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STATISTICAL CHARACTERIZATIONS OF GAUSSIAN MEASURES ON A HILBERT SPACE

BY

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Abstract. Let $X_1, \ldots, X_n$ be i.i.d. random vectors with values in a real separable Hilbert space. We consider the problem of estimating the mean of X_1 under quadratic loss and discuss analogues of characteristic properties of normally distributed real random variables. It is shown that there exists an equivariant sufficient linear statistic iff X_1 is Gaussian. Further the optimality of the sample mean $\bar{X}$ in the class of all equivariant or unbiased estimators is a characteristic property of Gaussian random vectors.

1. Equivariant sufficient statistics and Pitman estimators. Let H be a real separable infinite-dimensional Hilbert space. Denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the norm in H. For fixed $n \in N$ let

$$\mathscr{E} = \big((H^n, \mathfrak{B}(H^n)), \{P_\vartheta\colon \vartheta \in H\}\big)$$

be the translation experiment uniquely defined by a probability measure P_0 on $\mathfrak{B}(H^n)$, the Borel σ-algebra on H^n, where $P_\vartheta = P_0(\cdot - \vartheta)$, $\vartheta \in H$, and $x+y = (x_1+y, \ldots, x_n+y)$, $x \in H^n$, $y \in H$. We consider the problem of estimating ϑ under the loss function

$$H \times H \to \boldsymbol{R}_+, \quad (\vartheta, y) \mapsto |y-\vartheta|^2.$$

An *estimator* is a Borel measurable statistic $S\colon H^n \to H$; $E_\vartheta |S-\vartheta|^2$ is called the *risk* of the estimator S when the true value of the parameter is ϑ, where E_ϑ stands for the expectation with respect to P_ϑ. Let $L_H^2(P_\vartheta)$ denote the space of all estimators S such that $E_\vartheta |S|^2 < \infty$. An estimator S satisfying the condition $S(x+y) = S(x)+y$ for all $x \in H^n$, $y \in H$, is called *equivariant*. For such estimators the risk $E_\vartheta |S-\vartheta|^2 = E_0 |S|^2$ does not depend on $\vartheta \in H$. We say that an equivariant estimator S is a *Pitman estimator*, if $E_0 |S|^2 \leqslant E_0 |S_1|^2$ for all equivariant estimators S_1. A statistic $S\colon H^n \to Y$ for some set Y is called *invariant*, if $S(x+y) = S(x)$ for all $x \in H^n$, $y \in H$. Let

$$\mathfrak{A}(H^n) = \{A \in \mathfrak{B}(H^n)\colon 1_A \text{ is invariant}\}.$$

In this section we observe that the Pitman estimator may be characterized analogously to the case $H = \mathbb{R}$. We need the following information. The first lemma will also be useful for a characterization of Gaussian measures by sufficiency.

LEMMA 1.1. *If $S\colon H^n \to H$ is an equivariant Borel measurable statistic, then the following statements are equivalent*:

(i) *S is sufficient for $\mathscr{E}$.*

(ii) *$S^{-1}(\mathfrak{B}(H))$ and $\mathfrak{A}(H^n)$ are independent under P_0, where $\mathfrak{B}(H)$ denotes the Borel σ-algebra on H.*

Proof. (i)$\Rightarrow$(ii). Let $A \in \mathfrak{A}(H^n)$, $B \in \mathfrak{B}(H)$ and let $f\colon H \to \mathbb{R}$ be a Borel measurable function with $f \circ S = E_\vartheta(1_A|S)$ P_ϑ-a.s. for all $\vartheta \in H$. Since, for any $B' \in \mathfrak{B}(H)$, $y \in H$,

$$\int_{S^{-1}(B')} f \circ S \, dP_0 = \int_{S^{-1}(B')} f(S+y)\, dP_0,$$

we obtain by Fubini's theorem

$$P_0(A \cap S^{-1}(B)) = \int_{S^{-1}(B)} \int_{H^n} f\big(S(x)+S(z)\big)\, dP_0(z)\, dP_0(x)$$
$$= \int_{S^{-1}(B)} \int_{H^n} f\big(S(z)\big)\, dP_0(z)\, dP_0(x) = P_0(A) \cdot P_0\big(S^{-1}(B)\big).$$

