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STABLE AND MULTIPLY SELF-DECOMPOSABLE POINT PROCESSES

BY

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Abstract. Using the thinning operator there are defined stable and multiply self-decomposable point processes. A representation for generating functional of such processes is given. Moreover, stable and Poisson processes are characterized as solutions of some differential equations.

0. Introduction, notation and preliminaries. In general distribution theory there exists a dual relationship between discrete probabilities and continuous ones. In particular, discrete stable, self-decomposable (self-dec.) and n-times ($n = 1, 2, \ldots$) self-dec. distributions ([9] and [1]) share the basic properties with their continuous counterparts (cf. [6], p. 322, and [16]). Thus, it is natural to study discrete α-times $(0 < \alpha \leqslant \infty)$ self-dec. distributions as analogues of continuous α-times self-dec. ones. The latter class of distributions was introduced by the second author in several ways (cf. [11-14]).

In this paper we study stable and α-times $(0 < \alpha \leqslant \infty)$ seld-dec. point processes (p. proc.). Since the distributions on $Z_+ = \{0, 1, 2, \ldots\}$ can be interpreted as p. proc. on one-point space, our results are multi-dimensional extensions of that of Steutel and van Harn [9] and Berg and Forst [1] and stand for a multi-dimensional analogy of that in [12]. Moreover, we develop a new *differential method* for the representation of α-times self-dec. p. proc. Namely, we first prove that each α-times self-dec. p. proc. is α-differentiable (see Def. 0) and then obtain the representation (Theorem 2.2) by an inverse operation.

In the sequel we shall use the basic terminology concerning p. proc. and random measures (r.m.) in [4] and [7]. Further, we introduce the following notation:

X – a complete separable metric space; $\mathscr{B}$ – bounded Borel subsets of X; $\mathscr{F}$ – continuous non-negative functions on X with compact support; $\mathscr{H}$ – functions $f\colon X \to [0, 1]$ such that $l-f \in \mathscr{F}$; M, N – random and counting measures on X, respectively, equipped with the vague convergence; $\mathscr{P}(M)$, $\mathscr{P}(N)$ – r.m. and p. proc. on X, respectively, equipped with the weak convergence; $L_P(\cdot)$ – the Laplace transform of $P \in \mathscr{P}(M)$; $G_P(\cdot)$ – the generating functional of $P \in \mathscr{P}(N)$ defined by

$$G_P(f) = L_P(-\log f) \quad (f \in \mathscr{H});$$

$T_c P$ $(c \geqslant 0)$ – the image of $P \in \mathscr{P}(M)$ under the mapping T_c defined by $T_c \mu = c\mu$ $(\mu \in M)$; D_c $(0 < \quad < 1)$ – the thinning operator defined on $\mathscr{P}(N)$ (cf. [7], p. 91, and [4], p. 9); $L_0 := L_0(N)$ – the infinitely divisible (i.d.) p. proc.; $\tilde{P}$ – the canonical measure of an i.d. p. proc. P (cf. [4], p. 39); $A[P, t]$ – the r.m. P^t, where $P \in \mathscr{P}(M)$, $t \geqslant 0$, and the power is taken in the convolution sense (we assume that P is i.d. if t is a non-integer number).

Definition 0.1 (cf. [12]). A p. proc. P on X is said to be *α-times self-dec.* $(0 < \alpha < \infty)$ if for every $0 < c < 1$ there exists an i.d. p. proc. $V_{\alpha,c}$ such that

$$P = \mathop{*}_{k=0}^{\infty} D_{c^k} A\left[P_{\alpha,c}, \binom{\alpha+k-1}{k}\right], \tag{0.1}$$

where $*$ denotes the convolution operation. Further, P is called *conpletely self-dec.* if it is α-times self-dec. for each $\alpha > 0$.

Let $L_\alpha := L_\alpha(N)$ denote the class of all α-times $(0 < \alpha \leqslant \infty)$ self-dec. p. proc. on X.

