PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 7, Fasc. 1 (1986), p. 13-26

SUP-NORM CONVERGENCE OF THE EMPIRICAL PROCESS INDEXED BY FUNCTIONS AND APPLICATIONS

BY

MIKLÓS CSÖRGÖ[1] (OTTAWA), SÁNDOR CSÖRGÖ[2] (SZEGED), LAJOS HORVÁTH[3] (SZEGED) AND DAVID M. MASON[4] (NEWARK)

Abstract. A new approximation of the uniform empirical and quantile processes results in a weak invariance principle indexed by functions for the general empirical process. Consequences of this result are weak convergence of empirical moment generating, Hall, moment and generalized mean processes.

1. Introduction. Let $U_{1,n} \leqslant \ldots \leqslant U_{n,n}$ denote the order statistics of the first n of independent uniform-(0, 1) $(U(0, 1))$ random variables (rv) $U_1, U_2, \ldots$ with the corresponding *uniform empirical distribution function* $G_n(\cdot)$, defined to be right continuous and *uniform empirical quantile function*

$$U_n(s) := U_{k,n}, \quad (k-1)/n < s \leqslant k/n \ (k = 1, \ldots, n),$$

where $U_n(0) := U_{1,n}$. We define the *uniform empirical process*

$$\alpha_n(s) := n^{1/2}(G_n(s) - s), \quad 0 \leqslant s \leqslant 1,$$

and the *uniform quantile process*

$$u_n(s) := n^{1/2}(s - U_n(s)), \quad 0 \leqslant s \leqslant 1.$$

In our paper [2] we showed that with an appropriate sequence of

---

[1] Research partially supported by an NSERC Canada grant at Carleton University, Ottawa.

[2] Research done while at the University of California, San Diego.

[3] Research done while at Carleton University, also partially supported by NSERC Canada grants of M. Csörgö and D. A. Dawson, and by an EMR Canada grant of M. Csörgö.

[4] Research done while at Universität München, supported by the Alexander von Humboldt Foundation.

Brownian bridges $\{B_n(s); 0 \leqslant s \leqslant 1\}$ on an appropriately constructed probability space we have (cf. Theorem 1.1 of [2])

$$P\{\sup_{0 \leqslant s \leqslant d/n} |u_n(s) - B_n(s)| \geqslant n^{-1/2}(a \log d + x)\} \leqslant be^{-cx}, \tag{1.1}$$

whenever $n_0 \leqslant d \leqslant n$, $0 \leqslant x \leqslant d^{1/2}$, where n_0, a, b and c are suitably chosen positive constants. A similar inequality holds true in the neighbourhood of one. When $d = n$, (1.1) reduces to Theorem 1 of [1]. The latter inequality as well as that of (1.1) is based on a similar inequality of Komlós, Major and Tusnády [9] on approximating partial sums of i.i.d. rv by a Wiener process. Our inequality (1.1) leads to Brownian bridge approximations for weighted uniform empirical and quantile processes with rates in probability. Several applications of these are given in [2]. One of them is the area of invariance principles indexed by functions of which one is going to be used in the sequel. In order to state this result, we have to introduce some notations.

When $0 < a < b < 1$ and g is a left continuous and f is a right continuous function, then

$$\int_a^b f dg = \int_{[a,b)} f dg \quad \text{and} \quad \int_a^b g df = \int_{(a,b]} g df,$$

whenever these integrals make sense as Lebesgue–Stieltjes integrals. In this case the usual integration by parts formula

$$\int_a^b f dg + \int_a^b g df = g(b) f(b) - g(a) f(a)$$

is valid.

For any Brownian bridge $\{B(s); 0 \leqslant s \leqslant 1\}$, and with $0 < a < b < 1$ and the functions f and g as above we define the stochastic integral

$$\int_a^b f(s) dB(s) := f(b) B(b) - f(a) B(a) - \int_a^b B(s) df(s)$$

and the same formula for g replacing f.

If g or f are not finite at at least one of the endpoints, then the corresponding integrals are meant as improper integrals whenever they are finite in the nonstochastic case and almost surely finite in the stochastic case.

Let $\mathscr{L}$ denote any class of functions l defined on (0, 1) such that

(L′.1) Each l can be written as $l = l_1 - l_2$, where l_1 and l_2 are nondecreasing left continuous functions defined on (0, 1).

Let L be a positive nonincreasing function defined on (0, 1/2] slowly varying near zero and define

$$N(\delta) = \sup_{l \in \mathscr{L}} \sup_{0 < s \leqslant \delta} \{(|l_1(s)| + |l_2(s)| + |l_1(1-s)| + |l_2(1-s)|) s^{1/2}/L(s)\}.$$

Now we are in the position of quoting Corollary 3.2 of [2].

