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AN ASYMPTOTIC FIXED-PRECISION CONFIDENCE INTERVAL FOR THE MINIMUM OF A QUADRATIC REGRESSION FUNCTION

BY

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Abstract. A confidence interval for the minimum of the quadratic regression $Ey = a+bx+cx^2$ with and without assumption that $c > 0$ is constructed. The solution is based on recursive estimators for regression parameters.

1. Introduction. Given a sequence of real numbers $(x_i, i \geqslant 1)$ and a sequence of i.i.d. random variables $(\varepsilon_i, i \geqslant 1)$ such that $E\varepsilon_i = 0$, $E\varepsilon_i^2 = \sigma^2$ (unknown), consider two regression models:

Model A: $y_i = a+bx_i+cx_i^2+\varepsilon_i$, $a, b \in R$, $c > 0$.

Model B: $y_i = a+bx_i+cx_i^2+\varepsilon_i$, $a, b, c \in R$.

The problem is to estimate the point $x_0 = -b/2c$ at which the regression function has its minimum. We are interested in a fixed-precision estimation: given a positive d and $\gamma \in (0, 1)$, construct a confidence interval I for x_0 such that $P(x_0 \in I) = \gamma$ and the length of I is not greater than d.

If $\hat{b}$ and $\hat{c}$ are unbiased estimators of b and c, respectively, we obtain the well-known problem of estimating the ratio of two means. Unfortunately, no satisfactory solution of the problem is known (a recent discussion for the Gaussian case can be found e.g. in Ogawa [4]).

The main idea of our solution consists in constructing simple recursive estimators B_k, C_k and S_k^2 of b, c and σ^2, respectively, which can be represented as sums of i.i.d. random variables. This enables us to apply the Chow-Robbins [2] and Anscombe [1] theories, and to construct an asymptotic solution.

In the Model A the minimum of the regression function is known to exist. The results concerning an appropriate stopping rule $N(d)$ and the confidence interval $I(N(d))$ are given in Theorem 1.

In the Model B it is not known in advance whether the regression function has any minimum at all. An appropriate modification of the problem is as follows. Call the *correct decision* the decision which is "stop sampling and construct a confidence interval" if $c > 0$ or "stop sampling without constructing any confidence interval" if $c \leqslant 0$. Our solution guarantees the probability $P(CD)$ of the correct decision to be not less than γ, asymptotically as d tends to 0, and the confidence interval to be not longer than d (if constructed).

2. Estimation of the regression parameters. The minimal number of experimental points needed for estimating all unknown parameters of the model equals four. Let us consider the sequence of quadruplets of experimental points $(x_{4k+1}, x_{4k+2}, x_{4k+3}, x_{4k+4})$, $k = 0, 1, \ldots$, and the corresponding sequence of quadruplets of observations $(y_{4k+1}, y_{4k+2}, y_{4k+3}, y_{4k+4})$, $k = 0, 1, \ldots$ Under the appropriate scaling we can choose, for the sake of computational simplicity, $x_{4k+1} = 1$, $x_{4k+2} = -1$, and $x_{4k+3} = x_{4k+4} = 0$. This choice has some optimal properties (for a discusion, see Męczarski [3]).

Given k, let $B_k = (y_{4k+1} - y_{4k+2})/2$, $C_k = (y_{4k+1} + y_{4k+2} - y_{4k+3} - y_{4k+4})/2$, $S_k^2 = (y_{4k+3} - y_{4k+4})^2/2$. Now B_k, C_k and S_k^2 are the least squares estimators of b, c and σ^2, respectively. Let us define

$$\hat{b}_n = \frac{1}{n}\sum_{k=1}^{n} B_k, \quad \hat{c}_n = \frac{1}{n}\sum_{k=1}^{n} C_k, \quad s_n^2 = \frac{1}{n}\sum_{k=1}^{n} S_k^2.$$

The estimators $\hat{b}_n$, $\hat{c}_n$ and s_n^2 are unbiased (since B_k, C_k and S_k^2 are unbiased), strongly consistent (by the strong law of large numbers) and recursively computable. The random variable

$$z_n(x_0) = \sqrt{n}\frac{\hat{b}_n + 2x_0\hat{c}_n}{(\frac{1}{2}s_n^2 + 4x_0^2 s_n^2)^{1/2}}$$

is asymptotically normal $N(0, 1)$ and satisfies the Anscombe condition. This enables us to construct the asymptotic confidence sets $I(n) = \{x_0\colon |z_n(x_0)| < \tau_n\}$, where $(\tau_n, n \geqslant 1)$ is a sequence of positive constants tending to

$$\tau = \Phi^{-1}\left(\frac{1+\gamma}{2}\right),$$

with Φ being the cumulative distribution function of $N(0, 1)$.

Observe that $I(n)$ is a bounded interval if $\hat{c}_n^2 - \tau_n^2 s_n^2/n > 0$, and its length $\delta(n)$ is given by the formula

$$\delta(n) = \frac{s_n}{\sqrt{n}}\tau_n\frac{\sqrt{\hat{b}_n^2 + \frac{1}{2}\hat{c}_n^2 - \frac{1}{2}\tau_n^2 s_n^2/n}}{c_n^2 - \tau_n^2 s_n^2/n}.$$

We define $\delta(n) = \infty$, if $\hat{c}_n \leqslant \tau_n s_n/\sqrt{n}$.