Remark 0.1. (i) For $\alpha = 1, 2$ and $X = \{x\}$ the concept of α-times self-dec. p. proc. reduces to that of discrete multiply self-dec. probabilities (cf. [1, 16]).

(ii) For $\alpha = 1, 2, \ldots$ Def. 0.1 is equivalent to the following:

$P \in L_\alpha$ iff for every $0 < \quad < 1$ there exist $P_1, \ldots, P_\alpha$ such that $P = D_c P * P_1$, $P_1 = D_c P * P_2, \ldots, P_{\alpha-1} = D_c P_{\alpha-1} * P_\alpha$ (cf. [12]).

Definition 0.2 (cf. [15]). A p. proc. P is said to be *α-differentiable* $(0 < \alpha < \infty)$ if $P \in L_\alpha$ and there exists a limit

$$D^\alpha P := \lim_{t \to 0+} A[P_{\alpha,c}, t^{-\alpha}], \tag{0.2}$$

where $P_{\alpha,c}$ is as in (0.1) and $t = -\log c$.

Definition 0.3. A p. proc. P is said to be *stable* if there exists a p. proc. Q and a sequence $\{c_n\} \subset (0, 1)$ such that $P = \lim D_{c_n} Q^n$.

Remark 0.2. (i) If $X = \{x\}$, then stable p. proc. are exactly discrete stable probabilities as defined by Steutel and van Harn [9].

(ii) $P \in \mathscr{P}(N)$ is stable iff either P is a Poisson p. proc. or

$$\exists_{a \in (0,1)} \forall_{c \in (0,1)} \widetilde{D_c P} = c^a \tilde{P}. \tag{0.3}$$

Remark 0.3. Replacing the operator D_c $(0 < c < 1)$ in Definitions 0.1, 0.2 and 0.3 by T_c we get the corresponding concepts for r.m. on X.

1. Stable point processes. Consider a stable r.m. W on X with canonical measure $[V, \lambda]$, where $\alpha \in M$ and λ is a measure on $M \setminus \{0\}$ (cf. [4], p. 39). By virtue of A. 6.1 [4] and Borel–Cantelli lemma it follows that there exists a strictly positive continuous function h on X such that hW is supported by the subset M_b of M consisting of finite measures. Moreover (cf. [4], Theorem 6.1),

$$\int_{M \setminus \{0\}} [1 - \exp(-\mu h)] \lambda(d\mu) < \infty. \tag{1.1}$$

Further, from [8] it follows that if $W \neq \varepsilon_\mu$ $(\mu \in M)$, then there exists an a, $0 < a < 1$, and a finite Borel measure G on $M \setminus \{0\}$ such that

$$\lambda(A) = \int_{M \setminus \{0\}} \int_0^\infty l_A(t\mu) t^{-a-1} dt (\mu h)^{-a} G(d\mu), \tag{1.2}$$

where A is a Borel subset of $M \setminus \{0\}$.

Conversely, if G is a finite Borel measure on $M \setminus \{0\}$, then (1.2) defines a canonical measure corresponding to a stable r.m. Hence and by Theorem 6.1 [4] we get the following

THEOREM 1.1. *Let W be a stable r.m. on X. Then there exists a measure $v \in M$ such that*

$$-\log L_W(f) = vf \quad (f \in \mathscr{F}) \tag{1.3}$$

or there exists a number a, $0 < a < 1$, and a finite Borel measure m on $M \setminus \{0\}$ such that

$$-\log L_W(f) = \int_{M \setminus \{0\}} (\mu f)^a (\mu h)^{-a} G(d\mu) \quad (f \in \mathscr{F}). \tag{1.4}$$

Conversely, for any v, a, G as mentioned above, formulas (1.3) *and* (1.4) *define some stable r.m. on X.*

Let Q_ϱ denote a *Poisson process* on X with intensity $\varrho \in M$. Given an r.m. W let Q_W denote the *Cox process* directed by W (cf. [5]). It is wasy to verify that if W is i.d. with canonical measures $[v, \lambda]$, then Q_W is i.d. and

$$\tilde{Q}_W = \int_{M \setminus \{0\}} Q_\varrho \lambda(d\varrho). \tag{1.5}$$

Further, from Def. 0.2 and Theorem 8.4 [4] it follows that each stable p. proc. is Coxian. By the obvious relation $D_c Q_W = Q_{T_c W}$ we infer that a Cox process Q_W is stable iff W is stable.