THEOREM A. *Let $\mathscr{L}$ be any class of functions l as above, satisfying* (L'.1) *and such that*

$$\lim_{\delta \downarrow 0} N(\delta) = 0. \tag{L'.2}$$

Then on the probability space of (1.1)

$$\sup_{l \in \mathscr{L}} \Big| \int_0^1 l(s)\, d\alpha_n(s) - \int_{1/n}^{1-1/n} l(s)\, dB_n(s) \Big| / L(1/n) = o_P(1). \tag{1.2}$$

In our paper [2] in Corollary 3.4 we replace the limits of integration $1/n$ and $(n-1)/n$ by 0 and 1 when integrating with respect to the Brownian bridges B_n in Theorem A.

One of the aims of this paper is to extend Theorem A to accommodate the general empirical process (cf. Theorem 1.1) and then to extend the limits of integration to the real line (cf. Theorem 1.2). These results then are going to be applied to produce invariance principles for the empirical moment generating, Hall, moment and generalized mean processes.

2. Sup-norm convergence of the general empirical process indexed by functions. Let F be a right continuous distribution function and Q be its left continuous inverse (quantile function). Let a right-continuous distribution function F be given with its left-continuous quantile function $Q(s) = \inf\{x: F(x) \geqslant s\}$, $Q(0) = Q(0+)$, $Q(1) = Q(1-)$. When wishing to accommodate the general empirical process $\beta_n(x) = \alpha_n(F(x)) = n^{1/2}(F_n(x) - F(x))$, $x \in (Q(0), Q(1))$, based on $Q(U_1), \ldots, Q(U_n)$, we let $\mathscr{G}_F$ be a class of real valued Borel measurable functions g defined on $(Q(0), Q(1))$, the support of F, so that the functions $l(s) := g(Q(s))$, $s \in (0, 1)$, should form an $\mathscr{L}$ class satisfying (L'.1). In terms of these $\mathscr{L}$ functions we define the corresponding $N(\delta)$ as before, and whenever g is such that $g(Q(\cdot))$ satisfies condition (L'.1), we define, for $-\infty \leqslant c < d \leqslant \infty$,

$$\int_c^d g(x)\, dB(F(x)) := \int_{F(c)}^{F(d-)} g(Q(s))\, dB(s), \tag{2.1}$$

where $\{B(s); 0 \leqslant s \leqslant 1\}$ is any Brownian bridge. Then we have the following analogue of Theorem A:

THEOREM 1.1. *Let $\mathscr{G}_F$, respectively $\mathscr{L}$, be the class of functions g, respectively l, as above, satisfying the corresponding* (L'.1) *and* (L'.2) *conditions. Then on the probability space of* (1.1) *we have Theorem* A *in terms of the latter $l(\cdot) := g(Q(\cdot)) \in \mathscr{L}$ and also*

$$\sup_{g \in \mathscr{G}_F} \Big| \int_{-\infty}^{\infty} g(x)\, d\alpha_n(F(x)) - \int_{Q(1/n)}^{Q(1-1/n)} g(x)\, dB_n(F(x)) \Big| / L(1/n) = o_P(1). \tag{2.2}$$

Proof. Assume without loss of generality that each $g(Q(\cdot))$ is a non-decreasing left continuous function. We have

$$\int_{-\infty}^{\infty} g(x)\,d\alpha_n(F(x)) = \int_0^1 g(Q(s))\,d\alpha_n(s), \text{ always,}$$

and, by (2.1),

$$\int_{Q(1/n)}^{Q(1-1/n)} g(x)\,dB_n(F(x)) := \int_{F(Q(1/n))}^{F(Q(1-1/n)-)} g(Q(s))\,dB_n(s).$$

Since (1.2) automatically holds and says

$$\sup_{g\in\mathscr{G}_F} \Big| \int_0^1 g(Q(s))\,d\alpha_n(s) - \int_{1/n}^{1-1/n} g(Q(s))\,dB_n(s)\Big| / L(1/n) = o_P(1),$$

in order to verify (2.2) we have to show only that

$$\text{(2.3)}\quad \sup_{g\in\mathscr{G}_F} \Big| \int_{1/n}^{1-1/n} g(Q(s))\,dB_n(s) - \int_{F(Q(1/n))}^{F(Q(1-1/n)-)} g(Q(s))\,dB_n(s)\Big| / L(1/n) = o_P(1).$$

Applying integration by parts to both integrals in (2.3), the problem of verifying (2.3) reduces to showing that

$$\sup_{g\in\mathscr{G}_F} \{|g(Q(1-1/n))\,B_n(1-1/n)| + |g(Q(1/n))\,B_n(1/n)|\}/L(1/n) = o_P(1)$$

and

$$\sup_{g\in\mathscr{G}_F} \{|g(Q(F(Q(1-1/n)-)))\,B_n(F(Q(1-1/n)-))| + \\ + |g(Q(F(Q(1/n))))\,B_n(F(Q(1/n)))|\}/L(1/n) = o_P(1).$$

