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MOMENTS AND GENERALIZED CONVOLUTIONS

BY

K. URBANIK (WROCŁAW)

Abstract. There are established basic inequalities for moments of generalized convolutions of probability measures. Moreover, some necessary and sufficient conditions for the existence of moments of the characteristic measure are given.

1. Generalized convolutions were introduced in [3]. Let us recall some concepts and definitions.

We denote by P the set of all probability measures defined on Borel subsets of the positive half-line R_+. The set P is endowed with the topology of weak convergence. For $\mu \in P$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1}E)$ for all Borel subsets E of R_+. By δ_c we denote the probability measure concentrated at the point c.

A continuous in each variable separately commutative and associative P-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_a $(a > 0)$ with δ_0 as the unit element. Moreover, the key axiom postulates the existence of norming constants c_n and a measure $\gamma \in P$ other than δ_0 such that $T_{c_n} \delta_1^{\circ n} \to \gamma$, where $\delta_1^{\circ n}$ is the n-th power of δ_1 under $\circ$. The measure γ is called the *characteristic measure* of $\circ$. It is defined uniquely up to a scale change T_a $(a > 0)$.

The set P with the operation $\circ$ and the operations of convex combinations is called a *generalized convolution algebra*. Generalized convolution algebras admitting a non-constant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations are called *regular*. All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular.

Given a positive real number α, for any $\mu \in P$ we put

$$m_\alpha(\mu) = \int_0^\infty x^\alpha \mu(dx), \qquad m_\alpha^*(\mu) = \int_0^\infty x^\alpha \log x \mu(dx).$$

By monotone convergence theorem the maps $\alpha \to m_\alpha(\mu)$ and $\alpha \to m_\alpha^*(\mu)$ are continuous on the left for any $\mu \in P$. Moreover,

$$(\alpha-\beta)^{-1}\big(m_\alpha(\mu)-m_\beta(\mu)\big) \to m_\alpha^*(\mu), \tag{1}$$

whenever $m_\beta(\mu) < \infty$ and $\beta \to \alpha-$. Furthermore, by formula (2.13) in [6],

$$m_\alpha(\mu \circ \nu) = \int_0^\infty \int_0^\infty m_\alpha(\delta_x \circ \delta_y)\,\mu(dx)\,\nu(dy), \tag{2}$$

$$m_\alpha^*(\mu \circ \nu) = \int_0^\infty \int_0^\infty m_\alpha^*(\delta_x \circ \delta_y)\,\mu(dx)\,\nu(dy) \tag{3}$$

for all $\mu, \nu \in P$.

We denote by P_α and P_α^* the subsets of P consisting of all μ fulfilling the conditions $m_\alpha(\mu) < \infty$ and $|m_\alpha^*(\mu)| < \infty$, respectively. It is clear that P_α and P_α^* are invariant under maps T_a $(a > 0)$.

A homeomorphic map $\mu \to \hat{\mu}$ from P into the set of continuous bounded real-valued functions on R_+ with the topology of uniform convergence on every compact subset is said to be a *characteristic function* of the generalized convolution $\circ$ if

$$(c\mu+(1-c)\nu)\hat{} = c\hat{\mu}+(1-c)\hat{\nu} \qquad (0 \leqslant c \leqslant 1),$$
$$\widehat{\mu \circ \nu} = \hat{\mu}\hat{\nu} \quad \text{and} \quad \widehat{T_a\mu(t)} = \hat{\mu}(at) \qquad (a > 0)$$

for all $\mu, \nu \in P$.

It has been proved in [3] (Theorem 6) that a generalized convolution algebra admits a characteristic function if and only if it is regular. By Theorem 2.1 in [5] the characteristic function is unique up to a scale change. Moreover, it is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\,\mu(dx)$$

with a continuous kernel Ω with the properties $|\Omega(t)| \leqslant 1$ $(t \in R_+)$ and $\Omega(t) = 1 - t^\varkappa L(t)$, where $\varkappa > 0$ and the function L is slowly varying at the origin. The constant $\varkappa$ is called the *characteristic exponent* of the generalized convolution $\circ$.

Changing the scale if necessary and taking into account Theorem 7 in [3] we may assume without loss of generality that the characteristic function of the characteristic measure γ is given by the formula

$$\hat{\gamma}(t) = \exp(-t^\varkappa). \tag{4}$$

Then, by Lemma 1 in [2],

$$\lim_{t\to 0+} \frac{1-\Omega(t)}{t^\varkappa} = m_\varkappa(\gamma)^{-1} \tag{5}$$

which yields

$$\lim_{t\to 0+}\frac{1-\hat{\mu}(t)}{t^{\varkappa}} = m_{\varkappa}(\mu)\,m_{\varkappa}(\gamma)^{-1} \tag{6}$$

for any $\mu\in P_{\varkappa}$ other than δ_0.