Let Q_W be a non-Poisson stable Cox process. By virtue of (1.2) and (1.5) it follows that

$$\tilde{Q}_W = \int_{M\setminus\{0\}} \int_0^\infty Q_{t\mu} t^{-a-1} dt (\mu t)^{-a} G(d\mu), \tag{1.6}$$

which, together with Theorem 6.1 [4], implies the following

THEOREM 1.2. *Let P be a stable p. proc on X. Then P is a Poisson process or there exists a number a, $0 < a < 1$, a strictly positive continuous function h on X and a finite Borel measure on $M\setminus\{0\}$ such that*

$$-\log G_P(f) = \int_{M\setminus\{0\}} [\mu(1-f)]^a (\mu h)^{-a} G(d\mu) \qquad (f \in \mathscr{K}). \tag{1.7}$$

Conversely, for any a, h, G as mentioned above, formula (1.7) *defines some stable p. proc. on X.*

Remark. The index a is the same as in (0.3). P is Poissonian iff it is a-stable with $a = 1$.

2. α-times self-decomposable point processes. Suppose that P is an α-times self-dec. $(0 < \alpha < \infty)$ p. proc. on X. Then, by Def. 0.1, it follows that for every $0 < \quad < 1$ there exists an i.d. p. proc. $V = V_{\alpha,c}$ on X such that

$$\log G_P(f) = \sum_{k=0}^{\infty} \binom{\ +k-l}{k} \log G_V(1 - c^k + c^k f) \qquad (f \in \mathscr{K}). \tag{2.1}$$

Hence we get

$$\log G_V(f) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \log G_P(1 - c^k + c^k f). \tag{2.2}$$

Let us fix a function φ on X such that $1 - e^{-\varphi}$ belongs to $\mathscr{K}$ and put

$$g(s) = -\log G_P(1 - r^{-\varphi+s}) \qquad (s \leqslant 0). \tag{2.3}$$

It is easy to see from (2.2) that if $P \in L_\alpha$, then

$$\Delta_t^\alpha g(s) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} g(s - kt) \geqslant 0 \tag{2.4}$$

for any $s \leqslant 0$ and $t \geqslant 0$.

Consider a particular case where X consists of d points $(d = 1, 2, \ldots)$. Then a p. proc. P on X can be identified with a distribution on Z_+^d. Let P be an α-times self-dec. distribution on Z_+^d. By an elementary argument we see that the function g defined by (2.3) is of class C^∞. Therefore there exists a limit

$$D^\alpha g(s) = \lim_{t\to 0+} t^{-\alpha} \Delta_t^\alpha g(s) \quad (s \leqslant 0), \tag{2.5}$$

which implies that there exists an i.d. distribution λ on Z_+^d such that

$$\log G_\lambda(f) = \lim_{t\to 0+} t^{-\alpha} \log G_V(f), \tag{2.6}$$

where $f \in [0, 1) \times \ldots \times [0, 1)$ (d times), $V = V_{\alpha,c}$ and $t = -\log c$. From (2.6) it follows that each α-times self-dec. distribution on Z_+^d is α-differentiable, i.e. there exists a limit

$$D^\alpha P = \lambda = \lim_{t\to 0+} A[V_{\alpha,c}, t^{-\alpha}], \tag{2.7}$$

where $t = -\log c$.