Now let $\gamma_n = \max\{1/n,\ F(Q(1/n)),\ 1-F(Q(1-1/n)-)\}$. Noting that $F(Q(1/n)) \geqslant 1/n$ and $1-F(Q(1-1/n)-) \geqslant 1/n$, we see that by the definition of $N(\delta)$ in our present context the sum of the left-hand side of the above two lines is less than or equal to

$$N(\gamma_n)\{|n^{1/2}B_n(1/n)| + |n^{1/2}B_n(1-1/n)| + |B_n(F(Q(1/n)))/(F(Q(1/n)))^{1/2}| + \\ + |B_n(F(Q(1-1/n)-))/(1-F(Q(1-1/n)-))^{1/2}|\},$$

which by (L'.2) equals to $o(1)O_P(1)$. This also completes the proof of (2.2).

In our next theorem we extend the limits of integration $Q(1/n)$ and $Q(1-1/n)$ in (2.2) to $-\infty$ and ∞. The present method of doing this is completely different from that of the mentioned extension of Theorem A in [2], Corollary 3.4.

THEOREM 1.2. *Let $\mathscr{G} = \mathscr{G}_F = \{g_t(\cdot);\ t\in[a,b]^d\}$ be a function class such that $[a,b]$ is a finite interval, $d \geqslant 1$ is an integer. Assume that $\mathscr{G}$ is as in*

Theorem 1.1, *satisfying its corresponding* (L′.1) *and* (L′.2) *conditions, the latter with* $L(\cdot) = 1$. *Assume that the function*

$$d_{\mathscr{G}}^2(s, t) = d_{\mathscr{G}_F}^2(s, t) = \int_{-\infty}^{\infty} (g_s(x) - g_t(x))^2 \, dF(x)$$

is continuous on $[a, b]^d \times [a, b]^d$ *and let* $N_{d_{\mathscr{G}}}([a, b]^d, \varepsilon)$ *be the minimum number of* $d_{\mathscr{G}}$*-balls with centres in* $[a, b]^d$ *and radii at most* $\varepsilon > 0$ *that cover* $[a, b]^d$, *where a* $d_{\mathscr{G}}$*-ball with centre* t *and radius* $\delta > 0$ *is the set* $B_{d_{\mathscr{G}}}(t, \delta) = \{s\colon d_{\mathscr{G}}(s, t) < \delta\}$. *If in addition to the* (L′.1) *and* (L′.2) *conditions the metric entropy condition*

$$J([a, b]^d, d_{\mathscr{G}}) = \int_0^{\hat{d}_{\mathscr{G}}} (\log N_{d_{\mathscr{G}}}([a, b]^d, \varepsilon))^{1/2} \, d\varepsilon < \infty$$

is also satisfied, where $\hat{d}_{\mathscr{G}} = \sup\{d_{\mathscr{G}}(s, t)\colon s, t \in [a, b]^d\}$ *is the* $d_{\mathscr{G}}$*-diameter of* $[a, b]^d$, *then, on the probability space of Theorem* A, *as* $n \to \infty$,

$$\sup_{t \in [a,b]^d} \Big| \int_{-\infty}^{\infty} g_t(x) \, d\alpha_n(F(x)) - \int_{-\infty}^{\infty} g_t(x) \, dB_n(F(x)) \Big| = o_P(1).$$

Remark 1.1. A well known sufficient condition of the metric entropy condition is

$$\int_0^{\delta} \big(\varphi_{\mathscr{G}_F}(h) / (h(\log h^{-1}))^{1/2}\big) \, dh < \infty \quad \text{for some } \delta > 0, \tag{2.4}$$

where

$$\varphi_{\mathscr{G}_F}(h) := \sup_{\substack{s, t \in [a,b]^d \\ \|s - t\|_d \le h}} d_{\mathscr{G}_F}(s, t),$$

and $\|\cdot\|_d$ stands for the maximum norm in $\mathbb{R}^d$. We note also that the metric entropy condition is Dudley's [4] sufficient condition for the sample continuity of the Gaussian process

$$\Big\{ \int_{-\infty}^{\infty} g_t \, dW(F(x)), \ t \in [a, b]^d \Big\},$$

where W is a standard Wiener process, and the stochastic integral itself is defined as that of (2.1). On the other hand, (2.4) is Fernique's [5] earlier and stronger sufficient condition for the sample continuity of the same process. See [6] for the relationship of the two conditions.