For every k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ from P we put

$$\Phi(\mu_1, \mu_2, \ldots, \mu_k; t) = \prod_{j=1}^{k}\left(1-\hat{\mu}_j(t)\right). \tag{7}$$

Since, by Proposition 1.3 in [5], for any $\mu, \nu\in P$,

$$\int_0^\infty \hat{\mu}(tx)\,\nu(dx) = \int_0^\infty \hat{\nu}(tx)\,\mu(dx),$$

we have, by (4),

$$\begin{aligned}\int_0^\infty \Phi(\mu_1, \mu_2, \ldots, \mu_k; tx)\,\gamma(dx) \\ = 1+\sum_{r=1}^{k}(-1)^r \sum_{i_1, i_2, \ldots, i_r}\int_0^\infty \exp(-t^{\varkappa}x^{\varkappa})(\mu_{i_1}\circ\mu_{i_2}\circ\ldots\circ\mu_{i_r})(dx),\end{aligned} \tag{8}$$

where the summation $\sum\limits_{i_1, i_2, \ldots, i_r}$ runs over all r-element subsets $\{i_1, i_2, \ldots, i_r\}$ of the set of indices $\{1, 2, \ldots, k\}$.

Given $\alpha\neq\varkappa, 2\varkappa, \ldots, (k-1)\varkappa$, $0<\alpha<k\varkappa$, we put for any k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ from P

$$\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k) = \frac{\varkappa}{\Gamma\left(-\dfrac{\alpha}{\varkappa}\right)}\int_0^\infty \Phi(\mu_1, \mu_2, \ldots, \mu_k; t)\,t^{-\alpha-1}\,dt. \tag{9}$$

If $\mu_1, \mu_2, \ldots, \mu_k\in P_{\varkappa}$, then, by (6),

$$|\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k)| < \infty \tag{10}$$

and, by formula

$$\lim_{\alpha\to r\varkappa-}\left(r-\frac{\alpha}{\varkappa}\right)\Gamma\left(-\frac{\alpha}{\varkappa}\right) = \frac{(-1)^r}{r!} \quad (r = 1, 2, \ldots), \tag{11}$$

$$\lim_{\alpha\to r\varkappa-}\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k) = 0 \quad (r = 1, 2, \ldots, k-1).$$

Moreover, if $\mu_1, \mu_2, \ldots, \mu_k\neq\delta_0$, then

$$\lim_{\alpha\to k\varkappa-}\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k) = (-1)^k k!\,m_{\varkappa}(\gamma)^{-k}\prod_{j=1}^{k} m_{\varkappa}(\mu_j).$$

The above relations enable us to define

$$\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k) \quad (\mu_1, \mu_2, \ldots, \mu_k \in P_\varkappa;\ \mu_1, \mu_2, \ldots, \mu_k \neq \delta_0)$$

for all α satisfying the inequality $0 < \alpha \leqslant k\varkappa$ by setting

$$\varphi_{r\varkappa}(\mu_1, \mu_2, \ldots, \mu_k) = 0 \quad (r = 1, 2, \ldots, k-1) \tag{12}$$

and

$$\varphi_{k\varkappa}(\mu_1, \mu_2, \ldots, \mu_k) = (-1)^k k!\, m_\varkappa(\gamma)^{-k} \prod_{j=1}^{k} m_\varkappa(\mu_j). \tag{13}$$

The map $\alpha \to \varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k)$ is then continuous on the left.

Suppose that $\mu, \nu \in P$. Since $\widehat{\mu \circ \nu} = \hat{\mu}\hat{\nu}$, we have, by (9) and (10),

$$\varphi_\alpha(\mu \circ \nu) = \varphi_\alpha(\mu) + \varphi_\alpha(\nu) - \varphi_\alpha(\mu, \nu) \quad (0 < \alpha < \varkappa). \tag{14}$$

Moreover, $\varphi_\alpha(\delta_x) = x^\alpha m_\alpha(\delta_1)$ and, by Fubini's Theorem,

$$\varphi_\alpha(\mu) = \int_0^\infty \varphi_\alpha(\delta_x)\, \mu(dx) = \varphi_\alpha(\delta_1)\, m_\alpha(\mu) \quad (0 < \alpha < \varkappa). \tag{15}$$

Consequently, by (4),

$$\varphi_\alpha(\delta_1)\, m_\alpha(\gamma) = \varphi_\alpha(\gamma) = \frac{\varkappa}{\Gamma\left(\frac{\alpha}{\varkappa}\right)} \int_0^\infty \frac{1 - \exp(-t^\varkappa)}{t^{1+\alpha}}\, dt = -1 \quad (0 < \alpha < \varkappa), \tag{16}$$

which yields

$$\varphi_\alpha(\delta_1) = -m_\alpha(\gamma)^{-1} \quad (0 < \alpha < \varkappa) \tag{17}$$

and $\gamma \in P$ $(0 < \alpha < \varkappa)$. Formula (16) has been proved in [1], p. 119.

