PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 7, Fasc. 1 (1986), p 77-93

NECESSARY AND SUFFICIENT CONDITIONS FOR EXTENDED CONVERGENCE OF SEMIMARTINGALES

BY

LESZEK SŁOMIŃSKI (TORUŃ)

Abstract. The extended convergence (in sense of Aldous [1]) of processes with filtrations is considered. There are examined the cases where the well-known conditions, sufficient for the weak convergence of semimartingales, are exactly equivalent to the extended convergence.

1. INTRODUCTION

Let S be a Polish space. In this paper we adapt the following notation:

$\mathscr{B}(S)$ – the σ-algebra of Borel subsets of S;

$\mathscr{P}(S)$ – the space of probability measures on $\mathscr{B}(S)$ equipped with the topology of weak convergence;

$D(S)$ – the space of mappings $x\colon \boldsymbol{R}^+ \to S$ which are right-continuous and admit left-hand side limits (it is well known that $D(S)$ endowed with the Skorokhod topology is metrisable as a Polish space, see [2]).

Definition 1. Let (Ω, F, P) be a probability space. The pair $(\mathscr{X}, \mathscr{F})$ is a *process with filtration* iff $\mathscr{X} = \{X(t)\}_{t\in\boldsymbol{R}^+}$ is a family of S-valued random elements defined on (Ω, F, P) and $\mathscr{F} = \{F(t)\}_{t\in\boldsymbol{R}^+}$ is a filtration in this space (i.e. nondecreasing family of sub-σ-algebras of F) such that:

(H_1) almost all trajectories $\omega \to \big(\mathscr{X}(\cdot, \omega)\colon \boldsymbol{R}^+ \to S\big)$ belong to $D(S)$;

(H_2) $X(t)$ is $F(t)$-measurable, $t\in\boldsymbol{R}^+$;

(H_3) $F(t) = \bigcap_{u>t} F(u)$, $t\in\boldsymbol{R}^+$.

Let $\operatorname{Cont}\mathscr{X} \overset{\text{df}}{=} \{t\in\boldsymbol{R}^+\colon P\big(X(t) \neq X(t-)\big) = 0\}$.

It has been proved by Aldous [1] that for every process with filtration $(\mathscr{X}, \mathscr{F})$ there exists a unique prediction process $(\mathscr{Z}, \mathscr{F})$ with values in the space $\mathscr{P}(D(S))$ such that, for every $t\in\boldsymbol{R}^+$ and every $A\in\mathscr{B}(D(S))$, $Z(t)(A) = P(\mathscr{X}\in A\,|\,F(t))$.

We define the extended distribution of the process $(\mathscr{X}, \mathscr{F})$ as the distribution of the random element

$$\Omega\ni\omega\to\big(\mathscr{X}(\cdot,\omega), \mathscr{Z}(\cdot,\omega)\big)\in D\big(S\times P(D(S))\big).$$

Definition 2 (Aldous [1]). Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N\cup\{\infty\}}$ be a sequence of processes with filtrations. We say that the sequence $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *converges extendly to* $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ (and write $(\mathscr{X}^n, \mathscr{F}^n)\to(\mathscr{X}^\infty, \mathscr{F}^\infty)$) if the extended distributions of $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ are weakly convergent to the extended distribution of $(\mathscr{X}^\infty, \mathscr{F}^\infty)$.

It is possible to characterise the extended convergence by the weak convergence of some families of bounded martingales. Let $(\mathscr{X}, \mathscr{F})$ be a real process with filtration. For every $m\in N$, every $T=(t_1,\ldots,t_m)\in\boldsymbol{R}^m$, $t_1<t_2<\ldots<t_m$, $\Theta=(\theta_1,\ldots,\theta_m)$ we denote by $\mathscr{X}_{T,\Theta,m}$ the regular version of the martingale of the form

$$\{E(\exp i\sum_{k=1}^{m}\theta_k X(t_k)|F(t))\}_{t\in\boldsymbol{R}^+}. \tag{1}$$

In Section 4 we shall prove the following

PROPOSITION 1. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N\cup\{\infty\}}$ be a sequence of real processes with filtrations, $X^n(0)=0$, $n\in N\cup\{\infty\}$. The sequence $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ converges extendly to $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ iff two following conditions are satisfied*:

(i) $\big(X^n_{T,\Theta^1,m}(t_1),\ldots,X^n_{T,\Theta^m,m}(t_m)\big)\xrightarrow[\mathscr{D}]{}\big(X^\infty_{T,\Theta^1,m}(t_1),\ldots,X^\infty_{T,\Theta^m,m}(t_m)\big)$;

(ii) $\{(\mathscr{X}^n, \mathscr{X}^n_{T,\Theta^i,m})\}_{n\in N}$ *is tight in* $D(C^2)$, $i=1,2,\ldots,m$, *for every* $m\in N$, *every* $T\in[\mathrm{Cont}\,\mathscr{X}^\infty\cap\mathrm{Cont}\,\mathscr{Z}^\infty]^m$, $\Theta^i\in\boldsymbol{R}^m$, $i=1,2,\ldots,m$.