(ii)$\Rightarrow$(i). We define $T\colon H^n \to H^n$ by $T(x) = x - S(x)$. Then T is a maximal invariant, Borel measurable statistic with $T^{-1}(\mathfrak{B}(H^n)) = \mathfrak{A}(H^n)$. Hence, S and T are independent under P_0. Then it is easily seen that S and T are independent under P_ϑ, $\vartheta \in H$. Furthermore, T is an ancillary statistic for $\mathscr{E}$. For any $A \in \mathfrak{B}(H^n)$, $B \in \mathfrak{B}(H)$ and $\vartheta \in H$ this yields

$$P_\vartheta\big(A \cap S^{-1}(B)\big) = \int_{H^n} 1_B\big(S(x)\big) \cdot 1_A\big(S(x)+T(x)\big)\, dP_\vartheta(x)$$
$$= \int_H 1_B(y) \int_{H^n} 1_A(y+t)\, dP_0^T(t)\, dP_\vartheta^S(y)$$
$$= \int_H 1_B(y) \cdot P_0^T(A-y)\, dP_\vartheta^S(y)$$
$$= \int_{S^{-1}(B)} P_0^T\big(A - S(x)\big)\, dP_\vartheta(x),$$

thus $E_\vartheta(1_A|S) = P_0^T(A-S)$ P_ϑ-a.s. This proves the assertion.

Remark. The proof of Lemma 1.1 shows that a Borel measurable, equivariant statistic $S\colon H^n \to H$ is sufficient for $\mathscr{E}$ if and only if for every $A \in \mathfrak{A}(H^n)$ there exists a version of $E_\vartheta(1_A|S)$ independent of $\vartheta \in H$.

LEMMA 1.2. *Let $S \in L_H^2(P_0)$ be equivariant. Then S is a Pitman estimator if and only if $E_0\langle S, g\rangle = 0$ for all invariant estimators $g \in L_H^2(P_0)$.*

Proof. *The "if" part.* Let $S_1 \in L_H^2(P_0)$ be another equivariant estimator.

Then $g = S_1 - S \in L_H^2(P_0)$ is an invariant estimator and we have

$$E_0 |S_1|^2 = E_0 |S|^2 + 2E_0 \langle S, g\rangle + E_0 |g|^2 \geqslant E_0 |S|^2$$

since, by assumption, $E_0 \langle S, g\rangle = 0$.

The "only if" part. Let $g \in L_H^2(P_0)$ be an invariant estimator. For any $\lambda \in \boldsymbol{R}$, $S_1 = S + \lambda g$ is an equivariant estimator. Further

$$E_0 |S_1|^2 = E_0 |S|^2 + 2\lambda E_0 \langle S, g\rangle + \lambda^2 E_0 |g|^2 \geqslant E_0 |S|^2,$$

which implies $E_0 \langle S, g\rangle = 0$.

Given a P_ϑ-Bochner integrable statistic S: $H^n \to H$ and a $(\mathfrak{B}(H^n), \mathfrak{B}(Y))$-measurable statistic T: $H^n \to Y$ for some measurable space $(Y, \mathfrak{B}(Y))$, then the T-conditional expectation $E_\vartheta(S|T)$: $H^n \to H$ of S is the P_ϑ-a.e. unique, P_ϑ-Bochner integrable, $(T^{-1}(\mathfrak{B}(Y)), \mathfrak{B}(H))$-measurable statistic such that

$$\int_A E_\vartheta(S|T)\, dP_\vartheta = \int_A S\, dP_\vartheta \quad \text{for all } A \in T^{-1}(\mathfrak{B}(Y));$$

integration and expectation will always be considered in the sense of Bochner. Since H has the Radon-Nikodym property, the usual proof of the existence of the conditional expectation works with H replacing $\boldsymbol{R}$. We have

$$\langle y, E_\vartheta(S|T)\rangle = E_\vartheta(\langle y, S\rangle | T) \quad P_\vartheta\text{-a.s. for all } y \in H,$$

$$E_\vartheta E_\vartheta(S|T) = E_\vartheta S \quad P_\vartheta\text{-a.s.},$$

$$|E_\vartheta(S|T)|^2 \leqslant E_\vartheta(|S|^2 | T) \quad P_\vartheta\text{-a.s. for } S \in L_H^2(P_\vartheta)$$

(cf. [5], Chap. V–2).

THEOREM 1.3. *If $S \in L_H^2(P_0)$ is equivariant, then*

$$S_0 = S - E_0(S|T)$$

is the (up to P_0-equivalence uniquely determined) Pitman estimator, where $T(x) = x - S(x)$, $x \in H^n$.