Now suppose that P is an α-times self-dec. p. proc. on a separable Polish space X. Let $E_1, \ldots, E_d$ be disjoint sets in $\mathscr{B}$. Denote by Q the image of P under the mapping $N \ni \ \mapsto (\mu E_1, \ldots, \mu E_d) \in Z_+^d$. Then Q is an α-times self-dec. distribution on Z_+^d. Hence (2.7) holds with P replaced by Q. Finally, by Lemma 5.1 [4] it follows that (0.3) holds. Thus we have proved the following

THEOREM 2.1. *Every α-times self-dec. $(0 < \alpha < \infty)$ p. proc. on X is α-differentiable.*

In the sequel we shall need the following lemma:

LEMMA 2.1. *For each $Q \in L_0$*

$$\int_N \log^\alpha(1+\mu A) Q(d\mu) < \infty \quad (A \in \mathscr{B}) \tag{2.8}$$

iff

$$\int_{N\setminus\{0\}} \log^\alpha(1+\mu A) \tilde{Q}(d\mu) < \infty \quad (A \in \mathscr{B}). \tag{2.9}$$

Proof is similar to that of Lemma 2.5 [14] and is omitted.

The following theorem gives a representation of α-times self-dec. p. proc. on X:

THEOREM 2.2. *$P \in L_\alpha$ $(0 < \alpha < \infty)$ iff there is $Q \in L_0$ such that condition* (2.8) *is satisfied and*

$$\log G_P(f) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} \log G_Q(1-e^{-u}+e^{-u}f)\, du \quad (f \in \mathscr{K}). \tag{2.10}$$

Moreover, we get the formula

$$Q = D^{\alpha} P. \tag{2.11}$$

Proof. Given $P \in L_{\alpha}$, define a function g by means of (2.3), where φ is fixed. Since P is α-differentiable (Theorem 2.1) there exists a limit

$$k(s) := \lim_{t \to 0+} t^{-\alpha} \Delta_t^{\alpha} g(s) = D^{\alpha} g(s) \quad (s \leqslant 0). \tag{2.12}$$

Hence (see e.g. [2])

$$g(s) = I^{\alpha} k(s) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{s} (s-u)^{\alpha-1} k(u)\, du. \tag{2.13}$$

By (2.2)-(2.4) and Theorem 2.1 we infer that the function k defined by (2.12) is of the form

$$k(s) = -\log G_Q(1-e^{-\varphi+s}) \quad (s \leqslant 0), \tag{2.14}$$

where $Q = D^{\alpha} P$. Hence and by (2.13) we get the formula

$$g(s) = -\frac{1}{\Gamma(\alpha)} \int_{-\infty}^{s} (s-u)^{\alpha-1} \log G_Q(1-e^{-\varphi+u})\, du \quad (s \leqslant 0). \tag{2.15}$$

In particular, for $s = 0$ and $f = 1-e^{-\varphi}$ formulas (2.15) and (2.3) together imply (2.10).

Note that

$$\int_{N \setminus \{0\}} (1-e^{-\mu(A)}) \tilde{P}(d\mu) < \infty$$

for every $A \in \mathscr{B}$ (cf. [4], Theorem 6.1).

The last condition together with (2.10) implies that

(2.16)

$$\int_{N \setminus \{0\}} \int_{0}^{\infty} u^{\alpha-1} [1-(1-\exp(-u)+\exp(-u-1))^{\mu(A)}]\, du Q(d\mu) < \infty \quad (A \in \mathscr{B}),$$

or, equivalently,

$$\int_{N \setminus \{0\}} \log^{\alpha}(1+\mu(A)) Q(d\mu) < \infty. \tag{2.17}$$

Hence and by Lemma 2.1 the "only if" part of the theorem is proved. The "if part" is clear (cf. Theorem 6.1 [4]).

