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CHARACTERIZATION OF THE MULTIVARIATE MARSHALL–OLKIN EXPONENTIAL DISTRIBUTION

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Abstract. The paper is concerned with characterizations of the Marshall–Olkin exponential distribution based on an integrated lack of memory equation.

Suppose that $X_1, X_2, \ldots, X_n$ are nonnegative random variables and $G(x_1, x_2, \ldots, x_n)$, $x_k \geqslant 0$ $(k = 1, 2, \ldots, n)$, is their survival distribution function (s.d.f.), i.e.

$$G(x_1, x_2, \ldots, x_n) \equiv \mathrm{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n).$$

It is well known [1, 4] that, if G has the lack of memory property (LMP) of the type

$$G(x_1+t, x_2+t, \ldots, x_n+t) = G(t, t, \ldots, t)\, G(x_1, x_2, \ldots, x_n) \tag{1}$$

and if all marginal distributions of G satisfy equalities of type (1), then G is an s.d.f. of the multivariate Marshall–Olkin (M–O) distribution. The requirement that the marginal distributions of G should satisfy equalities analogous to (1) can be replaced by the following: all marginal distributions of G are M–O distributed.

Denote by e and x the vectors $e = (1, 1, \ldots, 1)$ and $x = (x_1, x_2, \ldots, x_n)$, respectively. Now equation (1) can be written briefly as

$$G(x+te) = G(te)\, G(x). \tag{2}$$

If an s.d.f. satisfies equation (2), we say that *it has the LMP*. We give here a weaker definition of the LMP.

Let $a_i = 0$ or 1, $i = 1, 2, \ldots, n$, and $a = (a_1, a_2 \ldots, a_n)$. Denote by E

the set

$$E = \{a\colon \text{ only one } a_i \text{ is } 0, \text{ the other } (n-1) \quad a_i \text{ are } 1\}.$$

Definition. The s.d.f. $G(x)$ has a *weak LMP* (WLMP) if

$$G(te+\boldsymbol{x}\circ\boldsymbol{a}) = G(te)\,G(\boldsymbol{x}\circ\boldsymbol{a}) \tag{3}$$

for all $t \geqslant 0$ and $\boldsymbol{a} \in E$, where $\boldsymbol{x}\circ\boldsymbol{a} \equiv (x_1 a_1, x_2 a_2, \ldots, x_n a_n)$.

The difference between equations (3) and (2) is that in (3) at least one of the coordinates of $\boldsymbol{x}\circ\boldsymbol{a}$ is zero, whereas in (2) all coordinates of $\boldsymbol{x}$ can be different from zero. Evidently, (3) follows from (2). The inverse statement that (2) follows from (3) is contained in the following lemma and requires an additional condition for the function G.

LEMMA. *If an s.d.f. $G(x)$ has a WLMP and $G(te) = e^{-\lambda t}$, $\lambda > 0$, then G also has an LMP.*

Proof. Let $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, $y_k \geqslant 0$, and $u \geqslant 0$ be arbitrary. Suppose that y_i is the smallest coordinate of $\boldsymbol{y}$, i.e. $y_i = \min(y_1, y_2, \ldots, y_n)$. Then, for a fixed $\boldsymbol{y}$, $G(ue+\boldsymbol{y})$ can be written as

$$G(ue+\boldsymbol{y}) = G(te+\boldsymbol{x}\circ\boldsymbol{e}), \tag{4}$$

where we write $t = u+y_i$ for a fixed i, and the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ has the following coordinates:

$$x_k = \begin{cases} y_k - y_i & \text{for } k \neq i, \\ 0 & \text{for } k = i. \end{cases}$$

As $G(te+\boldsymbol{x}\circ\boldsymbol{e}) = G(te+\boldsymbol{x}\circ\boldsymbol{a})$, where $\boldsymbol{a} \in E$ and the i-th coordinate of $\boldsymbol{a}$ is zero, equation (4) becomes $G(ue+\boldsymbol{y}) = G(te)\,G(\boldsymbol{x}\circ\boldsymbol{a})$.This equation, using (3), gives $G(ue+\boldsymbol{y}) = G(te)\,G(\boldsymbol{x}\circ\boldsymbol{a})$ or, after taking into account that $G(te) = e^{-\lambda t}$,

$$G(ue+\boldsymbol{y}) = G(ue)\,G(y_i e)\,G(\boldsymbol{x}\circ\boldsymbol{a}). \tag{5}$$

The last two factors on the right-hand side of (5), again according to (3), give $G(y_i e)\,G(\boldsymbol{x}\circ\boldsymbol{a}) = G(y_i e+\boldsymbol{x}\circ\boldsymbol{a})$. This way equation (5) becomes

$$G(ue+\boldsymbol{y}) = G(ue)\,G(y_i e+\boldsymbol{x}\circ\boldsymbol{a}). \tag{6}$$

In fact, since $y_i e+\boldsymbol{x}\circ\boldsymbol{a} = \boldsymbol{y}\circ\boldsymbol{e} = \boldsymbol{y}$, equation (6) is $G(ue+\boldsymbol{y}) = G(ue)\,G(\boldsymbol{y})$, which is an LMP according to (2).