Standard arguments show that in (i) and (ii) it is sufficient to consider only the countable number of conditions.

We can obtain the similar result for the S-valued processes with filtrations. In this case we use the new bounded martingales

$$\{E\big(f_m(X(t_1),\ldots,X(t_m))|F(t)\big)\}_{t\in\boldsymbol{R}^+} \tag{2}$$

instead of (1) for every $m\in N$, every $(t_1,\ldots,t_m)\in\boldsymbol{R}^m$ and bounded and continuous functions $f_m\colon S^m\to\boldsymbol{R}$.

In this paper we consider the extended convergence of real semimartingales to the continuous in probability process with independent increments. Jacod has observed in [9] that the convergence in probability of the local predictable characteristics of semimartingales is "essentially equivalent" to the extended convergence. In [12] we have proved that the sufficient conditions of convergence, discussed by Brown [3], Liptser and Shiryaev [16], Grigelionis and Mikulevicius [4] and Jacod, Kłopotowski and

Memin [11], are in fact stronger that the extended convergence of semimartingales. Now we examine the cases where these conditions are exactly equivalent to the extended convergence.

As a consequence we generalise the well known results by Liptser and Shiryaev [16], [17] (Theorem 2, Section 2, and Theorem 4, Section 3) and Aldous [1] (Theorem 1, Section 2, and Theorem 3, Section 3).

Finally, let us add that similar problem was recently discussed by Grigelionis, Mikulevicius and Kubilius [6] (see also [14] and [13]). They characterise sufficient conditions of convergence using the finite-dimensional convergence of extended distributions. However, we hope that our method is simpler than that mentioned above.

2. MAIN RESULTS

Let (Ω, F, P) be a complete probability space and $(\mathscr{X}, \mathscr{F})$ be a semimartingale (relatively to P) such that $X(0) = 0$ and the filtration $\mathscr{F}$ satisfies the completeness assumptions, i.e. $F(0)$ contains all P-null sets of $\mathscr{F}$.

Let $h\colon \boldsymbol{R} \to [-1, 1]$ be a continuous function satisfying $h(x) = x$ for $|x| \leqslant 1$ and $h(x) = 0$ for $|x| \geqslant 2$. Consider the process $\mathscr{X}^h$ defined by

$$X^h(t) = X(t) - \sum_{s \leqslant t} \big(\Delta X(s) - h(\Delta X(s))\big), \qquad t \in \boldsymbol{R}^+. \tag{4}$$

The process with filtration $(\mathscr{X}^h, \mathscr{F})$ is a special semimartingale. It can be uniquely decomposed into the sum $\mathscr{X}^h = B^h + M$, where $(B^h, \mathscr{F})$ is a predictable process with bounded variation and $(M, \mathscr{F})$ is a locally square integrable martingale.

Let $(\mathscr{X}^c, \mathscr{F})$ be the unique continuous martingale part of the semimartingale $(\mathscr{X}, \mathscr{F})$. Define

$$\sigma^2(t) = \langle \mathscr{X}^c \rangle (t), \qquad t \in \boldsymbol{R}^+. \tag{4}$$

Let $\nu = \nu(dt \times dx)$ be the dual predictable projection of the jump-measure $N(dt \times dx)$ of the process $(\mathscr{X}, \mathscr{F})$,

$$N\big((0, t] \times A\big) = \sum_{s \leqslant t,\, \Delta X(s) \neq 0} I\big(\Delta X(s) \in A\big), \qquad A \in \mathscr{B}(\boldsymbol{R}). \tag{5}$$

We say that the triplet (B^h, σ^2, ν) is a *system of local predictable characteristics of the semimartingale* $(\mathscr{X}, \mathscr{F})$.