3. Completely self-decomposable point processes and random measures. In this section we apply the extreme point method to give a representation of infinitely self-dec. p. proc. and r.m. First we prove the following

PROPOSITION 3.1. *Every p. proc. in L_∞ is Coxian.*

Proof. Step 1. X is a d-point space ($d = 1, 2, \ldots$). Then a p. proc. $P \in L_\infty$ can be identified with a distribution on Z_+^d.

Let $C = Z_+^d \cup \{\infty\}$ denote a compactification of Z_+^d and H the class of all probability distributions λ on C such that $\lambda|_{C\setminus\{0,\infty\}}$ is a canonical measure of some completely self-dec. distribution on Z_+^d. Obviously, H is convex and compact. Let λ be an extreme point of H. Then, either $\lambda = \varepsilon_\infty$, $\lambda = \varepsilon_0$ or $\lambda = \sum_{n \neq 0} \lambda_n \varepsilon_n$ with $\lambda_n \geqslant 0$ and $\sum_n \lambda_n = 1$. In the later case we put

$$\alpha := \alpha(c) = 1 - D_c \lambda\{\varepsilon_0\} = 1 - \sum_{n \neq 0} \lambda_n (1-c)^{|n|},$$

where $|n| = n_1 + \ldots + n_d$ whenever $n = (n_1, \ldots, n_d)$. It is clear that $0 < \alpha < 1$. Next we define

$$\gamma_1 = \frac{1}{\alpha} D_c \lambda|_{Z_+^d\setminus\{0\}} \quad \text{and} \quad \gamma_2 = \frac{1}{1-\alpha}(\lambda - D_c\lambda)|_{Z_+^d\setminus\{0\}}.$$

It is easily seen that $\gamma_1, \gamma_2 \in H$ and, moreover, $\lambda = \alpha\gamma_1 + (1-\alpha)\gamma_2$, which implies that $\lambda = \gamma_1 = \gamma_2$. Thus we have proved that, for each $0 < c < 1$, there exists an $\alpha(c)$ such that $D_c \lambda|_{Z_+^d\setminus\{0\}} = \alpha(c)\lambda$, which implies that there exists a number a, $0 < a < 1$, such that $\alpha(c) = c^a$.

Hence and by Remark 0.2 it follows that λ is a canonical measure of a stable distribution on Z_+^d. On the other hand, every $\gamma \in H$, concentrated on $Z_+^d \setminus \{0\}$, is the convex combination of extreme points of H concentrated on $Z_+^d \setminus \{0\}$. Consequently, every completely self-dec. distribution P on Z_+^d is the limit of finite convolutions of stable distributions. Therefore, P is a Cox distribution.

Step 2. X is an arbitrary separable Polish space. Let P belong to L_∞. Then, for any finite sequence $B_1, \ldots, B_d$ of pairwise disjoint subsets of $\mathscr{B}$, the image of P on Z_+^d under the mapping $M \ni \ \mapsto (\mu B_1, \ldots, \mu B_d)$ is completely self-dec. By Step 1 and by a simple argument it follows that P is Coxian.

Consider a completely self-dec. r.m. $W = [\alpha, \lambda]$ on X, where α, λ are canonical measures of W, $\alpha \in M$ and λ is a measure on $M \setminus \{0\}$. From the proof of Theorem 6.1 [4] it follows that there exists a strictly positive continuous function h on X such that

$$\int_{M\setminus\{0\}} (1 - e^{-\mu h}) \lambda(d\mu) < \infty. \tag{3.1}$$

Thus, defining $m(d\mu) = (1 - e^{-\mu h}) \lambda(d\mu)$, we get a finite measure on M.