Proof. In view of (2.2) of Theorem 1.1, whose $L(:)$ now is assumed to be 1, in order to verify our statement it suffices to show that

$$\sup_{t \in [a,b]^d} \Big| \int_{A_n} g_t(x) \, dB(F(x)) \Big| = o_P(1), \quad n \to \infty, \tag{2.5}$$

with a fixed Brownian bridge $\{B(s);\ 0 \leqslant s \leqslant 1\} \stackrel{\mathscr{D}}{=} \{W(s)-sW(1);\ 0 \leqslant s \leqslant 1\}$, where

$$A_n := \{x : x < Q(1/n)\} \cup \{x : x \geqslant Q(1-1/n)\}.$$

We note that the continuity of $d_{\mathscr{G}}^2$ implies (via the Schwarz and Minkowski inequalities) that

$$\int_{-\infty}^{\infty} g_t(x)\,dF(x), \qquad \int_{-\infty}^{\infty} |g_t(x)|\,dF(x), \qquad \int_{-\infty}^{\infty} g_t^2(x)\,dF(x)$$

and, *a fortiori*,

$$\int_{A_n} |g_t(x)|\,dF(x), \qquad \int_{A_n} g_t^2(x)\,dF(x), \qquad n = 1, 2, \ldots,$$

are all continuous functions on $[a, b]^d$. Hence by Dini's theorem (cf. [11], p. 66)

$$W(1) \sup_{t \in [a,b]^d} \int_{A_n} |g_t(x)|\,dF(x) = o_P(1), \qquad n \to \infty.$$

Hence, in order to verify (2.5), it suffices to show that

$$\sup_{t \in [a,b]^d} |\Gamma_n(t)| = o_P(1), \qquad n \to \infty, \tag{2.6}$$

where

$$\Gamma_n(t) := \int_{A_n} g_t(x)\,dW(F(x)).$$

Set

$$d_n^2(s, t) = E(\Gamma_n(s) - \Gamma_n(t))^2 = \int_{A_n} (g_s(x) - g_t(x))^2\,dF(x).$$

Since $d_n^2(s, t) \downarrow 0$ as $n \to \infty$ for any pair $s, t \in [a, b]^d$, again by Dini's theorem, we have

$$\hat{d}_n = \sup_{s,t \in [a,b]^d} d_n(s, t) \to 0 \qquad \text{as } n \to \infty. \tag{2.7}$$

Since $d_n(s, t) \leqslant d_{\mathscr{G}}(s, t)$ for $s, t \in [a, b]^d$ and $n \geqslant 1$, we have $B_{d_{\mathscr{G}}}(t, \delta) \subset B_{d_n}(t, \delta)$ for any $t \in [a, b]^d$, $\delta > 0$ and $n \geqslant 1$. Whence

$$N_{d_n}([a, b]^d, \varepsilon) \leqslant N_{d_{\mathscr{G}}}([a, b]^d, \varepsilon) \qquad \text{for all } \varepsilon > 0 \text{ and } n \geqslant 1.$$

Thus

$$J([a, b]^d, d_n) = \int_0^{\hat{d}_n} (\log N_{d_n}([a, b]^d, \varepsilon))^{1/2}\,d\varepsilon$$

$$\leqslant \int_0^{\hat{d}_n} (\log N_{d_{\mathscr{G}}}([a, b]^d, \varepsilon))^{1/2}\,d\varepsilon \to 0 \qquad \text{as } n \to \infty,$$

by our metric entropy condition and (2.7). Now a convenient form of Dudley's theorem (Theorem 3.1 in [10]) as applied to the process Γ_n gives that there exists an absolute constant $K > 0$ such that

$$\begin{aligned} E\{\sup_{t\in[a,b]^d} |\Gamma_n(t)|\} &\leqslant (E\{\sup_{t\in[a,b]^d} |\Gamma_n(t)|^2\})^{1/2} \\ &\leqslant K\{(E|\Gamma_n(\boldsymbol{a})|^2)^{1/2} + \hat{d}_n + J([a, b]^d, d_n)\} \\ &= K\{(\int_{A_n} g_{\boldsymbol{a}}^2(x)\,dF(x))^{1/2} + \hat{d}_n + J([a, b]^d, d_n)\}, \end{aligned}$$

where $\boldsymbol{a} = (a, \ldots, a) \in \mathbb{R}^d$. This bound goes to zero by (2.7) and (2.8) as $n \to \infty$. Hence (2.6) follows by the Markov inequality.

Remark 1.2. It is possible to state Theorem 1.2 for more general classes of functions $\mathscr{G}_F$, not necessarily indexed by a Euclidean parameter. Using the general form of Dudley's theorem in Theorem 3.1 of [10], the proof remains the same. The reason for stating Theorem 1.2 in the above simple "Euclidean" form is that it is the one that finds applications of interest in the sequel.