Consider an array $\tau = \{t_{nk}\}$ of nonnegative numbers such that for each n the sequence $\{t_{nk}\}_{k \in \boldsymbol{N} \cup \{0\}}$ forms a partitions of $\boldsymbol{R}^+$ for which

$$0 = t_{n0} < t_{n1} < \ldots, \qquad \lim_{k \to \infty} t_{nk} = +\infty, \tag{6}$$

and

(7) $$\max_{k \leq r_n(t)} (t_{nk} - t_{n,k-1}) \to 0,$$

where $r_n(t) = \max[k\colon t_{nk} \leqslant t]$, $t \in \boldsymbol{R}^+$, $n \in N$.

Now let $(\mathscr{Y}, \mathscr{F})$ be a process with filtration from the class $S_g(\tau)$ defined by Jacod [10], i.e. $\mathscr{Y} = \mathscr{X} + B$, where $(\mathscr{X}, \mathscr{F})$ is a semimartingale with the triplet of local predictable characteristics $(0, \sigma^2, \nu)$, $\nu(\{t\} \times (\boldsymbol{R} \setminus \{0\})) = 0$, $t \in \boldsymbol{R}^+$, $X(0) = 0$. The process with filtration $(B, \mathscr{F})$, $B(0) = 0$, has continuous trajectories and the following property:

(8) $$\sup_{t \leqslant q} \Big| \sum_{k=1}^{r_n(t)} E\big(h(B(t_{nk}) - B(t_{n,k-1}))F(t_{n,k-1})\big) - B(t)\Big| \underset{P}{\to} 0, \quad q \in \boldsymbol{R}^+.$$

It is proved in [20] that for every process $(\mathscr{Y}, \mathscr{F}) \in S_g(\tau)$ there exists a unique triplet of local predictable characteristics (B, σ^2, ν).

Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n \in N}$ be a sequence of semimartingales. For every $(\mathscr{X}^n, \mathscr{F}^n)$ let us denote the system of local predictable characteristics by $(B^{h,n}, \sigma^{2,n}, \nu^n)$, $n \in N$.

THEOREM 1. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n \in N}$ be a sequence of semimartingales. Let $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ belong to $S_g(\tau)$ and have the triplet of characteristics $(B^\infty, \sigma^{2,\infty}, \nu^\infty)$. Assume that*

(A) $$\big((\mathscr{X}^n, B^{h,n}), \mathscr{F}^n\big) \to \big((\mathscr{X}^\infty, B^\infty), \mathscr{F}^\infty\big).$$

If we write $M^\infty = \mathscr{X}^{h,\infty} - B^\infty$, then the following conditions are satisfied:

(i) $(M^n, [M^n], \langle M^n \rangle) \underset{\mathscr{D}}{\to} (M^\infty, [M^\infty], \langle M^\infty \rangle)$ *in* $D(\boldsymbol{R}^3)$;

(ii) $$\Big(\int_0^{\boldsymbol{\cdot}} \int_{\boldsymbol{R}} f(x) N^n(ds \times dx), \int_0^{\boldsymbol{\cdot}} \int_{\boldsymbol{R}} f(x) \nu^n(ds \times dx)\Big) \underset{\mathscr{D}}{\to} \Big(\int_0^{\boldsymbol{\cdot}} \int_{\boldsymbol{R}} f(x) N^\infty(ds \times dx), \int_0^{\boldsymbol{\cdot}} \int_{\boldsymbol{R}} f(x) \nu^\infty(ds \times dx)\Big)$$

in $D(\boldsymbol{R}^2)$ for every $f \in C_{v(0)}$, where $C_{v(0)}$ is a family of bounded and continuous functions vanishing in an open neighbourhood of 0.

The proof of Theorem 1 is given in Section 4.

For the sake of brevity we will say that $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *satisfies* (H_∞) iff $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ is continuous in probability process with independent increments.

THEOREM 2. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n \in N}$ be a sequence of semimartingales. Let the process $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ satisfy (H_∞) and have the triplet of characteristics $(B^\infty, \sigma^{2,\infty}, \nu^\infty)$. Suppose that*

(Sup B) $$\sup_{t \leqslant q} |B^{h,n}(t) - B^\infty(t)| \underset{P}{\to} 0, \quad q \in \boldsymbol{R}^+.$$

Then the following conditions (i) *and* (ii) *are equivalent*:

(i) $$(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty);$$

$$\text{(M)} \quad \sigma^{2,n}(t)+\sum_{s\leq t}\int_{\boldsymbol{R}} h^2(x)\,\nu^n(\{s\}\times dx)-\sum_{s\leq t}\Big(\int_{\boldsymbol{R}} h(x)\,\nu^n(\{s\}\times dx)\Big)^2$$

$$\text{(ii)} \qquad \to \sigma^{2,\infty}(t)+\sum_{s\leq t}\int_{\boldsymbol{R}} h^2(x)\,\nu^\infty(\{s\}\times dx), \qquad t\in \boldsymbol{R}^+,$$

$$\text{(N)} \quad \int_0^t\int_{\boldsymbol{R}} f(x)\,\nu^n(ds\times dx) \underset{P}{\to} \int_0^t\int_{\boldsymbol{R}} f(x)\,\nu^\infty(ds\times dx), \qquad t\in \boldsymbol{R}^+, f\in C_{v(0)}.$$

Proof. (ii)$\Rightarrow$(i). This implication readily follows from Theorem 2 and Proposition 2 from [12].

(i)$\Rightarrow$(ii). First note that $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ belongs to $S_g(\tau)$. Because of equalities

$$\langle M^n\rangle(t) = \sigma^{2,n}(t)+\sum_{s\leq t}\int_{\boldsymbol{R}} h^2(x)\,\nu^n(\{s\}\times dx)-\sum_{s\leq t}\Big(\int_{\boldsymbol{R}} h(x)\,\nu^n(\{s\}\times dx)\Big)^2$$

and

$$\langle M^\infty\rangle(t) = \sigma^{2,\infty}(t)+\sum_{s\leq t}\int_{\boldsymbol{R}} h^2(x)\,\nu^\infty(\{s\}\times dx),$$

and, by Theorem 1, the conclusions follow.

Let $(W, \mathscr{F}^W)$ be a standard Brownian motion (by $\mathscr{F}^{\mathscr{Y}}$ we denote the natural filtration of the process $\mathscr{Y}$). In [12] we have proved that under the condition (Sup B) there holds the equivalence: $\mathscr{X}^n \underset{\mathscr{D}}{\to} W$ iff $(\mathscr{X}^n, \mathscr{F}^n) \to (W, \mathscr{F}^W)$. Hence Theorem 2 is more general than the well-known result of Liptser and Shiryaev [16] on the necessary and sufficient conditions for the weak convergence of semimartingales to Brownian motion. We have also observed that the assumption (Sup B) is not necessary for the convergence $(\mathscr{X}^n, \mathscr{F}^n) \to (W, \mathscr{F}^W)$.

Now we will show that (Sup B) and the weak convergence $\mathscr{X}^n \to \mathscr{X}^\infty$, where $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ satisfies (H_∞), do not imply (ii).

Example. Let $(N, \mathscr{F}^N)$ be a simple Poisson process. Suppose that $\{(\mathscr{X}^n, \mathscr{F}^{\mathscr{X}^n})\}_{n\in \boldsymbol{N}}$ is a sequence of processes defined by the equalities: $X^n(t) = 2N(t)+(1/n)\,N(2t)$, $t\in \boldsymbol{R}^+$, $n\in \boldsymbol{N}$. Then condition (N) is not satisfied.

It is interesting that with the use of the concept of G-stable convergence, introduced by Grigelionis and Mikulevicius [5], it is possible to generalize slightly all results of our paper.

Definition 3 (Grigelionis and Mikulevicius [5]). Let G be a sub-σ-algebra of F. We say that the *sequence of processes* $\{(\mathscr{X}^n, \mathscr{Z}^n)\}_{n\in \boldsymbol{N}}$ *converges G-stably to* $(\mathscr{X}^\infty, \mathscr{Z}^\infty)$ (and write $(\mathscr{X}^n, \mathscr{Z}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty)$ G-stably) if

$$\int_A f(\mathscr{X}^n, \mathscr{Z}^n)\,dP \to \int_A f(\mathscr{X}^\infty, \mathscr{Z}^\infty)\,dP$$

for every $A\in G$ and every continuous and bounded function $f\colon D\big(\boldsymbol{R}\times \mathscr{P}(D(\boldsymbol{R}))\big) \to \boldsymbol{R}$.

We give an example of such a generalisation.