Given a subset K of M we put $\tau(K) = \{t\mu\colon\ t > 0,\ \mu \in K\}$. Further, using $1-e^{-\mu h}$ as a weight function on M one can prove the following (cf. [11], Lemma 3.4)

LEMMA 3.2. *Let λ be a canonical measure of a completely self-dec. r.m. on X. Then there exists a decomposition $\lambda = \sum_k \lambda_k$, where each λ_k is a canonical measure of a completely self-dec. r.m., λ_k is concentrated on $\tau(M_k)$, M_k is a compact subset of M, $0 \notin M_k$ and the sets $\tau(M_k)$, $k = 1, 2, \ldots$, are disjoint.*

This lemma reduces the study of canonical measures λ of completely self-dec. r.m. to the case of measures concentrated on $\tau(E)$, where E is a compact subset of M and $0 \notin E$.

Let $\bar{\tau}(E)$ be a compactification of $\tau(E)$ and H the class of all probability measures m on $\bar{\tau}(E)$ such that if γ is the restriction of m to $\tau(E)$, then the measure $\lambda(d\mu) := [\Phi(\mu)]^{-1}\gamma(d\mu)$, where $\Phi(\mu) = 1-\exp(-\mu h)$, $\mu \in \tau(E)$, is a canonical measure of a completely self-dec. r.m.

By the same method as in the proof of Theorem 6.4 [11] one can show that if m is an extreme point of H concentrated on $\tau(E)$, then the measure $\lambda(d\mu) = [\Phi(\mu)]^{-1} m(d\mu)$ is a canonical measure of a stable r.m. Therefore, by Theorem 1.1, there exist a number a, $0 < a < 1$, and $\mu \in \tau(E)$ such that

$$\lambda = (\mu h)^{-a} \int_0^\infty 1_A(t\mu)\, t^{-a-1}\, dt, \tag{3.2}$$

where A is a Borel subset of $\tau(E)$ and h is the same as in (3.1). Consequently, by Krein–Milman–Choquet Theorem ([8], Chapter 3), it follows that if γ is a canonical measure of a completely self-dec. r.m. and γ is concentrated on $\tau(E)$, then there exists a finite Borel measure G on $\tau(E) \times (0, 1)$ such that, for every γ-integrable function φ on $\tau(E)$, we get

$$\int_{\tau(E)} \varphi(\mu)\gamma(d\mu) = \int_{\tau(E)\times(0,1)} \int_0^\infty (\mu h)^{-a} \varphi(t\mu)\, t^{-a-1}\, dt G(d\mu, da). \tag{3.3}$$

We now turn our considerations to an arbitrary canonical measure λ of a completely self-dec. r.m. By Lemma 3.2 and by the above arguments it follows that there exist finite measures G_k $(k = 1, 2, \ldots)$ such that each G_k is concentrated on $\tau(M_k) \times (0, 1)$ and

$$\begin{aligned} \int_{M\setminus\{0\}} \varphi(\mu)\lambda(d\mu) &= \sum_k \int_{\tau(M_k)} \varphi(\mu)\lambda_k(d\mu) \\ &= \sum_k \int_{\tau(M_k)\times(0,1)} \int_0^\infty (\mu h)^{-a} \varphi(t\mu)\, t^{-a-1}\, dt\, G_k(d\mu, da) \\ &= \int_{M\setminus\{0\}} \int_0^1 \int_0^\infty (\mu h)^{-a} \varphi(t\mu)\, t^{-a-1}\, dt G(d\mu, da), \end{aligned} \tag{3.4}$$

where $G = \sum_k G_k$.

Since

$$\int_{M\setminus\{0\}} (1-\exp(-\mu h))\lambda(d\mu) < \infty,$$

we infer that G is a finite measure on $[M\setminus\{0\}]\times(0, 1)$.

Conversely, it is easy to prove that for every finite measure G formula (3.4) defines a canonical measure of a completely self-dec. r.m.