3. The empirical moment generating function. Let $X_1, X_2, \ldots$ be independent nondegenerate rv with common distribution F, whose moment generating function

$$m(t) = \int_{-\infty}^{\infty} e^{tx}\,dF(x)$$

exists in a nondegenerate interval J which has one of the following forms: (c, d), $[c, d]$, $(c, d]$ or $[c, d)$, where $-\infty \leqslant c \leqslant 0 \leqslant d \leqslant \infty$ and $c < d$. It is well known that in such a case m has derivatives of all orders in the interior of J.

The *empirical moment generating function* based on $X_1, \ldots, X_n$ is defined to be

$$m_n(t) = \sum_{i=1}^{n} \exp(tX_i)/n,$$

and the corresponding *empirical moment generating process* to be

$$M_n(t) = n^{1/2}(m_n(t) - m(t)) = \int_{-\infty}^{\infty} \exp(tx)\,d\alpha_n(F(x)).$$

Also, whenever $m(2t) < \infty$, we define the *moment generating function process* to be

$$M(t) = \int_{-\infty}^{\infty} \exp(tx)\,dB(F(x)).$$

Csörgö [3] proved that M_n converges weakly to M in the Banach space of continuous functions defined on a suitable bounded closed subinterval $I = [a, b] \subseteq J$, where the endpoints of I are as follows:

$$a = \begin{cases} \text{arbitrary negative number} & \text{if } c = -\infty, \\ c/2+\varepsilon & \text{if } -\infty < c < \infty, \\ 0 & \text{if } c = 0, \end{cases}$$

and

$$b = \begin{cases} \text{arbitrary positive number} & \text{if } d = \infty, \\ d/2-\delta & \text{if } 0 < d < \infty, \\ 0 & \text{if } d = 0, \end{cases}$$

where $\varepsilon = 0$ if $c \in J$ and ε is an arbitrarily small positive number if $c \notin J$, and, in a similar fashion $\delta = 0$ if $d \in J$ and $\delta > 0$ if $d \notin J$. We will show here that this result is also a consequence of our Theorem 1.2. In this case

$$g_t(x) = \exp(tx) \quad \text{for } a \leqslant t \leqslant b,$$

and

$$m(2t) = \int_{-\infty}^{\infty} g_t^2(x)\,dF(x) < \infty \quad \text{for } a \leqslant t \leqslant b.$$

In order to demonstrate the said result, we only have to check that condition (2.4) holds. Let $m^{(k)}$ denote the kth order derivative of m in the interior of I, and let

$$f_t(u) = m(2t)+m\big(2(t+u)\big)-2m(2t+u)$$

for $0 \leqslant u \leqslant b-t$ and $a \leqslant t \leqslant b$. It is easy to verify that $f_t(u)$ is monotone in u for each t, and hence, if $0 \leqslant h \leqslant b-a$,

$$\varphi_{\mathscr{G}_F}(h) = \sup_{0 \leqslant t \leqslant b-h} (f_t(h))^{1/2}.$$

Applying a Taylor expansion around zero, we get that for $0 \leqslant h \leqslant b-a$ and $a \leqslant t \leqslant b-h$

$$f_t(h) = \sum_{k=2}^{\infty} (h^k/k!)(2^k-2)\,m^{(k)}(2t)$$

on noticing that $f_t(0) = 0$, which implies that

$$\limsup_{h \downarrow 0} h^{-1} \varphi_{\mathscr{G}_F}(h) \leqslant \sup_{2a \leqslant t \leqslant 2b} m^{(2)}(t) < \infty.$$

The latter inequality immediately implies (2.4) in our present context. Thus we have obtained the following

THEOREM 2.1. *If a and b are defined as above, then on the probability space of* (1.1), *as $n \to \infty$,*

$$\sup_{a \leqslant t \leqslant b} \Big|M_n(t) - \int_{-\infty}^{\infty} \exp(tx)\,dB_n\big(F(x)\big)\Big| = o_P(1).$$

4. Empirical Hall functions. Let X be an rv with finite pth moment. Hall [7] proved the interesting result that if $p > 0$ is not an even integer, then the distribution function F of X is completely determined by the translated moments function

$$h(t) = h(t;\, p) := \int_{-\infty}^{\infty} |x+t|^p \, dF(x) = E|X+t|^p, \qquad -\infty < t < \infty. \tag{4.1}$$

The empirical counterpart h_n of (4.1) is then of statistical interest, and it is defined by

$$h_n(t) := \int_{-\infty}^{\infty} |x+t|^p \, dF_n(x) = n^{-1} \sum_{i=1}^{n} |X_i+t|^p.$$