Proof. Note first that a Borel measurable statistic g: $H^n \to H$ is $(\mathfrak{A}(H^n), \mathfrak{B}(H))$-measurable if and only if g is invariant. Therefore, $E_0(S|T)$ is invariant, so $S_0 \in L_H^2(P_0)$ is an equivariant estimator. Let $g \in L_H^2(P_0)$ be an invariant estimator. Then

$$\langle E_0(S|T), g\rangle = E_0(\langle S, g\rangle | T) \quad P_0\text{-a.s.}$$

To see this let Q: $H^n \times \mathfrak{B}(H) \to [0, 1]$ be the regular T-conditional distribution of S under P_0. For any $z \in H$ we have

$$\langle z, E_0(S|T)\rangle = E_0(\langle z, S\rangle | T) = \int_H \langle z, y\rangle Q(T(\cdot), dy)$$

$$= \langle z, \int_H y Q(T(\cdot), dy)\rangle \quad P_0\text{-a.s.},$$

which implies

$$E_0(S|T) = \int_H y\,Q(T(\cdot), dy) \qquad P_0\text{-a.s.}$$

Therefore, for any $A \in \mathfrak{A}(H^n)$ we obtain

$$\int_A \langle E_0(S|T), g\rangle\, dP_0 = \int_A \langle \int_H y\,Q(t, dy), g(t)\rangle\, dP_0^T(t)$$

$$= \int_A \int_H \langle y, g(t)\rangle\, Q(t, dy)\, dP_0^T(t) = \int_{H\times A} \langle y, g(t)\rangle\, dP_0^{(S,T)}(y, t)$$

$$= \int_A \langle S, g\rangle\, dP_0 = \int_A E_0(\langle S, g\rangle | T)\, dP_0,$$

which proves the statement.

We conclude that

$$E_0\langle S_0, g\rangle = E_0\langle S, g\rangle - E_0 E_0(\langle S, g\rangle | T) = 0,$$

hence, by Lemma 1.2, S_0 is a Pitman estimator.

To prove the uniqueness of the Pitman estimator assume that $S_1 \in L^2_H(P_0)$ is another Pitman estimator. Then $g = S_1 - S_0 \in L^2_H(P_0)$ is an invariant estimator and we have by Lemma 1.2

$$E_0|g|^2 = E_0\langle S_1, g\rangle - E_0\langle S_0, g\rangle = 0,$$

which implies $S_1 = S_0$ P_0-a.s.

COROLLARY 1.4. (a) *If $S \in L^2_H(P_0)$ is equivariant and sufficient for $\mathscr{E}$, then $S_0 = S - E_0 S$ is the Pitman estimator.*

(b) *If $S_1, S_2 \in L^2_H(P_0)$ are equivariant and sufficient for $\mathscr{E}$, then $S_1 = S_2 + y$ P_0-a.s. for some $y \in H$.*

Proof. The assertions follow immediately from Lemma 1.1. and Theorem 1.3.

2. Characterizations of Gaussian measures. In the sequel X_i: $H^n \to H$ denotes the i-th projection. We assume that $X_1, \ldots, X_n$ are i.i.d. under P_0. A probability measure μ on $\mathfrak{B}(H)$ is *Gaussian* if the continuous linear functionals $\langle y, \cdot\rangle$, $y \in H$, are normally distributed (possibly degenerate) when considered as random variables on the probability space $(H, \mathfrak{B}(H), \mu)$. The H-valued random vector X_i is Gaussian under P_0 if $P_0^{X_i}$ is a Gaussian measure, i.e. $\langle y, X_i\rangle$, $y \in H$, are normally distributed under P_0. Let

$$T = (X_1 - \bar{X}, \ldots, X_n - \bar{X}).$$

The following theorems extend well known characterizations of the normality of real random variables.

THEOREM 2.1. *Assume $n \geqslant 2$. The following statements are equivalent:*

(i) *There exists a sufficient statistic*

$$S = \sum_{i=1}^{n} c_i X_i, \qquad c_i \in \mathbf{R},$$

for $\mathscr{E}$ *with* $\sum_{i=1}^{n} c_i \neq 0$.

(ii) X_1 *is Gaussian under* P_0.

If (i) *and therefore* (ii) *are valid, then* $\bar{X}$ *is the essentially unique (up to* P_0-*equivalence and up to an additive constant) equivariant sufficient statistic for* $\mathscr{E}$ *in* $L^2_H(P_0)$.