Finally, by (3.4) and Theorem 6.1 [4], we get

THEOREM 3.3. *Let W be a completely self-dec. r.m. on X. Then there exists a measure $v \in M$, a strictly positive continuous function h on X and a finite Borel measure G on $[M\setminus\{0\}]\times(0, 1)$ such that*

$$(3.5)\qquad -\log L_W(f) = vf + \int_{M\setminus\{0\}} \int_0^1 (\mu f)^a (\mu h)^{-a} G(d\mu, da) \qquad (f \in \mathscr{F}).$$

Conversely, for any v, h, G as mentioned above, formula (3.5) *defines some completely self-dec. r.m.*

Since every p. proc. $P \in L_\infty$ is Coxian (Proposition 3.1), we infer that there exists a completely self-dec. r.m. W such that $P = Q_W$. Hence and by Theorem 3.3 we get the following

THEOREM 3.4. *Let P be a completely self-dec. p. proc. on X. Then there exists a strictly positive continuous function h on X and a finite Borel measure G on $[M\setminus\{0\}]\times(0, 1)$ such that*

$$(3.6)\qquad -\log G_P(f) = \int_{M\setminus\{0\}} \int_0^1 [\mu(1-f)]^a (\mu h)^{-a} G(d\mu, da) \qquad (f \in \mathscr{K}).$$

Conversely, for any h, G as mentioned above, formula (3.6) *defines a completely self-dec. p. proc. on X.*

A simple consequence of the above theorems is the following

COROLLARY. *The class of all completely self-dec. p. proc.* (*r.m.*) *is the smallest class containing all stable p. proc.* (*r.m.*) *and closed under convolution operation and weak convergence.*

4. Concluding remarks. In the same way as in [12] one can prove the following

THEOREM 4.1. (i) *For each α, $0 \leqslant \alpha \leqslant \infty$, L_α is closed under convolution operation and weak convergence.*

(ii) *For any α and β, $0 \leqslant \alpha < \beta \leqslant \infty$,*

$$(4.1)\qquad L_\beta \subsetneqq L_\alpha,$$

$$(4.2)\qquad L_\beta = \bigcap_{0<\gamma<\beta} L_\gamma,$$

$$(4.3)\qquad L_\alpha = \overline{\bigcup_{\gamma>\alpha} L_\gamma},$$

where the bar denotes the closure in the weak topology.

The following theorem is a discrete analogy of Theorem 4.2 [15]:

THEOREM 4.2. *A p. proc.* $P \in L_\alpha$, $0 < \alpha < \infty$, *is stable iff, for some* a, $0 < a \leqslant 1$,

$$D^\alpha P = A[P, a^\alpha]. \tag{4.4}$$

In particular, $P \in L_\alpha$ *is a Poisson process iff it is a fixed point of* D^α.

Proof. It is clear that if P is stable, then $P \in L_\alpha$ and (4.4) holds. Conversely, suppose that $P \in L_\alpha$ and (4.4) holds. Then P belongs to L_∞. Therefore, if its generating functional is given by (3.6), then we get the equation

$$\int_{M\setminus\{0\}} \int_0^1 [\mu(l-f)]^x (\mu h)^{-x} x^\alpha G(d\mu, dx)$$
$$= \int_{M\setminus\{0\}} \int_0^1 [\mu(l-f)^x (\mu h)^{-x} a^\alpha G(d\mu, dx) \qquad (f \in \mathscr{K}),$$

which implies that the measure G is concentrated on $[M\setminus\{0\}] \times (\{a\})$. Consequently, G_P is of the form (1.10) and P is a-stable. In particular, for $a = 1$, $D^\alpha P = P$ iff P is a Poisson process.

By the same *differential method* as in Section 2 one can prove the following

THEOREM 4.3. *An r.m.* W *on* X *is* α*-times self-dec.* $(0 < \alpha < \infty)$ *iff there exists an i.d. r.m.* V *such that*

$$\int_M \log^\alpha(1+\mu A)\, V(d\mu) \qquad (A \in \mathscr{B})$$

and

$$\log L_W f = \frac{1}{\Gamma(\alpha)} \int_0^\infty \log L_V(e^{-t} f)\, t^{-\alpha-1}\, dt \qquad (f \in \mathscr{F}).$$

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