2. THEOREM 1. *Let* $\mu, \nu \in P$. *Then*

$$m_\alpha(\mu \circ \nu) \leqslant m_\alpha(\mu) + m_\alpha(\nu) \quad \text{if } 0 < \alpha < \varkappa, \tag{18}$$

$$m_\varkappa(\mu \circ \nu) = m_\varkappa(\mu) + m_\varkappa(\nu) \tag{19}$$

and

$$m_\alpha(\mu \circ \nu) \geqslant m_\alpha(\mu) + m_\alpha(\nu) \quad \text{if } \alpha > \varkappa. \tag{20}$$

Proof. Suppose that $0 < \alpha < \varkappa$. Then, by (9), $\varphi_\alpha(\delta_x, \delta_y) \leqslant 0$ for all x, y and, by (16), $\varphi_\alpha(\delta_1) < 0$. Since, by (14) and (15),

$$m_\alpha(\delta_x \circ \delta_y) = x^\alpha + y^\alpha - \frac{\varphi_\alpha(\delta_x, \delta_y)}{\varphi_\alpha(\delta_1)}, \tag{21}$$

we have

$$m_\alpha(\delta_x \circ \delta_y) \leqslant x^\alpha + y^\alpha,$$

which, by (2), yields inequality (18). Further, by (5),

$$\int_0^\infty \varphi(\delta_x, \delta_y)\, t^{-\alpha-1}\, dt$$

is bounded on the interval $0 < \alpha < \varkappa$. Since

$$\int_0^\infty (1-\Omega(t))\, t^{-\alpha-1}\, dt \to \infty \qquad \text{as } \alpha \to \varkappa-$$

([4], p. 61), we have, by (9), $\varphi_\alpha(\delta_x, \delta_y)/\varphi_\alpha(\delta_1) \to 0$ as $\alpha \to \varkappa-$. Consequently, from (21) we get the equation

(22) $$m_\varkappa(\delta_x \circ \delta_y) = x^\varkappa + y^\varkappa,$$

which, by (2), yields (19).

Consider the case $\alpha > \varkappa$. We have then the inequality

$$m_\alpha(\delta_x \circ \delta_y)^{1/\alpha} \geqslant m_\varkappa(\delta_x \circ \delta_y)^{1/\varkappa},$$

which, by (22), implies $m_\alpha(\delta_x \circ \delta_y) \geqslant x^\alpha + y^\alpha$. Now (20) is a direct consequence of (2). This completes the proof.

From Theorem 1 we get the following statement:

COROLLARY 1. *If* $0 < \alpha \leqslant \varkappa$, *then the sets* P_α *are closed under the generalized convolution* $\circ$.

PROPOSITION 1. *Let* $\mu, \nu \in P_\varkappa^*$ *and* $\mu, \nu \neq \delta_0$. *Then*

$$m_\varkappa^*(\mu \circ \nu) = m_\varkappa^*(\mu) + m_\varkappa^*(\nu) + m_\varkappa(\gamma) \int_0^\infty \Phi(\mu, \nu; t)\, t^{-\varkappa-1}\, dt.$$

Proof. Since $P_\varkappa \subset P_\varkappa^*$, $\hat{\mu}(t) < 1$ and $\hat{\nu}(t) < 1$ for small enough positive t ([3], Theorem 5), we infer, by virtue of (6) and (7), that

$$0 < \int_0^\infty \Phi(\mu, \nu; t)\, t^{-\varkappa-1}\, dt < \infty.$$

By Theorem 1 (formula (19)) for $\alpha < \varkappa$ we have

$$m_\alpha(\mu \circ \nu) - m_\alpha(\mu) - m_\alpha(\nu)$$
$$= m_\alpha(\mu \circ \nu) - m_\varkappa(\mu \circ \nu) + m_\varkappa(\mu) - m_\alpha(\mu) + m_\varkappa(\nu) - m_\alpha(\nu)$$

which, by (1), yields

(23) $$(\varkappa-\alpha)^{-1}\big(m_\alpha(\mu \circ \nu) - m_\alpha(\mu) - m_\alpha(\nu)\big)$$
$$\to m_\varkappa^*(\mu) + m_\varkappa^*(\nu) - m_\varkappa^*(\mu \circ \nu) \qquad \text{as } \alpha \to \varkappa-.$$

On the other hand, by (14), (15) and (17), we have

$$m_\alpha(\mu\circ\nu)-m_\alpha(\mu)-m_\alpha(\nu)=m_\alpha(\gamma)\,\varphi_\alpha(\mu,\nu).$$

Since, by (9) and (11),

$$(\varkappa-\alpha)^{-1}\varphi_\alpha(\mu,\nu)\to-\int_0^\infty \Phi(\mu,\nu;t)\,t^{-\varkappa-1}\,dt \qquad \text{as } \alpha\to\varkappa-,$$

the above equation together with (23) yield our assertion, which completes the proof.