THEOREM 1. *If an s.d.f. $G(\boldsymbol{x})$ has a WLMP and $G(te)$ is an exponential function of t and all marginal distributions of $G(\boldsymbol{x})$ are* M–O *distributed, then G is* M–O *distributed.*

Proof. With respect to the Lemma, G has an LMP and, because the marginal distributions of G are M–O distributed, the distribution of G is also an M–O one.

The next characterizations of the multivariate M–O distribution are based on the integrated by t or by $\boldsymbol{x}$ equation (2). In the 1-dimensional case equation (2), integrated with respect to the Borel measure $\nu(t)$ on $(0, \infty)$, gives

$$(7)\quad \int_0^\infty G(t+x)\mu(dt) = G(x) \quad \text{for every } x \geqslant 0, \text{ where } \mu = \frac{\nu(t)}{\int_0^\infty G(t)\mu(dt)}.$$

Naturally, we suppose that

$$\int_0^\infty G(t)\mu(dt) < \infty.$$

Lau and Rao [2] proved recently that, if $\mu(t)$ has an infinite support, the unique solution of equation (7) is $G(t) = e^{-\alpha t}$, where $\alpha \geqslant 0$ is synonymously determined from

$$\int_0^\infty e^{-\alpha t}\mu(dt) = 1.$$

An analogical statement (for the case of a 2-dimentional function G) is proved in [5]. Namely, if $G(x_1, x_2)$ is an s.d.f. with exponential marginal distributions, if μ is a Borel measure on $(0, \infty)$ with infinite support, and if

$$(8)\quad \int_0^\infty G(t+x_1, t+x_2)\mu(dt) = G(x_1, x_2),$$

then G is M–O distributed.

We give here a general proof of this statement in the n-dimensional case.

THEOREM 2. *If $G(x_1, x_2, \ldots, x_n)$, $x_k \geqslant 0$, $k = 1, 2, \ldots, n$, is an s.d.f. whose marginal distributions are all of the* M–O *type and G satisfies the equation*

$$(9)\quad \int_0^\infty G(t+x_1, t+x_2, \ldots, t+x_n)\mu(dt) = G(x_1, x_2, \ldots, x_n)$$

for some Borel measure $\mu(t)$ on $(0, \infty)$ with infinite support, then G is an n-dimensional M–O *distribution.*

Proof. First write equation (9) in a more compact form:

$$(10)\quad \int_0^\infty G(t\boldsymbol{e}+\boldsymbol{x})\mu(dt) = G(\boldsymbol{x}).$$

For $\boldsymbol{x} = \boldsymbol{O}$ it follows from (10) that

$$\int_0^\infty G(t\boldsymbol{e})\mu(t) = G(\boldsymbol{O}) = 1,$$

while, for $x = xe$ it follows that the function $G_1(x) = G(xe)$ satisfies equation (10) in the 1-dimensional case. According to the result of Lau and Rao [2], $G_1(x) = e^{-\alpha x}$, i.e.

$$G(xe) = e^{-\alpha x}.$$

Suppose that $x = (x_1, x_2, \ldots, x_n)$ and let us fix all $n-1$ coordinates $x_2, x_3, \ldots, x_n$ of x. Put

$$G_{n-1}(u) = \frac{G(u, u+x_2, u+x_3, \ldots, u+x_n)}{G(0, x_2, x_3, \ldots, x_n)} = \frac{G(ue+x \circ a)}{G(x \circ a)}, \tag{11}$$

where $a \in E$ with $a_1 = 0$.

Since G satisfies equation (10), it is easy to check that $G_{n-1}(u)$ satisfies equation

$$\int_0^\infty G_{n-1}(u+t)\,\mu(dt) = G_{n-1}(u) \tag{12}$$

and $G_{n-1}(0) = 1$. Thus $G_{n-1}(t) = \exp(-\beta t)$, where $\beta = \beta(x_2, x_3, \ldots, x_n)$. However, we have seen that (12) has a solution $G_1(t) = \exp(-\alpha t)$ and, because the solution is unique, we have

$$\beta(x_2, x_3, \ldots, x_n) = \alpha, \qquad G_{n-1}(t) = e^{-\alpha t}. \tag{13}$$

By (11), from (13) we get

$$G(te+x \circ a) = e^{-\alpha t} G(x \circ a) = G_1(t)\, G(x \circ a) = G(te)\, G(x \circ a),$$

i.e. equation (3) is fulfilled for all $a \in E$ with the first coordinate $a_1 = 0$. We can analogically establish that equation (3) is satisfied as well for $a \in E$ with $a_2 = 0$ and, in general, for every $a \in E$. Therefore the function G has the WLMP. Besides, we have seen that $G(te)$ is exponential with respect to t. Thus, according to Theorem 1, $G(x)$ is an n-dimensional M–O distribution.

