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ON MINIMUM BIAS AND VARIANCE ESTIMATION FOR PARAMETRIC MODELS WITH SHRINKING CONTAMINATION

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Abstract. A close relationship is derived between optimal M-estimation and optimal robust testing for shrinking contaminations. Explicit formulas are given for solutions when the loss is defined as convex combination of asymptotic maximum bias span and variance. Neighbourhoods are described in terms of a general class of special capacities.

1. Introduction. The basic goal of this paper is to present a close relationship between the problem of optimal M-estimation, as viewed by Hampel [5] and the M-estimation induced by certain optimal statistics in a local asymptotic robust test problem of Rieder [7] and Bednarski [2]. Such a relationship is in fact clear at least by the form of the IC functions of Rieder and Hampel. Rieder [8] has already shown that one can construct estimates based on optimal IC-test statistics and that these estimates have asymptotically minimax property in the sense of good covering the unknown parameter value.

We shall consider M-estimates which minimize convex combinations of asymptotic variance and maximum bias span for a large class of contaminations induced by capacities. Controlling both bias and variance of estimates, in the case of parametric models with shrinking contamination, is an intuitive criterion. The criterion in fact stems from some aspects of asymptotic robust test problems.

An approach similar to the one presented here, but restricted to a different family of neighbourhoods, has been given by Bickel [4] who constructs M-estimates optimal in the sense of asymptotic minimax risk, where the risk is induced by convex loss functions. Bickel obtains his solutions by variational methods while here we utilize certain optimal test statistics obtained in a direct way in connection with robust asymptotic test problems (see Bednarski [3]).

Section 2 introduces basic results and definitions. In the next section we give a special formulation of Hampel's Lemma 8 [5]. Its assumptions are expressed in terms of maximum bias for ε-contamination neighbourhoods. Such a formulation suggests a definition of optimal robust estimates as those which minimize asymptotic variance given a bound for the maximal asymptotic bias and it also gives the possibility of studying optimization problems under more general class of neighbourhoods. In Section 4, by relating asymptotic robust test problems to estimation problems, we give results which show how to construct such optimal estimates. It is shown that their M-functions can be obtained from asymptotic minimax test problems for special capacities. One of the conclusions that we obtain is that identity of Hampel's and Rieder's solutions is due to the same type of neighbourhoods employed, even though in the original Hampel's lemma the neighbourhoods are not actually present. Finally, we conclude that the family of optimal estimates containes estimates optimal for other asymptotic risk functions, like the mean square error.

2. Preliminaries. In the sequel we shall consider a real parametric family $\{P_\theta\colon \theta \in \Theta\}$ of probability measures (p.m.) defined on a Borel σ-field $\mathscr{B}$ of a Polish space Ω, with Θ an open subset of R.

Let $\mathscr{M}$ be the set of all probability measures on $\mathscr{B}$.

The following regularity conditions are assumed to hold throughout the paper:

(2.1) For every $\theta \in \Theta$ there exists an exponential family $\{Q_{\theta,z}\colon |z| \leqslant \tau(\Theta)\}$ defined on $\mathscr{B}$ such that:

(i) $dQ_{\theta,z} = c_\theta(z) \exp z\Delta_\theta \, dP_\theta$ for some random variable Δ_θ;

(ii) the distribution function F_θ of Δ_θ under P_θ has a density with respect to Lebesque measure and the distribution has convex support;

(iii) $\lim\limits_{n} \sup\limits_{|z| \leqslant \tau(\theta)} nH^2(P_{\theta 4z/\sqrt{n}}, Q_{\theta,z/\sqrt{n}}) = 0$, where H stands for the Hellinger distance.

If for $\theta_0 \in \Theta$ the square root of dP_θ/dP_{θ_0} is differentiable at θ_0 in quadratic mean and an exponent of the derivative is integrable with respect to P_{θ_0}, then conditions (i) and (iii) are fulfilled in θ_0. This, in fact, is the case for the commonly used models. A more detailed discussion of these conditions is presented in [3], Sections 2 and 6. In the "very regular" cases Δ_θ is simply the derivative of loglikelihood ratio.

The basic parametric model will be contaminated by shrinking neighbourhoods induced by special capacities (see [2]). Let f be a concave function from $[0, 1]$ to $[0, 1]$ such that $f(1) > 0$. The practical justification of the assumption $f(1) > 0$ is given in Bednarski [3].