As a consequence of Proposition 1 we obtain the following statements.

COROLLARY 2. *If* $\gamma\in P_\varkappa$, *then* $P_\varkappa^*$ *is closed under the generalized convolution* $\circ$.

COROLLARY 3. *If there exists a pair* $\mu,\nu\in P$ *such that* $\mu,\nu\neq\delta_0$ *and* $\mu\circ\nu\in P_\varkappa^*$, *then* $\gamma\in P_\varkappa$.

Corollaries 2 and 3 yield

COROLLARY 4. $\gamma\in P_\varkappa$ *if and only if the set* $P_\varkappa^*$ *is closed under the generalized convolution* $\circ$.

COROLLARY 5. *If there exists a pair* $\mu,\nu\in P$ *such that* $\mu,\nu\neq\delta_0$ *and* $\mu\circ\nu\in P_\varkappa^*$, *then* $P_\varkappa^*$ *is closed under the generalized convolution* $\circ$.

Let $k=1,2,\ldots$ and $\alpha\geqslant\varkappa$. For every integer r and every k-tuple $\mu_1,\mu_2,\ldots,\mu_k$ from P we put

$$m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k)=0 \tag{24}$$

if either $r<1$ or $r>k$ and

$$m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k)=\sum_{i_1,i_2,\ldots,i_r} m_\alpha(\mu_{i_1}\circ\mu_{i_2}\circ\ldots\circ\mu_{i_r}), \tag{25}$$

if $1\leqslant r\leqslant k$, where the summation is extended over all r-element subsets $\{i_1,i_2,\ldots,i_r\}$ of the set of indices $\{1,2,\ldots,k\}$. Obviously,

$$m_{\alpha,k}(\mu_1,\mu_2,\ldots,\mu_k)=m_\alpha(\mu_1\circ\mu_2\circ\ldots\circ\mu_k) \tag{26}$$

and, by Theorem 1 (formula (19)),

$$m_{\varkappa,r}(\mu_1,\mu_2,\ldots,\mu_k)=\binom{k-1}{r-1}m_\varkappa(\mu_1\circ\mu_2\circ\ldots\circ\mu_k) \qquad (1\leqslant r\leqslant k). \tag{27}$$

By Theorem 1 we have also the inequality

$$m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k)\leqslant\binom{k}{r}m_{\alpha,q}(\mu_1,\mu_2,\ldots,\mu_k) \qquad \text{if } r\leqslant q\leqslant k. \tag{28}$$

Further, by monotone convergence theorem the map

$$\alpha\to m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k)$$

is continuous on the left. Taking into account (2) we easily get the following equation:

$$(29)\quad m_{\alpha,r+1}(\mu_1,\mu_2,\ldots,\mu_k,\mu_{k+1})+m_{\alpha,r-1}(\mu_1,\mu_2,\ldots,\mu_{k-1})+ \\ +m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_{k-1}) \\ = m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_{k-1},\mu_k)+m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_{k-1},\mu_{k+1})+ \\ +\int_0^\infty m_{\alpha,r-1}(\mu_1,\mu_2,\ldots,\mu_{k-1},\delta_x)(\mu_k\circ\mu_{k+1})(dx).$$

LEMMA 1. *Given $\alpha \geqslant \varkappa$ and $k \geqslant 2$. Suppose that there exists a constant c_α such that*

$$(30)\quad \sum_{r=1}^{k}(-1)^r m_{\alpha,r}(\nu_1,\nu_2,\ldots,\nu_k) = c_\alpha\prod_{j=1}^{k} m_\varkappa(\nu_\gamma)$$

for every k-tuple $\nu_1,\nu_2,\ldots,\nu_k$ with $\nu_1\circ\nu_2\circ\ldots\circ\nu_k\in P_\alpha$. Then for every $n>k$

$$(31)\quad \sum_{r=1}^{n}(-1)^r m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_n) = 0$$

whenever $\mu_1\circ\mu_2\circ\ldots\circ\mu_n\in P_\alpha$.

