in D$. By (2), the sequences $\{X(s_n)p\}$ and $\{X(t_n)p\}$ are Cauchy in norm, hence $X(s_n)p \to a$, $X(t_n)p \to b$ in norm for some $a, b \in \mathscr{A}$. Observe that condition $(*)$ implies $X(s_n) \to X(s)$, $X(t_n) \to X(t)$ in measure. Using the continuity of algebraic operations in measure topology, we get

$$X(s_n)p \to X(s)p, \qquad X(t_n)p \to X(t)p \text{ in measure,}$$

so that

$$X(s_n)p \to X(s)p, \qquad X(t_n)p \to X(t)p \text{ in norm.}$$

Finally,

$$\|(X(s)-X(t))p\| \leqslant \|X(s)p - X(s_n)p\| + \|(X(s_n)-X(t_n))p\| + \|X(t_n)p - X(t)p\|$$

and, passing to the limit, we obtain

$$\|(X(s)-X(t))p\| \leqslant \limsup \|(X(s_n)-X(t_n))p\| \leqslant C \limsup |s_n - t_n|^{\gamma} = C|s-t|^{\gamma},$$

which concludes our proof.

Let X be such as in the theorem. We list a number of consequences of our result.

COROLLARY 1. *X is uniformly continuous in Segal's sense.*

COROLLARY 2. *If $\mathscr{A}$ is a finite direct sum of type I factors, then $\tilde{\mathscr{A}} = \mathscr{A}$*

and X satisfies the Hölder condition in norm; consequently, X is uniformly continuous in norm topology.

Proof. For the equality $\tilde{\mathscr{A}} = \mathscr{A}$, see [2]. To obtain the second part of the assertion, in the Hölder condition just proved let us take $\varepsilon > 0$ suffciently small to get the implication:

$$\text{if } \tau(p^{\perp}) < \varepsilon, \text{ then } p = 1;$$

this is always possible since $\tau(\operatorname{Proj} \mathscr{A})$ is a discrete set of positive numbers.

COROLLARY 3. *Suppose that X takes its values in $\mathscr{A}$, is bounded in norm on $[a, b]$ and satisfies condition $(*)$ with $\alpha > \beta$. Then X is globally Hölder in norm for each $1 \leqslant \gamma < \alpha/\beta$, i.e. there is some $C > 0$ such that*

$$\|(X(s) - X(t))\| \leqslant C|s-t|^{\gamma}, \qquad s, t \in [a, b].$$

In particular, X is uniformly continuous in norm.

Proof. Choose a sequence (ε_n) of strictly positive numbers satisfying $\sum \varepsilon_n < \infty$ and corresponding sequences (p_n) and (δ_n) as in the Hölder condition. Put $r_n = \bigwedge_{k=n}^{\infty} p_k$. Then $r_n \uparrow 1$ and

$$\|(X(s) - X(t)) r_n\| \leqslant C|s-t|^{\gamma} \quad \text{ for } |s-t| < \delta_n.$$

It is easily seen that this inequality is valid for arbitrary $s, t \in [a, b]$. Indeed, it suffices to take a partition $s = s_0 < s_1 < \ldots < s_l = t$ of diameter less than δ_n and observe that

$$\sum_{k=1}^{l} (s_k - s_{k-1})^{\gamma} \leqslant |s-t|^{\gamma}.$$

Now, for $\xi \in \bigcup_{n=1}^{\infty} r_n H$ (where H is a representation space of $\mathscr{A}$), $\|\xi\| \leqslant 1$, we have

$$\|(X(s) - X(t)) \xi\| \leqslant C|s-t|^{\gamma}.$$

Since $\bigcup_{n=1}^{\infty} r_n H$ is a dense subspace of H and X is uniformly bounded, the result follows.

COROLLARY 4. *Suppose that X takes its values in $\mathscr{A}$ and it is bounded in norm on $[a, b]$. Then X is uniformly continuous in $*$-ultrastrong topology.*

Proof. Taking r_n, δ_n as in the course of the proof of the preceeding corollary, we obtain the inequality $\|(X(s) - X(t)) \xi\| \leqslant C|s-t|^{\gamma}$ for $\xi \in r_n H$, $\|\xi\| \leqslant 1$ and $|s-t| < \delta_n$. That gives the uniform ultrastrong continuity of X on $\bigcup_{n=1}^{\infty} r_n H$. Taking into account the uniform boundedness of X, we get the uniform ultrastrong continuity of X on H. The $*$-case is treated similarly.

Remark. In the commutative case our theorem reduces to the classical Kolmogoroff theorem on the continuity of trajectories of a stochastic process. The method for such a reduction is standard (cf. [6], Ex. 1.1).

REFERENCES

[1] C. J. K. Batty, *The strong law of large numbers for states and traces of a W^*-algebra*, Z. Wahrsch. verw. Gebiete 48 (1979), p. 177-191.

[2] S. K. Berberian, *Von Neumann algebras admitting no unbounded measurable operator*, Math. Japon. 25 (1980), p. 61-64.

[3] R. Jajte, *On curves in von Neumann algebras*, Bull. Acad. Polon. Sci., Ser. Sci. Math. 30 (1982), p. 251–254.

[4] E. C. Lance, *Martingale convergence in von Neumann algebras*, Math. Proc. Cambridge Philos. Soc. 84 (1978), p. 47-56.

[5] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. 15 (1974), p. 103-116.

[6] I. E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. 57 (1953), p. 401-457.

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