For every $\theta \in \Theta$ we shall take the extended model

$$\mathscr{P}_{\theta,n} = \{W \in \mathscr{M}\colon\ W(A) \leqslant [P_\theta(A) + (1/\sqrt{n}) f \circ P_\theta(A)] \wedge 1,\ \forall_{A \in \mathscr{B}}\}.$$

We shall say that an estimate *is studied at θ in the extended model* if its asmptotic behaviour is studied for sequences of distributions from $\mathscr{P}_{\theta,n}^{\otimes n}$ that are contiguous to $P_\theta^{\otimes n}$, where n is the number of independent observations.

The M-functions $\varphi(x, \theta)$ which we shall consider are assumed to satisfy the following conditions under the parametric model.

Let $\hat{\theta}_n$ be an M-estimate corresponding to φ. For every $\theta \in \Theta$

$$\sqrt{n}[\hat{\theta}_n - \theta - A^{-1}(\theta)\sum_1^n \varphi(x_i, \theta)] \to 0 \text{ in } P_\theta^{\otimes n}, \tag{2.2}$$

where $A(\theta) = \int \varphi(x, \theta)\Delta_\theta(x)dP_\theta > 0$ and $\int \varphi(x, \theta)dP_\theta = 0$.

By $\mathscr{F}$ we denote the class of all bounded M-functions for the model $\{P_\theta\colon \theta \in \Theta\}$ which satisfy (2.2) and such that

$$\sup_x \varphi(x, \theta) = \sup_x \operatorname{ess}_{P_\theta} \varphi(x, \theta) \quad \text{and} \quad \inf_x \varphi(x, \theta) = \inf_x \operatorname{ess}_{P_\theta} \varphi(x, \theta).$$

General conditions for φ to be in $\mathscr{F}$ can be found in [9], Chapter 5.

With the extended parametric model there is associated a local asymptotic minimax test problem (cf. [6], [7] and [3]). It turns out that asymptotically minimax tests for the sequences of hypotheses and alternatives $\mathscr{P}_{\theta-\tau/\sqrt{n},n}^{\otimes n}$ and $\mathscr{P}_{\theta-\tau/\sqrt{n},n}^{\otimes n}$, respectively, are based on a sum

$$\sum_{i=1}^n \varphi_\theta(\Delta_\theta(x_i))$$

of independent variables, where the function φ_θ is determined by minimization of

$$\int_{m(h)}^{M(h)} [f \circ P_\theta(h > t) + f \circ P_\theta(h \leqslant t)]\,dt - 2\tau \int h\Delta_\theta\,dP_\theta + (1/2)\int h^2\,dP_\theta \tag{2.3}$$

over the class of functions h, which satisfy $\int h dP_\theta = 0$ and

$$m(h) = \inf h = \inf_{P_\theta} \operatorname{ess} h, \qquad M(h) = \sup h = \sup_{P_\theta} \operatorname{ess} h.$$

Namely, if h minimizes (2.3) at θ, then $\varphi_\theta = h$ P_θ a.e. Further on, to simplify the notation, we shall write m and M instead of $m(h)$ and $M(h)$.

Expression (2.3) is obtained in the following way.

Let θ be fixed and put $P = P_\theta$, $\Delta = \Delta_\theta$. Moreover, let

$$g_\tau(A, t) = [f \circ P(A) + f \circ P(A^c)] - 2\tau \int_A \Delta\,dP + \begin{cases} tP(A) & \text{for } t \geqslant 0, \\ -tP(A^c) & \text{for } t < 0. \end{cases}$$

The function g_τ is a limit of properly normalized Bayes risks for testing $\mathscr{P}_{\theta-\tau/\sqrt{n},n}$ against $\mathscr{P}_{\theta+\tau/\sqrt{n},n}$ (see [3], Section 2). From Section 3 of [3] it follows that under condition (2.1) there is a unique (up to equivalence class) bounded function h of Δ so that for all $t \in [\inf_P \operatorname{ess} h, \sup_P \operatorname{ess} h]$ we have

$$g_\tau(h \geqslant t, t) = \inf_{A \in \mathscr{B}} g_\tau(A, t) \leqslant 0$$

and $h = \varphi_\theta$. Therefore, if we take any random variable h' minimizing

$$\int_m^M g(h' > t,\, t)\, dt,$$

then it has to be equivalent to h. It is easy to verify that for every bounded measurable function h we have

$$\int_m^M g(h > t,\, t)\, dt = \int_m^M [f \circ P(h > t) + f \circ P(h \leqslant t)]\, dt - 2\tau \int h\Delta\, dP + \frac{1}{2} \int h^2\, dP.$$

Explicit solutions to the minimization problem (2.3) are given in [3], Section 3.