Proof. (i) $\Rightarrow$ (ii). We may assume

$$\sum_{i=1}^{n} c_i = 1 \quad \text{and} \quad c_1 \neq 0.$$

Then S is equivariant and, by Lemma 1.1, S and $T_1 = X_1 - X_2$ are independent under P_0, thus for $y \in H$

$$\langle y, S\rangle = \sum_{i=1}^{n} c_i \langle y, X_i\rangle \quad \text{and} \quad \langle y, T_1\rangle = \langle y, X_1\rangle - \langle y, X_2\rangle$$

are independent under P_0. Therefore, it follows by the Skitovich-Darmois Theorem that $\langle y, X_1\rangle$ is normally distributed under P_0 (cf. [2], Theorem 3.1.1).

(ii) $\Rightarrow$ (i). We may assume that $E_0 X_1 = 0$ and the support of $P_0^{X_1}$ is all H. Let $C\colon H \to H$ denote the covariance operator of X_1 under P_0 which is determined by the relation

$$\langle Cy, z\rangle = E_0(\langle y, X_1\rangle \langle z, X_1\rangle), \qquad y, z \in H.$$

Then C is a linear compact injective operator which is positive, symmetric and trace class.

If $\vartheta \in C^{1/2}(H)$. Then $P_\vartheta^{X_1}$ is $P_0^{X_1}$-continuous and

$$\frac{dP_\vartheta^{X_1}}{dP_0^{X_1}} = \exp\left(L_\vartheta - \frac{1}{2}|C^{-1/2}\vartheta|^2\right)$$

(cf. [6], p. 83, Theorem 2), where L_ϑ is defined as follows. Let $\{e_i\colon i \in N\}$ be an orthonormal basis of H consisting of eigenvectors of C. Then $e_i \in C^{1/2}(H)$ for all $i \in N$ and the random variables $Z_i = \langle C^{-1/2} e_i, \cdot\rangle$ are i.i.d. $N(0, 1)$ under $P_0^{X_1}$. Hence, the sequence $(L_{\vartheta,k})_{k \in N}$ of continuous linear functionals defined by

$$L_{\vartheta,k} = \sum_{i=1}^{k} Z_i \langle e_i, C^{-1/2}\vartheta\rangle$$

is a martingale under $P_0^{X_1}$ and

$$\sup_{k\in N} E_0 |L_{\vartheta,k} \circ X_1| \leqslant |C^{-1/2}\vartheta| < \infty.$$

The martingale convergence theorem implies that $L_{\vartheta,k}$ converges to a limit L_ϑ $P_0^{X_1}$-a.s.

Observe that there is a Borel measurable subspace D_ϑ of H such that $P_0^{X_1}(D_\vartheta) = 1$ and $L_\vartheta | D_\vartheta$ is linear. This yields

$$\frac{dP_\vartheta}{dP_0} = \exp\left(nL_\vartheta \circ \bar{X} - \frac{n}{2}|C^{-1/2}\vartheta|^2\right),$$

thus $\bar{X}$ is sufficient for the subexperiment $\{P_\vartheta\colon \vartheta \in C^{1/2}(H)\}$ of $\mathscr{E}$.

$\bar{X}$ is Gaussian with $E_0 \bar{X} = 0$ and covariance operator $C_1 = (1/n)C$ under P_0. Since $C_1^{1/2}(H) = C^{1/2}(H)$ and $P_0^{\bar{X}}(\cdot - \vartheta) = P_\vartheta^{\bar{X}}$, the linear hull of

$$\left\{\frac{dP_\vartheta^{\bar{X}}}{dP_0^{\bar{X}}}\colon \vartheta \in C^{1/2}(H)\right\}$$

is norm dense in $L^1(P_0^{\bar{X}})$ (cf. [4], Theorem 4.1), thus $\bar{X}$ is a bounded complete statistic for $\{P_\vartheta\colon \vartheta \in C^{1/2}(H)\}$. Hence, by a well known result of Basu, $\bar{X}$ and T are independent under P_0. Since $T^{-1}(\mathfrak{B}(H^n)) = \mathfrak{A}(H^n)$, $\bar{X}$ is sufficient for $\mathscr{E}$ by Lemma 1.1.

The assertion concerning the uniqueness of $\bar{X}$ follows from Corollary 1.4 (b).

THEOREM 2.2. *Assume* $E_0 X_1 = 0$, $E_0 |X_1|^2 < \infty$ *and* $n \geqslant 3$. *Then the following statements are equivalent*:

(i) $\bar{X}$ *is the Pitman estimator.*

(ii) X_1 *is Gaussian under* P_0.