COROLLARY 1. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of semimartingales. Let* $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *be a process with conditionally independent increments given* G,

$$(*) \qquad G \subset \bigcap_{n=1}^{\infty} F^n(0),$$

such that $\nu^\infty(\{t\}\times(\boldsymbol{R}\setminus\{0\})) = 0$, $t\in\boldsymbol{R}^+$. *If we replace condition* (i) *in Theorem* 2 *by* $(\mathscr{X}^n, \mathscr{F}^n)\to(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *G-stably, then the conclusion of Theorem* 2 *is still true.*

In some special cases we can also omit the assumption (*). A more extensive discussion of this problem may be found in [19].

3. EXTENDED CONVERGENCE OF INCREASING PROCESSES AND LOCAL MARTINGALES

By $\mathscr{A}^+$ ($\mathscr{A}^+_{\text{loc}}$) we denote the family of processes with filtrations $(\mathscr{X}, \mathscr{F})$ which have nondecreasing trajectories and are integrable (locally integrable). Let $\mathscr{M}_{\text{loc}}$ denote the family of local martingales and let $(\tilde{\mathscr{X}}, \mathscr{F})\in\mathscr{A}^+_{\text{loc}}$ be a predictable compensator of the process $(\mathscr{X}, \mathscr{F})\in\mathscr{A}^+_{\text{loc}}$ (i.e. $(\tilde{\mathscr{X}}, \mathscr{F})$ is predictable and $(\mathscr{X}-\tilde{\mathscr{X}}, \mathscr{F})\in\mathscr{M}_{\text{loc}}$).

Definition 4. Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of processes with filtrations. We say that $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *satisfies the condition* (J^p) (*or* (J^p_E)), $p>0$, iff for every $t\in\boldsymbol{R}^+$ there exists a sequence $\{\tau_n\}_{n\in N}$ of $\mathscr{F}^n$-stopping times such that

$$\lim_{n\to\infty} P[\tau_n < t] = 0$$

and

(9) $\{\sup_{t\leqslant\tau_n}|\Delta X^n(t)^p\}_{n\in N}$ is uniformly integrable (or $\sup_n E\sup_{t\leqslant\tau_n}|\Delta X^n(t)|^p < +\infty$).

The two following results (proofs of which are given in Section 4) form a basis for the present section and are essential for the proof of Theorem 1. The first one is an extended version of the well-known theorem by Jacod [8].

PROPOSITION 2. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of processes from* $\mathscr{M}_{\text{loc}}$ *satisfying* (J^1_E). *Assume that* $(\mathscr{X}^n, \mathscr{F}^n)\to(\mathscr{X}^\infty, \mathscr{F}^\infty)$. *Then*

$$((\mathscr{X}^n, [\mathscr{X}^n]), \mathscr{F}^n)\to((\mathscr{X}^\infty, [\mathscr{X}^\infty]), \mathscr{F}^\infty).$$

PROPOSITION 3. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of processes from* $\mathscr{A}^+_{\text{loc}}$ *satisfying* (J^1). *Suppose that* $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *belongs to* $\mathscr{A}^+_{\text{loc}}$, $\nu^\infty(\{t\}\times(\boldsymbol{R}\setminus\{0\}))=0$, $t\in\boldsymbol{R}^+$. *If for a sequence of processes* $\{(\mathscr{Y}^n, \mathscr{F}^n)\}_{n\in N}$ *the convergence*

$$((\mathscr{Y}^n, \mathscr{X}^n), \mathscr{F}^n)\to((\mathscr{Y}^\infty, \mathscr{X}^\infty), \mathscr{F}^\infty)$$

holds, then also

$$(\mathscr{Y}^n, \mathscr{X}^n, \tilde{\mathscr{X}}^n) \underset{\mathscr{D}}{\to} (\mathscr{Y}^\infty, \mathscr{X}^\infty, \tilde{\mathscr{X}}^\infty)$$

in $D(\boldsymbol{R}^3)$.

We shall also need the concept of a domination between two processes. This notion was introduced by Lenglart [15]. Let $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{F})$ be two processes with filtrations. Suppose that $\mathscr{Y}$ is nondecreasing, $Y(0) = 0$. We say that $\mathscr{X}$ *is* $\mathscr{F}$*-dominated by* $\mathscr{Y}$ (and write $\mathscr{X} \prec \mathscr{Y}$) if, for every $\mathscr{F}$-stopping time τ, $E|X(\tau)| \leqslant EY(\tau)$. It is clear that Definition 4 and the inequalities of Rebolledo [18] imply

LEMMA 1. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *and* $\{(\mathscr{Y}^n, \mathscr{F}^n)\}_{n\in N}$ *be two sequences of processes with filtrations for which* $\mathscr{X}^n \prec \mathscr{Y}^n$, $n \in N$.