If the derivative f' of f exists, then

$$\varphi_\theta(\Delta_\theta) = \{f' \circ F_\theta(\Delta_\theta^*) - f' \circ (1 - F_\theta(\Delta_\theta^*))\} + 2\tau \Delta_\theta^*, \tag{2.4}$$

where $\Delta_\theta^* = d_0 \vee \Delta_\theta \wedge d_1$ with constants d_0, d_1 depending on τ and θ. In the case of a shift model the function φ_θ is, for fixed τ, a shift function with respect to θ.

3. Hampel's lemma. Let $IF(\mathscr{F})$ be a subclass of $\mathscr{F}$ which consists of M-functions that are influence functions for its M-estimates. By (2.2) one can easily see that if $\varphi \in IF(\mathscr{F})$ and $\hat{\theta}_n$ is its M-estimate, then under sequences from $\mathscr{P}_{\theta,n}^{\otimes n}$, contiguous to $P_\theta^{\otimes n}$, the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$, if the limit exists, is normal $N(c, \sigma^2)$, with $\sigma^2 = E_\theta \varphi_\theta^2$ and $c \in [a(\theta), b(\theta)]$, where

$$a(\theta) = \lim_n \inf_{W \in \mathscr{P}_{\theta,n}} (\sqrt{n}\, E_W\, \varphi_\theta), \qquad b(\theta) = \lim_n \sup_{W \in \mathscr{P}_{\theta,n}} (\sqrt{n}\, E_W\, \varphi_\theta).$$

If $\mathscr{P}_{\theta,n}$ is generated by ε-contamination capacity, that is $f(x) = \varepsilon(1-x)$, then one can easily obtain that maximum bias $(-a(\theta) \vee b(\theta))$ equals

$$\sup_x |\varphi(x, \theta)| = \varepsilon \cdot (\text{gross error sensitivity}).$$

By this relation we can state Hampel's lemma in the following form:

HAMPEL'S LEMMA [5]. *Let us consider the model with shrinking ε-contamination neighbourhoods. Then, in the class of functions from $IF(\mathscr{F})$ for which $[-a(\theta) \vee b(\theta)] \leqslant c/\varepsilon$, there exists a unique one which minimizes the asymptotic variance of its M-estimate.*

Hampel's solution has the property that the interval indicating the span of bias is symmetric about zero. Notice, however, that if an M-estimate $\hat{\theta}_n$ gives an interval $[a(\theta), b(\theta)]$ with ends bounded and continuous in $\bar{\theta}$, then we can use estimate $\hat{\theta}_n - (a(\bar{\theta}_n) + b(\bar{\theta}_n))/(2/\sqrt{n})$, which already has bias in a symmetric interval of the same length. For $\bar{\theta}_n$ we can take any consistent, under the extended model, estimate of θ. Further on we shall measure the bias of M-estimate by the length of such intervals, that is we shall deal with the bias span $b(\theta)-a(\theta)$.

Let us now return to the general family of neighbourhoods. Imitating the presented reasoning, we can define optimal robust M-estimates as those which minimize asymptotic variance given a bound on the bias span. This general definition reduces to Hampel's minimization problem if we deal with ε-contamination neighbourhoods. The next section will show that optimal M-functions, which are simple truncation of derivative of loglikelihood ratios, are associated with linear f only.

4. Results. Let $\mathscr{P}_{\theta,n}$ be generated by a sequence of special capacities of the form

$$v_{\theta,n} = [P_\theta + (1/\sqrt{n}) f \circ P_\theta] \wedge 1,$$

where f is a concave function from $[0, 1]$ to $[0, 1]$ and $f(1) > 0$.