3. Solution for Model A. Given $d > 0$, we define the stopping rule $N = N(d)$ as follows:

$$N = N(d) = \inf\{n \geqslant 1:\ \delta(n) < d\}.$$

THEOREM 1. *Under the above assumptions:*
(i) $P(N(d) < \infty) = 1$ *for all* $d > 0$ *and all* $c > 0$;
(ii) $\lim_{d \to 0} P(x_0 \in I(N(d))) = \gamma$ *for all* $c > 0$.

Proof. The proof is based on the following

LEMMA (Chow and Robbins [2]). *Let* $(y_n,\ n \geqslant 1)$ *be a sequence of random variables such that* $y_n > 0$ *a.s.,* $\lim_{n \to \infty} y_n = 1$ *a.s., and let*

$$N(t) = \inf\left\{n \geqslant 1:\ y_n \leqslant \frac{1}{t} f(n)\right\},$$

where $(f(n),\ n \geqslant 1)$ *is a sequence of positive reals diverging to* $+\infty$ *and such that* $f(n)/f(n-1) \to 1$ *as* $n \to \infty$. *Then*
(a) $P(N(t) < \infty) = 1$ *for every* t;
(b) $N(t) \to \infty$ *as* $t \to \infty$ *a.s.*;
(c) $\lim_{t \to \infty} \frac{f(N(t))}{t} = 1$ *a.s.*

Observe that

$$L = \lim_{n \to \infty} \sqrt{n}\,\delta(n) = \frac{\tau\sigma}{c^2}\sqrt{b^2 + \frac{1}{2}c^2}$$

and define $y_n = \sqrt{n}\,\delta(n)/L$. By part (a) of the Lemma we obtain (i). Using part (c) of the Lemma with $f(n) = \sqrt{n}/L$, and the Anscombe [1] theorem, we obtain (ii).

4. Solution for Model B. If $c \leqslant 0$, then minimum does not exist and, by the definition of $\delta(n)$, the stopping rule $N(d)$ is infinite with a positive probability. Define a new stopping rule $T = T(d)$ by a censoring of $N(d)$ as

$$T = T(d) = \min\{N(d),\ M(d)\}, \qquad d > 0,$$

where $d \to M(d)$, $d > 0$, is a function with values in the set of positive integers. Now we stop at the time T and the decision is "$x_0 \in I(N(d))$", if $T(d) = N(d)$, or "no minimum exists", if $T(d) < N(d)$. The probability of the correct decision is given by the formula

$$P(CD) = \begin{cases} P\{N(d) \leqslant M(d),\ x_0 \in I(N(d))\}, & \text{if } c > 0. \\ P\{N(d) > M(d)\}, & \text{if } c \leqslant 0. \end{cases}$$

Observe that, for $c > 0$, $P\{x_0 \in I(N(d))\} \geqslant P(CD)$.

THEOREM 2. *Let $M(d)$ be chosen in such a way that*

(a) $\lim_{d\to 0}[d^2 M(d)] = \infty$, (b) $\lim_{d\to 0}[d^2 \log\log M(d)] = 0$;

then

$$\lim_{d\to 0} P(CD) = \gamma \quad \textit{for } c > 0, \tag{1}$$

$$\lim_{d\to 0} P(CD) = 1 \quad \textit{for } c \leqslant 0. \tag{2}$$

Proof. Let $c > 0$. By part (c) of Lemma, $d^2 N(d) \to L^2$ a.s., and hence

$$\lim_{d\to 0} P(CD) = \lim_{d\to 0} P\{d^2 N(d) \leqslant d^2 M(d),\ x_0 \in I(N(d))\} = \lim_{d\to 0} P\{x_0 \in I(N(d))\},$$

which by Theorem 1 (ii) is equal to γ.

Let $c = 0$. Now

$$P(CD) = P\{N(d) > M(d)\} = P\{d^2 \log\log N(d) > d^2 \log\log M(d)\}$$
$$\geqslant P\{\delta^2(N(d)) \log\log N(d) > d^2 \log\log M(d)\}.$$

By the definition of $\delta(n)$, we have $\delta(n) = \infty$ on the set $\{\hat{c}_n \leqslant \tau_n s_n/\sqrt{n}\}$, and

$$\delta(n) = \frac{s_n \tau_n}{\sqrt{n}} \cdot \frac{\sqrt{\hat{b}_n^2 + \frac{1}{2}(\hat{c}_n^2 - \tau_n^2 s_n^2/n)}}{\hat{c}_n^2 - \tau_n^2 s_n^2/n} \geqslant \frac{s_n \tau_n}{\sqrt{2n\hat{c}_n^2}}$$

on the set $\{c_n > \tau_n s_n/\sqrt{n}\}$.

It follows that

$$\delta(n)\sqrt{\log\log n} \geqslant \frac{s_n \tau_n}{\sqrt{2}} \sqrt{\frac{\log\log n}{n\hat{c}_n^2}}$$

everywhere. By the law of the iterated logarithm for $\hat{c}_n$ we have

$$\limsup_n (n\hat{c}_n^2/\log\log n) = \beta \text{ a.s.}$$

for a positive β, hence

$$\liminf_n \delta(n)\sqrt{\log\log n} > 0 \text{ a.s.}$$

Using part (b) of the Lemma, we obtain

$$\lim_{d\to 0} P\{\delta^2(N(d)) \log\log N(d) > d^2 \log\log M(d)\} = 1,$$

and, consequently, $\lim_{d\to 0} P(CD) = 1$. The case $c < 0$ is easy to prove.

Remark. By the presented procedure we can easily obtain the fixed-precision confidence interval for the ratio of two means.

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