Proof. It suffices to prove (31) for $n=k+1$. Suppose that

$$\mu_1\circ\mu_2\circ\ldots\circ\mu_{k+1}\in P_\alpha.$$

Then, by Theorem 1 (formulae (19) and (20)), we have

$$\mu_1\circ\mu_2\circ\ldots\circ\mu_k\in P_\alpha,\qquad \mu_1\circ\mu_2\circ\ldots\circ\mu_{k-1}\circ\mu_{k+1}\in P_\alpha$$

and, by (2),

$$\mu_1\circ\mu_2\circ\ldots\circ\mu_{k-1}\circ\delta_x\in P_\alpha$$

for $\mu_k\circ\mu_{k+1}$-almost all x. Thus, by assumption (30),

$$\sum_{r=1}^{k}(-1)^r m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k) = c_\alpha\prod_{j=1}^{k} m_\varkappa(\mu_j),$$

$$\sum_{r=1}^{k}(-1)^r m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_{k-1},\mu_{k+1}) = c_\alpha m_\varkappa(\mu_{k+1})\prod_{j=1}^{k-1} m_\varkappa(\mu_j)$$

and

$$\sum_{r=1}^{k}(-1)^r m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_{k-1},\delta_x) = c_\alpha x^\varkappa\prod_{j=1}^{k-1} m_\varkappa(\mu_j)$$

for $\mu_k\circ\mu_{k+1}$-almost all x. Now applying (24) and (29) we get

$$\sum_{r=1}^{k+1}(-1)^r m_{\alpha,r}(\mu_1,\mu_2,\ldots,\mu_k,\mu_{k+1}) \\ = c_\alpha\big(m_\varkappa(\mu_k)+m_\varkappa(\mu_{k+1})\big)\prod_{j=1}^{k-1} m_\varkappa(\mu_j)-c_\alpha m_\varkappa(\mu_k\circ\mu_{k+1})\prod_{j=1}^{k-1} m_\varkappa(\mu_j)$$

which, by Theorem 1 (formula (19)), yields (31) for $n = k+1$. This completes the proof.

LEMMA 2. *Given $k \geqslant 2$ and $(k-1)\varkappa < \alpha \leqslant k\varkappa$. Then, for every k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ from P with the properties $\mu_1, \mu_2, \ldots, \mu_k \neq \delta_0$,*

$$\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_{(k-1)\varkappa}, \tag{32}$$

$$m_{\alpha,k-1}(\mu_1, \mu_2, \ldots, \mu_k) < \infty \quad \text{if } \alpha \neq 2\varkappa, \tag{33}$$

and

$$\mu_1 \circ \mu_2 \in P_\varkappa^* \quad \text{if } \alpha = 2\varkappa, \tag{34}$$

we have

$$0 < |\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k)| < \infty \tag{35}$$

and

$$\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k) m_\alpha(\gamma) = \sum_{r=1}^{k} (-1)^r m_{\alpha,r}(\mu_1, \mu_2, \ldots, \mu_k). \tag{36}$$

Proof. From assumption (32) and Theorem 1 it follows that $\mu_j \in P_\varkappa$ $(j = 1, 2, \ldots, k)$ which, by (10) and (13), yields $|\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k)| < \infty$. Since $\hat{\mu}_j(t) < 1$ $(j = 1, 2, \ldots, k)$ for small enough positive t ([3], Theorem 5), we infer, by (7) and (9), that $|\varphi_\alpha(\mu_1, \mu_2, \ldots, \mu_k)| > 0$ provided $(k-1)\varkappa < \alpha < k\varkappa$.

Finally, consider the case $\alpha = k\varkappa$. Then, by (33) and (34), $\mu_1 \circ \mu_2 \in P_\varkappa^*$ which, by Corollary 3, yields $\gamma \in P_\varkappa$. Consequently, we get from (13) the inequality $|\varphi_{k\varkappa}(\mu_1, \mu_2, \ldots, \mu_k)| > 0$ because $m_\varkappa(\mu_j) > 0$ $(j = 1, 2, \ldots, k)$. Inequality (35) is thus proved.

Taking into account (26) and (28), we have the inequalities

$$m_{j\varkappa,r}(\mu_1, \mu_2, \ldots, \mu_k) \leqslant \binom{k}{r} m_{j\varkappa}(\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k)$$

$$\leqslant \binom{k}{r} m_{(k-1)\varkappa}(\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k)^{j/(k-1)} \quad (j = 1, 2, \ldots, k-1;\ r = 1, 2, \ldots, k)$$

which, by (32), yield

$$m_{j\varkappa,r}(\mu_1, \mu_2, \ldots, \mu_k) < \infty \quad (j = 1, 2, \ldots, k-1;\ r = 1, 2, \ldots, k).$$

Put

$$a_j(\mu_1, \mu_2, \ldots, \mu_k) = \sum_{r=1}^{k} (-1)^r m_{j\varkappa,r}(\mu_1, \mu_2, \ldots, \mu_k) \qquad (j = 1, 2, \ldots, k-1). \tag{37}$$

By (27) the equation

$$a_1(\mu_1, \mu_2, \ldots, \mu_k) = 0 \tag{38}$$

holds.