LEMMA 4.1. *Let $\varphi \in \mathscr{F}$. Then we have*

$$\lim_n \inf_{W \in \mathscr{P}_{\theta,n}} [\sqrt{n} E_W \varphi_\theta] = -\int_m^M f \circ P_0(\varphi_\theta \leqslant t)\, dt$$

and

$$\lim_n \sup_{W \in \mathscr{P}_{\theta,n}} [\sqrt{n} E_W \varphi_\theta] = \int_m^M f \circ P_\theta(\varphi_\theta > t)\, dt.$$

Proof. To prove the first equality notice that since $\int \varphi_\theta dP_0 = 0$ we have (by Lemma 3.4 in [1])

$$\sqrt{n} \sup_{W \in \mathscr{P}_{\theta,n}} E_W \varphi_\theta = \sqrt{n}\Big[\int_0^\infty v_{\theta,n}(\varphi_\theta - m > t)\, dt + m\Big]$$
$$= \int_{t_n}^{M-m} f \circ P_0(\varphi_\theta - m > t)\, dt + \sqrt{n}\Big(t_n - \int_0^{t_n} P_\theta(\varphi_\theta - m > t)\, dt\Big),$$

where $t_n = \sup\{t\colon v_{\theta,n}(\varphi_\theta - m > t) = 1\}$ or $t_n = 0$ if the set of such t is empty. Since the second summand is majorized by

$$\int_0^{t_n} f \circ P_\theta(\varphi_\theta - m > t)\, dt,$$

and $t_n \to 0$ we obtain the desired limit. The other relation can be proved similarly.

Thus, the asymptotic bias span of M-estimates, corresponding to $\varphi \in \mathscr{F}$ at θ, is by (2.2) equal to

$$\int_m^M [f \circ P_\theta(\varphi_\theta > t) + f \circ P_\theta(\varphi_\theta \leqslant t)]\, dt \Big/ \int \varphi_\theta \Delta_\theta dP_\theta$$

and will further be denoted by $[b(\theta)-a(\theta)]_\varphi$.

Let us take now a smooth function $\tau(\theta)$ so that, for every θ, there is a solution h_θ minimizing (2.3) and such that $h \in \mathscr{F}$. It is easy to see that in the case of shift model $h(x, \theta) = h(x-\theta)$ and $h \in \mathscr{F}$ (see [9], Chapter 5), for $\tau(\theta) \equiv \tau$.

Now we shall ask about M-estimates which minimize

$$(4.1) \qquad [b(\theta)-a(\theta)]_\varphi + \alpha(\theta)\left[\int \varphi_\theta^2\, dP_0 / \left(\int \varphi_\theta \Delta_\theta\, dP_\theta\right)^2\right],$$

with $\alpha(\theta) = (1/2)\int \varphi_\theta \Delta_\theta\, dP_0$, that is estimates that minimize a combination of asymptotic bias and variance. By Lemma 3.1 of Bednarski [3] it follows that infimum of (2.3) is less than zero. Therefore we always have $\alpha(\theta) > 0$. Let us notice that (4.1) does not change if we take multiples of φ.

THEOREM 4.1. *Under the above assumptions, for every $\theta \in \Theta$ the solution h_θ of* (2.3) *minimizes* (4.1) *at θ. Moreover if $h' \in \mathscr{F}$ minimizes* (4.1) *at θ, then there is a multiple of h' so that it solves* (2.3) *at θ.*

Proof. Suppose there is a $\varphi \in \mathscr{F}$ which gives (4.1) the value smaller than h does at θ. Take a multiple of φ by a positive constant (cf. (2.2)), say φ', such that $\int \varphi_\theta' \Delta_\theta\, dP_\theta = 2\alpha(\theta)$. Multiplying now expression (4.1) by $2\alpha(\theta)$ we obtain that φ' is better than h for (2.3) at θ. On the other hand, if h minimizes (4.1) at θ and $\int h_\theta \Delta_\theta\, dP_\theta = 2\alpha(\theta)$, then clearly it also minimizes (2.3) at θ.

The scope of the correspondence between robust estimation and testing described in Theorem 4.1 depends also on the range of $\alpha(\tau)$; notice that, at fixed θ, α is a function of τ. We shall consider this problem now.

Let $\theta \in \Theta$ be fixed and put $P = P_\theta$, $\Delta = \Delta_\theta$. Let $M(\tau)$ and $m(\tau)$ denote the supremum and infimum of h, the solution of (2.3) for a given τ. Let τ^* be the least value of τ for which there is a solution of (2.3) (for the existence of solution in the context of asymptotic robust test problems see [3]).