Proof. (i)$\Rightarrow$(ii). Let $y \in H$. According to Theorem 1.3 we have $E_0(\bar{X}|T) = 0$, thus $E_0(\langle y, \bar{X}\rangle | T) = 0$ P_0-a.s. If we define $y^n\colon H^n \to \boldsymbol{R}^n$ by $y^n(x) = (\langle y, x_1\rangle, \ldots, \langle y, x_n\rangle)$, then

$$E_0(\langle y, \bar{X}\rangle | y^n \circ T) = E_0 E_0(\langle y, \bar{X}\rangle | T) | y^n \circ T) = 0 \qquad P_0\text{-a.s.}$$

Therefore, it follows from a theorem of Kagan-Linnik-Rao that $\langle y, X_1\rangle$ is normally distributed under P_0 (cf. [2], p. 155).

(ii)$\Rightarrow$(i). By Theorem 2.1, $\bar{X}$ is sufficient for $\mathscr{E}$. Hence, the assertion follows from Corollary 1.4.

Theorem 2.2 may also be formulated as follows:

COROLLARY 2.3. *In the situation of Theorem* 2.2

$$E_0(\bar{X}|T) = 0 \qquad P_0\text{-a.s}$$

holds if and only if X_1 *is Gaussian.*

THEOREM 2.4. *Assume* $E_0 X_1 = 0$, $E_0|X_1|^2 < \infty$ *and* $n \geqslant 3$. *Then the following statements are equivalent*:

(i) $\bar{X}$ *is admissible in the class of all unbiased estimators of* ϑ.

(ii) X_1 *is Gaussian under* P_0.

If (i) *and, therefore,* (ii) *are valid, then* $\bar{X}$ *is the (up to* P_ϑ*-equivalence uniquely determined,* $\vartheta \in H$*) optimal unbiased estimator of* ϑ.

Proof. (i)$\Rightarrow$(ii). According to Theorem 1.3, $S = \bar{X} - E_0(\bar{X}|T)$ is the Pitman estimator. Since $\bar{X}$ is equivariant, we obtain $E_0|S|^2 \leqslant E_0|\bar{X}|^2$. Further we have

$$E_\vartheta S = E_\vartheta \bar{X} - E_\vartheta E_0(\bar{X}|T) = \vartheta - E_0 E_0(\bar{X}|T)$$
$$= \vartheta - E_0 \bar{X} = \vartheta \quad \text{for all } \vartheta \in H.$$

The admissibility of $\bar{X}$ yields $E_0|S|^2 = E_0|\bar{X}|^2$. Hence, the assertion follows from Theorem 2.2.

(ii)$\Rightarrow$(i). We shall show that $\bar{X}$ is optimal in the class of all unbiased estimators for every $n \in N$. According to Theorem 2.1 and Theorem 4.4 of Kozek and Wertz [3] it suffices to prove this claim for $n = 1$. We may assume that the support of P_0 is all H. Let C denote the (injective) covariance operator of P_0. Further let $\{e_i: i \in N\}$ be an orthonormal basis of H consisting of eigenvectors of C and let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0$ be the corresponding eigenvalues of C (each written as many times as is its multiplicity). Then

$$E_0|X_1|^2 = \sum_{i=1}^{\infty} \lambda_i < \infty.$$

Let $c = \sum_{i=1}^{\infty} \lambda_i$.

Now suppose that X_1 is not optimal in the class of all unbiased estimators. Then there exist $\vartheta_0 \in H$ and an unbiased estimator S_0 such that

$$E_{\vartheta_0}|S_0 - \vartheta_0|^2 \leqslant c - \varepsilon$$

for some $\varepsilon > 0$. Choose m such that

$$\sum_{i=1}^{m} \lambda_i \geqslant c - \varepsilon/2.$$

For

$$\vartheta_1 = \sum_{i=1}^{m} \langle e_i, \vartheta_0 \rangle e_i$$

and the unbiased estimator $S_1 = S_0(\cdot + \vartheta_0 - \vartheta_1) - \vartheta_0 + \vartheta_1$ we obtain

$$E_{\vartheta_1}|S_1 - \vartheta_1|^2 = E_{\vartheta_0}|S_0 - \vartheta_0|^2 \leqslant c - \varepsilon.$$