We introduce the corresponding Hall process H_n as

$$H_n(t) := n^{1/2}\big(h_n(t) - h(t)\big) = \int_{-\infty}^{\infty} |x+t|^p \, d\alpha_n\big(F(x)\big),$$

and show that the weak convergence of $H_n(t)$ in the space of continuous functions over an arbitrary finite interval $[a, b]$ is a consequence of Theorem 1.2 if $p > 1$ and $E|X|^{2p} < \infty$. It is easy to see that condition (L′.1) of Theorem 1.2 holds. If $E|X|^{2p} < \infty$, then

$$\lim_{s \downarrow 0} s^{1/(2p)} |Q(s)| = \lim_{s \uparrow 1} (1-s)^{1/(2p)} |Q(s)| = 0$$

and, therefore, (L′.2) of Theorem 1.2 is also satisfied for the function $g_t(x) := |x+t|^p$. An elementary calculation shows that if $p > 1$, then

$$\int_{-\infty}^{\infty} |x+t|^{2p} \, dF(x) \leqslant E|X|^{2p} + \max(|a|^{2p}, |b|^{2p}). \tag{4.2}$$

In order to check for condition (2.4) of Remark 1.1, we have to estimate the following integral:

$$d^2_{\mathscr{G}_F}(s, t) := \int_{-\infty}^{\infty} (|x+t|^p - |x+s|^p)^2 \, dF(x).$$

Using a one term Taylor expansion we get that

$$\begin{aligned} d^2_{\mathscr{G}_F}(s, t) &= \int_{-\infty}^{\infty} \big((|x+t| - |x+s|)\, p\theta_{t,s}^{p-1}(x)\big)^2 \, dF(x) \\ &= \int_{-\infty}^{\infty} p^2 \theta_{t,s}^{2p-2}(x) (|x+t| - |x+s|)^2 \, dF(x) \\ &\leqslant |t-s|^2 \int_{-\infty}^{\infty} p^2 \theta_{t,s}^{2p-2}(x) \, dF(x), \end{aligned}$$

where $|\theta_{t,s}(x)| \leqslant \max(|x+t|, |x+s|)$. Hence, by (4.2),

$$d^2_{\mathscr{G}_F}(s, t) \leqslant |t-s|^2 p^2 \big(E|X|^{2p-2} + \max(|a|^{2p-2}, |b|^{2p-2})\big).$$

Consequently, in our present context, $\varphi_{\mathscr{G}_F}(h)$ of Remark 1.1 is such that $\varphi_{\mathscr{G}_F}(h) \leqslant Ch$ with some constant $C > 0$. Whence also the finiteness of the integral of (2.4) in our present context. Thus we proved the following

THEOREM 4.1. *If* $E|X|^{2p} < \infty$, $p > 1$, *then on the probability space of* (1.1), *as* $n \to \infty$,

$$\sup_{a \leqslant t \leqslant b} \left|H_n(t) - \int_{-\infty}^{\infty} |x+t|^p \, dB_n(F(x))\right| = o_P(1)$$

for any finite interval $[a, b]$.

Hall [7] also considered another translated moments function

$$\tilde{h}(t) = \tilde{h}(t; p) := \int_{-\infty}^{\infty} |x+t|^p \operatorname{sgn}(x+t) \, dF(x) = E|X+t| \operatorname{sgn}(X+t),$$

and proved that if p is not an odd integer, then $\tilde{h}$ determines F of X uniquely. The corresponding empirical version of $\tilde{h}$ is

$$\tilde{h}_n(t) := n^{-1} \sum_{i=1}^{n} |X_i + t| \operatorname{sgn}(X_i + t).$$

Introducing

$$\tilde{H}_n(t) := n^{1/2}\left(\tilde{h}_n(t) - \tilde{h}(t)\right) = \int_{-\infty}^{\infty} |x+t|^p \operatorname{sgn}(x+t) \, d\alpha_n(F(x)),$$

one can prove similarly as above that if $p > 1$ and $E|X|^{2p} < \infty$, then on the probability space of (1.1)

$$\sup_{a \leqslant t \leqslant b} \left|\tilde{H}_n(t) - \int_{-\infty}^{\infty} |x+t|^p \operatorname{sgn}(x+t) \, dB_n(F(x))\right| = o_P(1) \quad \text{as } n \to \infty.$$