Let us introduce the notation

(39) $$f(\mu_1, \mu_2, \ldots, \mu_k; t) = \sum_{r=1}^{k} (-1)^r \sum_{i_1, i_2, \ldots, i_r} \int_0^\infty \left(\exp(-t^\varkappa x^\varkappa) - \sum_{j=0}^{k-1} (-1)^j \frac{t^{j\varkappa} x^{j\varkappa}}{j!}\right) (\mu_{i_1} \circ \mu_{i_2} \circ \ldots \circ \mu_{i_r})(dx)$$

where the summation $\sum_{i_1, i_2, \ldots, i_r}$ runs over all r-element subsets $\{i_1, i_2, \ldots, i_r\}$ of the set of indices $\{1, 2, \ldots, k\}$. By (8) we have

(40) $$\int_0^\infty \Phi(\mu_1, \mu_2, \ldots, \mu_k; t)\gamma(dx) = f(\mu_1, \mu_2, \ldots, \mu_k; t) + \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} a_j(\mu_1, \mu_2, \ldots, \mu_k) t^{j\varkappa}.$$

Further, by (28), (33) and (34),

(41) $$m_{\alpha,r}(\mu_1, \mu_2, \ldots, \mu_k) < \infty \qquad (r = 1, 2, \ldots, k-1).$$

Consequently, for any β satisfying the condition $(k-1)\varkappa < \beta < \alpha$ we get

(42) $$\int_0^\infty f(\mu_1, \mu_2, \ldots, \mu_k; t) t^{-\beta-1}\, dt = \frac{1}{\varkappa} \Gamma\left(-\frac{\beta}{\varkappa}\right) \sum_{r=1}^{k} (-1)^r m_{\beta,r}(\mu_1, \mu_2, \ldots, \mu_k).$$

We shall prove (36) by induction with respect to k. First consider the case $k = 2$. Since, by (38), $a_1(\mu_1, \mu_2) = 0$, formula (40) can be rewritten in the form

$$\int_0^\infty \Phi(\mu_1, \mu_2; tx)\gamma(dx) = f(\mu_1, \mu_2; t)$$

which, by (9) and (42), gives

$$\varphi_\beta(\mu_1, \mu_2) m_\beta(\gamma) = m_{\beta,2}(\mu_1, \mu_2) - m_{\beta,1}(\mu_1, \mu_2)$$

for any β from the interval $\varkappa < \beta < \alpha$. Since all maps $\beta \to \varphi_\beta(\mu_1, \mu_2)$, $\beta \to m_\beta(\gamma)$ and $\beta \to m_{\beta,r}(\mu_1, \mu_2)$ are continuous on the left, the above equation and (41) yield (as $\beta \to \alpha-$) formula (36) for all α from the interval $\varkappa < \alpha \leqslant 2\varkappa$.

Suppose now that $k > 2$ and (36) is true for all indices less than k. Then in particular, for any j-tuple $\nu_1, \nu_2, \ldots, \nu_j$ from P satisfying the conditions $\nu_1, \nu_2, \ldots, \nu_j \neq \delta_0$ and $\nu_1 \circ \nu_2 \circ \ldots \circ \nu_j \in P_{j\varkappa}$, we have

$$\varphi_{j\varkappa}(\nu_1, \nu_2, \ldots, \nu_j) m_{j\varkappa}(\gamma) = \sum_{r=1}^{j} (-1)^r m_{j\varkappa,r}(\nu_1, \nu_2, \ldots, \nu_j) \qquad (j = 2, 3, \ldots, k-1)$$

which, by (13), can be rewritten in the form

$$\sum_{r=1}^{j}(-1)^r m_{j\varkappa,r}(\nu_1, \nu_2, \ldots, \nu_j)$$
$$= (-1)^j j!\, m_\varkappa(\gamma)^{-j}\prod_{i=1}^{j} m_\varkappa(\nu_i) \qquad (j = 2, 3, \ldots, k-1).$$

Taking into account (32) and applying Lemma 1 we get the following equations:

$$\sum_{r=1}^{k}(-1)^r m_{j\varkappa,r}(\mu_1, \mu_2, \ldots, \mu_k) = 0 \qquad (j = 2, 3, \ldots, k-1).$$

Thus, by (37), (38) and (40),

$$\int_0^\infty \Phi(\mu_1, \mu_2, \ldots, \mu_k; tx)\gamma(dx) = f(\mu_1, \mu_2, \ldots, \mu_k; t)$$

which, by (9) and (42), yields

$$\varphi_\beta(\mu_1, \mu_2, \ldots, \mu_k)m_\beta(\gamma) = \sum_{r=1}^{k}(-1)^r m_{\beta,r}(\mu_1, \mu_2, \ldots, \mu_k)$$

for any β from the interval $(k-1)\varkappa < \beta < \alpha$. Now taking into account (41) and the continuity on the left of maps $\beta \to \varphi_\beta(\mu_1, \mu_2, \ldots, \mu_k)$, $\beta \to m_\beta(\gamma)$ and $\beta \to m_{\beta,r}(\mu_1, \mu_2, \ldots, \mu_k)$ we get (36) for all α from the interval $(k-1)\varkappa < \alpha \leqslant k\varkappa$, which completes the proof.