The next assertion characterizes also the n-dimensional M–O, but this time through an integrated equation obtained from (2) after an integration with respect to x. Unfortunately, the functions of the class G, in which we look for a solution, are not all s.d.f. It is necessary to consider a narrower class. We restrict ourself to the multivariate distributions of the class IFR. More precisely, we shall use one of the possible definitions (see [3]) about the distributions with a monotone failure-rate (IFR — increasing failure-rate).

Definition. An s.d.f. $G(x)$, $x = (x_1, x_2, \ldots, x_n)$, $x_i \geqslant 0$, *belongs to the class* IFR if

$$R(x) = \frac{G(te+x)}{G(x)} \tag{14}$$

is decreasing (in a broad sense) with respect to x for all $t \geqslant 0$, and $G(x) > 0$ for every x.

THEOREM 3. *Let $G(x)$ belong to the class IFR and $\mu(t)$ be a Borel measure on $R_n^+ = \{t\colon t_k \geqslant 0,\ k = 1, 2, \ldots, n\}$ such that*

$$\int_{R_n^+(a)} d\mu(t) \neq 0 \tag{15}$$

for every $a = (a_1, a_2, \ldots, a_n)$, where $R_n^+(a) = \{s\colon t_k \geqslant a_k,\ 1 \leqslant k \leqslant n\}$. If G satisfies equation

$$\int_{R_n^+} G(te+x)\, d\mu(w) = G(te), \qquad t \geqslant 0, \tag{16}$$

and if all marginal distributions of G are M–O *distributed, then $G(x)$ is* M–O *distributed.*

Proof. For $t = 0$ it follows from (16) that

$$\int_{R_n^+(a)} G(x)\, d\mu(x) = 1, \tag{17}$$

whence

$$I(a) = \int_{\bar{R}_n^+(a)} G(x)\, d\mu(x) = 1 - \int_{R_n^+(a)} G(x)\, d\mu(x). \tag{18}$$

In view of (15), $I(a) \neq 0$ for every a. Equation (16) becomes now

$$\int_{\bar{R}_n^+(a)} R(x) G(x)\, d\mu(x) + \int_{R_n^+(a)} R(x) G(x)\, d\mu(x) = G(te). \tag{19}$$

We replace the integrand $R(x)$ in the first integral of the left of (19) with $R(O) \geqslant R(x)$ for $x \in \bar{R}_n^+(a)$ and in the second – with $R(a) \geqslant R(x)$ for $x \in R_n^+(a)$. This way we obtain

$$R(O) \int_{\bar{R}_n^+(a)} G(x)\, d\mu(x) + R(a) \int_{R_n^+(a)} G(x)\, d\mu(x) \geqslant G(te). \tag{20}$$

From (20), using (18), we have

$$R(O) I(a) + R(a)[1 - I(a)] \geqslant G(te) = R(O),$$

from where $R(a) \geqslant R(O)$ for arbitrary a. But $R(x)$ is decreasing and, therefore, $R(a) = R(O)$. The last equation is in fact

$$G(te + a) = G(te) G(a),$$

which indicates that G has the LMP. Since (by assumption) all marginal distributions of G are M–O distributed, $G(x)$ is also M–O distributed.

The random variables $X_1, X_2, \ldots, X_n$ have a joint "exponential minima" distribution if

$$P(\min_{i \in I} X_i > t) = \exp(-\theta_i t), \qquad \theta_i > 0,$$

for every non-empty subset $I \subset \{1, 2, \ldots, n\}$.

THEOREM 4. *Let, for every fixed* $x \geqslant 0$, *the quotient*

$$\lambda(te) = \frac{G(te+xe)}{G(te)} \tag{21}$$

do not increase with respect to $t \geqslant 0$, *and all marginal distributions of* G *be of the "exponential minima" type. If for some Borel measure* μ *on* $(0, \infty)$ *with infinite support we have*

$$\int_0^\infty G(xe+te)\,\mu(dt) = G(xe) \tag{22}$$

for every $x \geqslant 0$, *then* $G(x)$ *is "exponential minima" distributed.*

Proof. Analogically to the proof of Theorem 3 we use here the fact that $\lambda(te)$ does not increase in t and obtain $\lambda(\delta e) = \lambda(0)$ for every $\delta \geqslant 0$. The last equation is

$$G(\delta e+xe) = G(\delta e)\,G(xe). \tag{23}$$

Using now Theorem 5.4.2 of Galamgos and Kotz [1], it follows from (23) and from the assumption about the "exponential minima" type of the marginal distributions that the joint distribution G is also of the type "exponential minima".

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