The following part of the proof is similar to the proof of the minimax character of X_1 given by Berger and Wolpert in [1]. Define α: $H \to \mathbf{R}^m$ by

$$\alpha(x) = (\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle)$$

and β: $\mathbf{R}^m \to H$ by

$$\beta(r) = \sum_{i=1}^{m} r_i e_i.$$

Then

$$P^{\alpha}_{\beta(\gamma)} = P^{\alpha}_0(\cdot - \gamma) = N(\gamma, \Sigma),$$

where Σ is an $m \times m$ diagonal matrix with $\Sigma_{ii} = \lambda_i$ and $\gamma \in \mathbf{R}^m$. We proceed by constructing an unbiased estimator U: $\mathbf{R}^m \to \mathbf{R}^m$ of γ in the translation experiment $((\mathbf{R}^m, \mathfrak{B}(\mathbf{R}^m)), \{N(\gamma, \Sigma)\colon \gamma \in \mathbf{R}^m\})$ with

$$\sum_{i=1}^{m} \int_{\mathbf{R}^m} (U_i - \gamma_{1i})^2 \, dN(\gamma_1, \Sigma) \leqslant \sum_{i=1}^{m} \int_{\mathbf{R}^m} (V_i - \gamma_{1i})^2 \, dN(\gamma_1, \Sigma) - \varepsilon/2 = \sum_{i=1}^{m} \lambda_i - \varepsilon/2,$$

where $V = Id_{\mathbf{R}^m}$ and $\gamma_1 = \alpha(\vartheta_1)$, contradicting the optimality of the estimator V in the class of all unbiased estimators of $\gamma \in \mathbf{R}^m$.

α is sufficient for the subexperiment $\{P_{\beta(\gamma)}\colon \gamma \in \mathbf{R}^m\}$ of $\mathscr{E}$, because $\beta(\gamma) \in C^{1/2}(H)$,

$$\frac{dP_{\beta(\gamma)}}{dP_0} = \exp\left(L_{\beta(\gamma)} - \frac{1}{2}|C^{-1/2}\beta(\gamma)|^2\right),$$

$$L_{\beta(\gamma)} = \sum_{i=1}^{m} \gamma_i L_{e_i} = \sum_{i=1}^{m} \gamma_i \lambda_i^{-1} \langle e_i, \cdot\rangle \qquad P_0\text{-a.s.}$$

for all $\gamma \in \mathbf{R}^m$; see the proof of Theorem 2.1. Let f: $\mathbf{R}^m \to H$ be a Borel measurable statistic with $f \circ \alpha = E_{\beta(\gamma)}(S_1 | \alpha)$ $P_{\beta(\gamma)}$-a.s. for all $\gamma \in \mathbf{R}^m$ (cf. [3], Lemma 4.2) and define the unbiased estimator U of γ by $U = \alpha \circ f$. Then we obtain for any $\gamma \in \mathbf{R}^m$

$$\begin{aligned}
\sum_{i=1}^{m} \int_{\mathbf{R}^m} (U_i - \gamma_i)^2 \, dN(\gamma, \Sigma) &= \sum_{i=1}^{m} E_{\beta(\gamma)} (U_i \circ \alpha - \gamma_i)^2 \\
&= \sum_{i=1}^{m} E_{\beta(\gamma)} (E_{\beta(\gamma)} \langle e_i, S_1\rangle | \alpha) - \gamma_i)^2 \\
&\leqslant \sum_{i=1}^{m} E_{\beta(\gamma)} E_{\beta(\gamma)} ((\langle e_i, S_1\rangle - \gamma_i)^2 | \alpha) \\
&\leqslant E_{\beta(\gamma)} |S_1 - \beta(\gamma)|^2.
\end{aligned}$$

Since $\beta(\gamma_1) = \vartheta_1$, this yields for γ_1

$$\sum_{i=1}^{m} \int_{R^m} (U_i - \gamma_{1i})^2 \, dN(\gamma_1, \Sigma) \leqslant E_{\vartheta_1} |S_1 - \vartheta_1|^2 \leqslant c - \varepsilon \leqslant \sum_{i=1}^{m} \lambda_i - \varepsilon/2.$$

The assertion concerning the uniqueness of $\bar{X}$ is an immediate consequence of the following extension of the covariance method of Lehmann-Scheffé and Rao. Let

$$S \in \bigcap_{\vartheta \in H} L_H^2(P_\vartheta)$$

be an unbiased estimator of ϑ. Then S is an optimal unbiased estimator if and only if $E_\vartheta \langle S, g \rangle = 0$ for all $\vartheta \in H$ and

$$g \in \bigcap_{\vartheta \in H} L_H^2(P_\vartheta)$$

with $E_\vartheta g = 0$, $\vartheta \in H$.

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