As a direct consequence of Lemma 2 we get the following statement:

Corollary 6. *Let* $k \geqslant 3$, $(k-1)\varkappa < \alpha \leqslant k\varkappa$ *and* $\gamma \in P_\alpha$. *If* $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_{(k-1)\varkappa}$ *and* $m_{\alpha,k-1}(\mu_1, \mu_2, \ldots, \mu_k) < \infty$, *then* $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_\alpha$.

In fact, the k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ fulfils then conditions of Lemma 2. Since the left-hand side of (36) and

$$\sum_{r=1}^{k-1}(-1)^r m_{\alpha,r}(\mu_1, \mu_2, \ldots, \mu_k)$$

are finite, we infer that $m_{\alpha,k}(\mu_1, \mu_2, \ldots, \mu_k) < \infty$, which, by (26), yields our assertion.

Furthermore, from Lemmas 1 and 2 and definition (13) we get the following

Corollary 7. *Let* $k \geqslant 2$ *and* $n > k$. *If* $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_n \in P_{k\varkappa}$ *and* $\mu_1, \mu_2, \ldots, \mu_n \neq \delta_0$, *then*

$$\sum_{r=1}^{n}(-1)^r m_{k\varkappa,r}(\mu_1, \mu_2, \ldots, \mu_n) = 0.$$

THEOREM 2. *Let $k \geqslant 2$ and $(k-1)\varkappa < \alpha \leqslant k\varkappa$. Then $\gamma \in P_\alpha$ if and only if there exists a k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ from P such that $\mu_1, \mu_2, \ldots, \mu_k \neq \delta_0$ and $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_\alpha$*

Proof. Suppose that $\gamma \in P_\alpha$. Then, by (4), $\gamma^{\circ k} = T_c \gamma$, where $c = k^\varkappa$. Consequently, $\gamma^{\circ k} \in P_\alpha$ which proves the necessity of the condition. To prove the sufficiency let us assume that $\mu_1, \mu_2, \ldots, \mu_k \neq \delta_0$ and $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_\alpha$. It is clear that the k-tuple $\mu_1, \mu_2, \ldots, \mu_k$ fulfils the conditions of Lemma 2 and, by (26) and (28), the right-hand side of (36) is finite. Thus, by (35), $\gamma \in P_\alpha$ which completes the proof.

THEOREM 3. *Let $\varkappa < \alpha \leqslant 2\varkappa$. Then $\gamma \in P_\alpha$ if and only if the set P_α is closed under the generalized convolution $\circ$.*

Proof. The sufficiency of our condition follows immediately from Theorem 2. To prove the necessity we assume that $\gamma \in P_\alpha$. Consequently, $\gamma \in P_\varkappa$ and, by Corollary 2, $P_\varkappa^*$ is closed under the generalized convolution $\circ$. Thus $\delta_x \circ \delta_y \in P_\varkappa^*$ for any $x, y \in R_+$. This shows that the pair δ_x, δ_y $(x, y > 0)$ fulfils the conditions of Lemma 2, which yields the equation

$$\varphi_\alpha(\delta_x, \delta_y)\, m_\alpha(\gamma) = m_\alpha(\delta_x \circ \delta_y) - x^\alpha - y^\alpha$$

for positive x and y. If either $x = 0$ or $y = 0$, then the above equation is evident. Hence, by virtue of (2), we get the formula

$$m_\alpha(\mu \circ \nu) = m_\alpha(\mu) + m_\alpha(\nu) + \varphi_\alpha(\mu, \nu)\, m_\alpha(\gamma)$$

for any pair μ, ν from P_α. This shows that P_α is closed under $\circ$ which completes the proof.

From Theorems 2 and 3 we get

COROLLARY 8. *Let $\varkappa < \alpha \leqslant 2\varkappa$. If there exists a pair μ, ν with the properties $\mu, \nu \neq \delta_0$ and $\mu \circ \nu \in P_\alpha$, then P_α is closed under the generalized convolution $\circ$.*

THEOREM 4. *Let $k \geqslant 2$ and $(k-1)\varkappa < \alpha \leqslant k\varkappa$. If $\mu^{\circ k} \in P_\alpha$, then $\mu^{\circ n} \in P_\alpha$ for all $n = 1, 2, \ldots$*