5. Empirical moment and generalized mean function. Let $X, X_1, X_2, \ldots$ be independent rv with common distribution function F. We assume that $P\{X > 0\} = 1$ and consider the *empirical moment function*

$$k_n(t) = \frac{1}{n} \sum_{i=1}^{n} X_i^t$$

with theoretical counterpart $k(t) = EX^t$ assumed to be finite for t values in a finite interval $[a, b]$, where $a < 0 < b$. We introduce the *empirical moment process* as

$$K_n(t) = n^{1/2}\left(k_n(t) - k(t)\right)$$

together with the *empirical generalized mean process*

$$D_n(t) = n^{1/2}\left(d_n(t) - d(t)\right),$$

where $d_n(t) = \left(k_n(t)\right)^{1/t}$ and $d(t) = \left(k(t)\right)^{1/t}$ are the empirical and theoretical *generalized mean functions*. Notice that $d_n(1)=(X_1+\ldots+X_n)/n$, the arithmetic mean, $d_n(-1)=\big((X_1^{-1}+\ldots+X_n^{-1})/n\big)^{-1}$, the harmonic mean, and $d_n(0)$ exists as

$$d_n(0):=\lim_{t\to 0} d_n(t)=(X_1\cdot\ldots\cdot X_n)^{1/n},$$

the geometric mean. Also, $d(0)$ exists as

$$d(0):=\lim_{t\to 0} d(t)=\exp(E\log X)$$

(cf. [8], p. 201). This is the motivation for including zero in our interval $[a, b]$ and, hence, the assumed positivity of the underlying rv. From our method of proof it will be clear to the reader how to derive versions of our theorem below on other intervals not containing zero. Finally we note that both $d_n(t)$ and $d(t)$ are non-decreasing functions of t and

$$\lim_{t\to\infty} d_n(t)=\max(X_1,\ldots, X_n), \qquad \lim_{t\to-\infty} d_n(t)=\min(X_1,\ldots, X_n).$$

THEOREM 5.1. *If* $EX^{2a}+EX^{2b}<\infty$, *then on the probability space of* (1.1), *as* $n\to\infty$,

$$\sup_{a\leqslant t\leqslant b}\Big|K_n(t)-\int_0^\infty x^t\,dB_n(F(x))\Big|=o_P(1)$$

and

$$\sup_{a\leqslant t\leqslant b}\left|D_n(t)-\frac{d(t)}{k(t)}\int_0^\infty \frac{x^t-1}{t}\,dB_n(F(x))\right|=o_P(1).$$

Proof. Writing $x^t=\exp(t\log x)$ we see that our moment condition ensures that the moment generating function of the rv $\log X$ is finite in the interval $[2a, 2b]$. Hence the first statement is a special case of Theorem 2.1.

Turning to the proof of the second statement we first note that $d_n(t)\to d(t)$ almost surely as $n\to\infty$ at each fixed $t\in[a, b]$. This follows from the law of large numbers. Since d_n and d are all nondecreasing continuous functions, it follows by Pólya's theorem that

$$\sup_{a\leqslant t\leqslant b}|d_n(t)-d(t)|\to 0 \text{ a.s.} \tag{5.1}$$

and similarly

$$\sup_{a\leqslant t\leqslant b}|k_n(t)-k(t)|\to 0 \text{ a.s.,} \tag{5.2}$$

as $n\to\infty$. For any $t\neq 0$ we have, by the Lagrange theorem,

$$D_n(t)=\frac{d(t)}{k(t)}\frac{K_n(t)}{t}+\frac{K_n(t)}{t}\left\{(k_n^*(t))^{1/t-1}-\frac{d(t)}{k(t)}\right\},$$

where $\min\{k(t), k_n(t)\}\leqslant k_n^*(t)\leqslant\max\{k(t), k_n(t)\}$.

By (5.1) and (5.2) it is enough to show that

$$(5.3)\qquad \Delta_n^{(1)} = \sup_{a \leqslant t \leqslant b} \left| \frac{K_n(t)}{t} - \int_0^\infty \frac{x^t - 1}{t} dB_n(F(x)) \right| = o_P(1)$$

and

$$(5.4)\qquad \Delta_n^{(2)} = \left| D_n(0) - d(0) \int_0^\infty \log x \, dB_n(F(x)) \right| = o_P(1).$$

Let δ be any number such that

$$(5.5)\qquad 0 < \delta < \min\left\{ \frac{b}{2b+1}, \frac{-a}{-2a+1} \right\}.$$

Then

$$\Delta_n^{(1)} \leqslant \sup_{t \in [a,b] \setminus [-\delta,\delta]} \left| \frac{K_n(t)}{t} - \int_0^\infty \frac{x^t - 1}{t} dB_n(F(x)) \right| +$$

$$+ \sup_{-\delta \leqslant t \leqslant \delta} \left| \frac{K_n(t)}{t} - \int_0^\infty \frac{x^t - 1}{t} dB_n(F(x)) \right|,$$

and, since

$$\int_0^\infty \frac{x^t - 1}{t} dB_n(F(x)) = \frac{1}{t} \int_0^\infty x^t \, dB_n(F(x)) = - \int_0^\infty B_n(F(x)) x^{t-1} \, dx,$$

the first term is $o_P(1)$ by the first statement of the theorem. On the other hand, the second term is