LEMMA 4.2. *Under condition* (2.1), *for $\tau > \tau^*$, the functions $M(\tau)$ and $m(\tau)$ are continuous. Moreover $M(\tau)$ is increasing and $m(\tau)$ is decreasing.*

Proof. *Monotonicity.* From Section 3 of [3] we get that for every $\tau > \tau^*$ there exist unique numbers $z^+(\tau)$ and $z^-(\tau)$ so that

$$g_\tau(\Delta > z^+(\tau), M(\tau)) = g_\tau(\Delta > z^-(\tau), m(\tau)) = 0$$

and

$$\inf_{A \in \mathscr{B}} g_\tau(A, M(\tau)) = \inf_{A \in \mathscr{B}} g_\tau(A, m(\tau)) = 0.$$

Therefore, for $\tau_1 > \tau > \tau^*$, the inequality $M(\tau_1) \leqslant M(\tau)$ implies

$$g_{\tau_1}(\Delta > z^+(\tau), M(\tau_1)) < g_\tau(\Delta > z^+(\tau), M(\tau)) = 0$$

which, since the left-hand side is nonnegative, is impossible. Arguments for $m(\tau)$ are similar.

Continuity. Let $\tau_n \nearrow \tau$. Since $M(\tau)$ is increasing, we have $M(\tau_n) \nearrow c$ for some $c > 0$. Suppose $c < M(\tau)$. Then, for $c < t < M(\tau)$, we have

$$\inf_{A \in \mathscr{B}} g_\tau(A, t) < 0.$$

Thus, for τ_n close to τ, we obtain

$$\inf_{A \in \mathscr{B}} g_{\tau_n}(A, t) < 0$$

which contradicts $M(\tau_n) \leqslant c$.

Suppose now that $\tau_n \searrow \tau$ and $M(\tau_n) \searrow c > M(\tau)$. Then we have

$$\inf_{A \in \mathscr{B}} g_{\tau_n}(A, t) < 0 \quad \text{for every fixed } t \in (M(\tau), c).$$

Therefore $\inf_{A \in \mathscr{B}} g_\tau(A, t) \leqslant 0$ which contradicts the equality

$$\inf_{A \in \mathscr{B}} g_\tau(A, M(\tau)) = 0.$$

Similar arguments hold for $m(\tau)$.

LEMMA 4.3. *Under condition* (2.1) *we have*:

(i) *If* $\tau \searrow \tau^*$, *then* $M(\tau) \searrow 0$ *and* $m(\tau) \nearrow 0$.

(ii) *If* $\sup_P \text{ess}(-\Delta) = \sup_P \text{ess}(\Delta) = \infty$, *then*

$$\lim_{\tau \to \infty} M(\tau)/\tau = -\lim_{\tau \to \infty} m(\tau)/\tau = \infty.$$

Proof. (i) One can easily verify that

$$\inf_{A \in \mathscr{B}} g_{\tau^*}(A, 0) = 0.$$

Suppose $M(\tau) \searrow \varepsilon > 0$. Then, as in the proof of Lemma 4.2, for every $\tau > \tau^*$ we have

$$\inf_{A \in \mathscr{B}} g_\tau(A, \varepsilon) < 0.$$

So, for some nonempty measurable set A_0 and $0 < \varepsilon' < \varepsilon$, we obtain $(g_{\tau^*} A_0, \varepsilon') < 0$. This, however, is contradictory to

$$\inf_{A \in \mathscr{B}} g_{\tau^*}(A, 0) = 0.$$

(ii) Let us assume that there is a sequence $\tau_n \nearrow \infty$ so that $M(\tau_n)/\tau_n \to c < \infty$. If $c > 0$, let $\varepsilon > 0$ be such that $\varepsilon < cP(\Delta > c)/2$. Then

$$g_{\tau_n}(\Delta > c, M(\tau_n))/\tau_n \leqslant -2 \int_{\Delta > c} \Delta \, dP + 2cP(\Delta > c)$$

if only $M(\tau_n)/\tau_n < 3c/2$ and $1/\tau_n \leqslant \varepsilon/2$. Since the right-hand side of this inequality is strictly less than 0 and

$$\inf_{A \in \mathscr{B}} g_{\tau_n}(A, M(\tau_n)) = 0,$$

we obtain the contradiction. In the case $c = 0$ we obviously have $g_{\tau_n}(\Delta > 0, M(\tau_n))/\tau_n < 0$ for n large enough. This completes the proof.