Proof. Suppose the contrary. Let k be the least integer for which our assertion is false for a certain α from the interval $(k-1)\varkappa < \alpha \leqslant k\varkappa$. By Corollary 8 and inequality (20) we conclude that

$$k \geqslant 3. \tag{43}$$

Let

$$\mu^{\circ k} \in P_\alpha \tag{44}$$

and

$$\mu^{\circ n} \notin P_\alpha. \tag{45}$$

Obviously $\mu \neq \delta_0$ and, by (20), $n > k$. Without loss of generality we may assume that

$$\mu^{\circ(n-1)} \in P_\alpha. \tag{46}$$

Since our statement is true for the integer $k-1$ and $P_{(k-1)\varkappa} \subset P_\alpha$, we have $\mu^{\circ n} \in P_{(k-1)\varkappa}$. Put $\mu_1 = \mu_2 = \ldots = \mu_{k-1} = \mu$ and $\mu_k = \mu^{\circ(n+1-k)}$. Then

$$\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k = \mu^{\circ n} \in P_{(k-1)\varkappa}. \tag{47}$$

Further, $m_\alpha(\mu_{i_1} \circ \mu_{i_2} \circ \ldots \circ \mu_{i_{k-1}}) \leqslant m_\alpha(\mu^{\circ(n-1)})$ for every $(k-1)$-element subset $\{i_1, i_2, \ldots, i_{k-1}\}$ of $\{1, 2, \ldots, k\}$. Thus, by (46),

$$m_{\alpha,k-1}(\mu_1, \mu_2, \ldots, \mu_k) < \infty. \tag{48}$$

From (44) and Theorem 2 it follows that $\gamma \in P_\alpha$. Consequently, by (43), (47), (48) and Corollary 6, we have $\mu^{\circ n} = \mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_\alpha$ which contradicts (45). The Theorem is thus proved.

The condition $\mu^{\circ k} \in P_\alpha$ of Theorem 4 cannot be replaced by the weaker one $\mu^{\circ(k-1)} \in P_\alpha$. In fact, for the generalized convolution $\circ_{1,1}$, defined in [5], example 1.6, we have $\varkappa = 1$ and

$$\gamma(E) = \int_E x^{-3} \exp(-x^{-1})\, dx.$$

Thus $m_\alpha(\gamma) = \Gamma(2-\alpha)$ if $0 < \alpha < 2$ and $m_\alpha(\gamma) = \infty$ otherwise. Taking $k = 2$ we have $\delta_1^{\circ(k-1)} = \delta_1 \in P_{k\varkappa}$ and, accordingly to Theorem 2, $\delta_1^{\circ k} \notin P_{k\varkappa}$.

THEOREM 5. *Let $k \geqslant 2$ and $\mu^{\circ k} \in P_{k\varkappa}$. Then for every $n > k$*

$$m_{k\varkappa}(\mu^{\circ n}) = \sum_{r=1}^{k} (-1)^{k+r} \binom{n}{r} \binom{n-r-1}{k-r} m_{k\varkappa}(\mu^{\circ r}). \tag{49}$$

Proof. Since for $\mu = \delta_0$ our statement is obvious, we may assume that $\mu \neq \delta_0$. Then, by Theorem 4, the n-tuple $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$ fulfils the conditions of Corollary 7. Thus

$$\sum_{r=1}^{n} (-1)^r \binom{n}{r} m_{k\varkappa}(\mu^{\circ r}) = 0 \qquad (n > k).$$

Solving this system of equations we obtain formula (49).

Example. Let $\mu \neq \delta_0$ and $\mu \circ \mu \in P_\varkappa^*$. Then, by Corollary 3, $\gamma \in P_\varkappa$. Put

$$c_n = \left(\frac{m_\varkappa(\gamma)}{n m_\varkappa(\mu)} \right)^{1/\varkappa} \qquad (n = 1, 2, \ldots).$$

By (6), $\left(T_{c_n} \mu^{\circ n}(t)\right)^\wedge \to \exp(-t^\varkappa)$ which, by (4), yields $T_{c_n} \mu^{\circ n} \to \gamma$. Moreover, by Theorem 1 (formula (19),

$$m_\varkappa(T_{c_n} \mu^{\circ n}) = m_\varkappa(\gamma) \qquad (n = 1, 2, \ldots).$$

Suppose in addition that $k \geqslant 2$ and $\mu^{\circ k} \in P_{k\varkappa}$. Then, by Theorem 2, $\gamma \in P_{k\varkappa}$ and, by Lemma 2 and (13),

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} m_{k\varkappa}(\mu^{\circ r}) = (-1)^k k!\, m_\varkappa(\gamma)^{-k} m_\varkappa(\mu)^k m_{k\varkappa}(\gamma). \tag{50}$$

Further, from Theorem 5 we get, as $n \to \infty$,

$$m_{k\varkappa}(T_{c_n}\mu^{\circ n}) \to \frac{(-1)^k m_\varkappa(\gamma)^k}{k!\, m_\varkappa(\mu)^k} \sum_{r=1}^{k} (-1)^r \binom{k}{r} m_{k\varkappa}(\mu^{\circ r})$$

which, by (50), yields

$$m_{k\varkappa}(T_{c_n}\mu^{\circ n}) \to m_{k\varkappa}(\gamma).$$

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Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

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