$$(5.6)\qquad \sup_{-\delta \leqslant t \leqslant \delta} \left| \int_0^\infty \big(B_n(F(x)) - \alpha_n(F(x))\big) x^{t-1} \, dx \right|$$

$$\leqslant \int_1^\infty |\alpha_n(F(x)) - B_n(F(x))| \, x^{\delta - 1} \, dx + \int_0^1 |\alpha_n(F(x)) - B_n(F(x))| \, x^{-\delta - 1} \, dx$$

$$= o_P(1) \left\{ \int_1^\infty (1 - F(x))^{1/2 - \delta} x^{\delta - 1} \, dx + \int_0^1 (F(x))^{1/2 - \delta} x^{-(1+\delta)} \, dx \right\}$$

by Theorem 4.2.1 in [2], using the Chibisov-O'Reilly function $w(t) = (t(1-t))^{1/2-\delta}$. It follows from our moment condition that the sum of these two integrals is not greater than a finite constant times

$$\int_1^\infty x^{-2b(1/2-\delta)} x^{\delta - 1} \, dx + \int_0^1 x^{-2a(1/2-\delta)} x^{-(1+\delta)} \, dx$$

$$= \int_0^\infty x^{-1-b+\delta(2b+1)} \, dx + \int_0^1 x^{-1-a-\delta(-2a+1)} \, dx < \infty$$

by the choice of δ in (5.5). This proves (5.3).

To prove (5.4), we have again, by Lagrange's theorem,

$$\Delta_n^{(2)} = \left|n^{1/2}\{\exp(n^{-1}\sum_{i=1}^{n}\log X_i)-\exp(E\log X)\}-d(0)\int_0^\infty \log x\,dB_n(F(x))\right|$$

$$\leqslant d(0)\left|n^{-1/2}\sum_{i=1}^{n}(\log X_i - E\log X)-\int_0^\infty \log x\,dB_n(F(x))\right|+R_n$$

$$= d(0)\left|\int_0^\infty \log x\,d\alpha_n(F(x))-\int_0^\infty \log x\,dB_n(F(x))\right|+R_n$$

$$\leqslant d(0)\left|\int_0^\infty \big(\alpha_n(F(x))-B_n(F(x))\big)x^{-1}\,dx\right|+R_n,$$

where $R_n \to 0$ almost surely by the law of large numbers and the first term is $o_P(1)$ as a special case ($t=0$) of (5.6). The theorem is proved.

Acknowledgement. A part of this work was done while Miklós Csörgő and David Mason were visiting the Bolyai Institute of Szeged University. They are grateful to Professor Károly Tandori for his hospitality during their stay and all the authors thank him for making this collaboration possible.

REFERENCES

[1] M. Csörgő and P. Révész, *Strong approximations of the quantile process*, Ann. Statist. 6 (1978), p. 882-894.

[2] M. Csörgő, S. Csörgő, L. Horváth and D. M. Mason, *Weighted empirical and quantile processes*, Ann. Probab. (to appear).

[3] S. Csörgő, *The empirical moment generating function*. In: Coll. Math. Soc. J. Bolyai 32, *Nonparametric Statistical Inference* (B. V. Gnedenko, M. L. Puri and and I. Vincze, eds.), p. 139-150, Amsterdam 1982.

[4] R. M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Funct. Anal. 1 (1967), p. 290-330.

[5] X. Fernique, *Continuité de processus gaussiens*, C. R. Acad. Sci. Paris 258 (1964), p. 6058-6060.

[6] — *Régularité des trajectoires des fonctions aléatoires gaussiennes*. In: Ecole d'été de probabilités de Sts.-Flour, IV – 1974. Lecture Notes in Math. 480 (1975), p. 1-96.

[7] P. Hall, *A distribution is completely determined by its translated moments*, Z. Wahrsch. verw. Gebiete 62 (1983), p. 355-359.

[8] E. Hewitt and K. Stromberg, *Real and abstract analysis*, Berlin 1965.

[9] J. Komlós, P. Major and G. Tusnády, *An approximation of partial sums of independent rv's and the sample df, II*, Z. Wahrsch. verw. Gebiete 34 (1976), p. 33-58.

[10] M. B. Marcus and G. Pisier, *Random Fourier series with applications to harmonic analysis*, Annals of Math. Studies 101 (1981).

[11] B. Sz.-Nagy, *Introduction to real functions and orthogonal expansions*, New York 1965.

M. Csörgö
Dept. of Mathem. and Statistics
Carleton University
Ottawa K1S 5B6
Canada

S. Csörgö and L. Horváth
Bolyai Institute
Szeged University
Aradi Vétanúk Tere 1
H-6720 Szeged
Hungary

D. M. Mason
Dept. of Math. Sciences
University of Delaware
Newark, Delaware 19716
U.S.A.

Received on 2. 7. 1985