From Lemma 4.3 we conclude the following

LEMMA 4.4. *Let the derivative of f exist so that h is given by* (2.3). *Then* $\{\alpha(\tau)\colon \tau > \tau^*\} = (0, \infty)$.

Thus, for every $\alpha \in (0, \infty)$, the problem (4.1) of optimal M-estimation is equivalent to the optimization in some asymptotic robust test problem (2.3). Let, for $\alpha \in (0, \infty)$, φ_α minimize (4.1) and $x(\alpha) = (b-a)\varphi_\alpha$. Moreover, let $y(\alpha) = \sigma_\alpha^2$ be the asymptotic variance of M-estimate induced by φ_α. Denote by $\mathscr{R}$ the set defined by

$$\mathscr{R} = \{(x, y)\colon\ x = x(\alpha),\ y = y(\alpha)\ \text{for}\ \alpha \in (0, \infty)\}.$$

LEMMA 4.5. *Let us assume that* $\inf_P \operatorname{ess}(-\Delta) = \sup_P \operatorname{ess}(\Delta) = \infty$ *and that the derivative of f exists. Then*

(i) $x(\alpha)$, $y(\alpha)$ *are continuous in* $\alpha \in (0, \infty)$;

(ii) $\sup\{x\colon (x, y) \in \mathscr{R}\} = \infty$;

(iii) $B\colon = \inf\{x\colon (x, y) \in \mathscr{R}\} > 0$;

(iv) $V\colon = \sup\{y\colon (x, y) \in \mathscr{R}\} < \infty$.

Proof. The part (i) follows from Lemma 4.2 and continuity of the integrals in the formulas for the asymptotic bias span and variance. The form of the solution h, given by (2.4) and Lemma 4.3, gives (ii).

To prove (iii) and (iv) notice that, in the minimization problem (4.1), we can restrict attention to M-functions φ for which $\sup|\varphi| = D$ for a given constant $D > 0$. Then (iii) follows from the fact that the numerator in the formula for the bias span is bounded from zero. To complete the arguments for (iv) assume by contradiction that $V = \infty$. *Then there is a sequence* $\alpha(n) \in (0, \infty)$, $\alpha(n) \to \alpha_0$, so that $\int \varphi_{\alpha(n)} \Delta dP \searrow 0$ and $y(\alpha(n)) \nearrow \infty$. In the case $\alpha_0 \in [0, \infty)$ we easily obtain the contradiction showing that infimum of (4.1) has to be bounded in a neighbourhood of α_0. For $\alpha_0 = \infty$ we also obtain contradiction showing that, for any $\alpha \in (0, \infty)$, φ_α gives better minimum risks than those obtained for $\varphi_{\alpha(n)}$ if n is sufficiently large.

Lemma 4.5 easily implies the following result which transfers Hampel's basic lemma to a more general setup:

THEOREM 4.2. *Assume condition* (2.1) *is fulfilled and f is differentiable. Then, for every constant $b \in (B, \infty)$, in the class of M-functions which have asymptotic bias span $\leqslant b$ there exists one, say φ, which minimizes asymptotic variance. The bias span induced by this M-function is exactly b. Also* $(b, \sigma_\varphi^2) \in \mathscr{R}$.

Finally, we would like to comment on solutions for other risk functions. Let $r_1 \geqslant 0$ and $r_2 \geqslant 0$ be strictly increasing real functions. We can ask about M-estimates minimizing

$$r_1[(b-a)_\varphi]+\alpha r_2[\sigma_\varphi^2] \tag{4.2}$$

for a given $\alpha > 0$.

If e.g. we take $\alpha = 1$, $r_1(x) = x^2$ and $r_2(x) = x$, then we obtain the asymptotic mean square error. Theorem 4.2 implies that if φ_α minimizes (4.2) and $(b-a)_{\varphi_\alpha} > B$, then $\big((b-a)_{\varphi_\alpha}, \sigma_\varphi^2\big) \in \mathscr{R}$. In other words, the class of solutions that we obtained for our original risk function contains optimal solutions for risk functions of the form (4.2).

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