If $\{(\mathscr{Y}^n, \mathscr{F}^n)\}_{n\in N}$ *satisfies* (J^1) *or is a sequence of predictable processes, then*

$$Y^n(\sigma_n) \underset{P}{\to} 0 \Rightarrow \sup_{t\leqslant\sigma_n} |X^n(t)| \underset{P}{\to} 0, \tag{10}$$

$$\lim_{\eta\to\infty} \overline{\lim_{n\to\infty}} P[Y^n(\sigma_n) \geqslant \eta] = 0 \Rightarrow \lim_{\eta\to\infty} \overline{\lim_{n\to\infty}} P[\sup_{t\leqslant\sigma_n} |X^n(t)| \geqslant \eta] = 0 \tag{11}$$

for every tight in $\boldsymbol{R}$ *sequence* $\{\sigma_n\}_{n\in N}$ *of* $\mathscr{F}^n$*-stopping times.*

Now we can formulate and prove our main results about the extended convergence of processes from $\mathscr{A}^+_{\mathrm{loc}}$ and $\mathscr{M}_{\mathrm{loc}}$. We assume that the limit process $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ has the following property:

$$\sup_{t\leqslant q} |\Delta X^\infty(t)| \leqslant 1, \quad q \in \boldsymbol{R}^+. \tag{H^1_∞}$$

THEOREM 3. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of processes from* $\mathscr{A}^+_{\mathrm{loc}}$. *Let the limit process* $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *satisfy* (H^1_∞) *and* (H_∞). *Assume that the sequence* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *has property* (J^1). *Then the following conditions* (i) *and* (ii) *are equivalent*:

$$(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty); \tag{i}$$

$$\begin{cases} \text{(C)} & \tilde{X}^n(t) \underset{P}{\to} \tilde{X}^\infty(t), \quad t \in \boldsymbol{R}^+, \\ \text{(N)} & \int_0^t \int_{\boldsymbol{R}} f(x)\,\nu^n(ds\times dx) \underset{P}{\to} \int_0^t \int_{\boldsymbol{R}} f(x)\,\nu^\infty(ds\times dx), \quad t \in \boldsymbol{R}^+, f \in C_{v(0)}. \end{cases} \tag{ii}$$

Proof. (ii)$\Rightarrow$(i). First we show that condition (Sup B) holds, i.e.

$$\tilde{X}^{h,n}(t) \underset{P}{\to} \tilde{X}^{h,\infty}(t) = \tilde{X}^\infty(t), \quad t \in \boldsymbol{R}^+. \tag{12}$$

Let $\{f_\varepsilon\}_{\varepsilon>0} \subset C_{v(0)}$ be a family of functions such that $|f_\varepsilon(x)| \leqslant 1$, $x \in \boldsymbol{R}$, and, for every $\varepsilon > 0$,

$$f_\varepsilon(x) \overset{\mathrm{df}}{=} \begin{cases} 0 & \text{for } |x| \leqslant 1, \\ 1 & \text{for } |x| \geqslant 1+\varepsilon. \end{cases} \tag{13}$$

By (N),

$$\int_0^t \int_{\mathbf{R}} f_\varepsilon(x)\, \nu^n(ds \times dx) \underset{P}{\rightarrow} 0, \qquad t \in \mathbf{R}^+,\ \varepsilon > 0.$$

In view of (10) we also have

$$\int_0^t \int_{\mathbf{R}} f_\varepsilon(x)\, N^n(ds \times dx) \underset{P}{\rightarrow} 0, \qquad t \in \mathbf{R}^+,\ \varepsilon > 0.$$

Now we define the new families of processes $\{(\alpha_n, \mathscr{F}_n)\}$ and $\{(\beta_n(\varepsilon), \mathscr{F}^n)\}$:

$$\alpha_n(t) \overset{\text{df}}{=} X^n(t) - X^{h,n}(t), \tag{14}$$

$$\beta_n(t, \varepsilon) \overset{\text{df}}{=} \sum_{s \leqslant t} \Delta X^n(s)\, I\big(\Delta X^n(s) > 1+\varepsilon\big), \tag{15}$$

where $n \in N$, $t \in \mathbf{R}^+$ and $\varepsilon > 0$.

Note that

$$\sum_{s \leqslant t} I\big(\Delta X^n(s) > 1+\varepsilon\big) \underset{P}{\rightarrow} 0, \qquad n \in N,\ t \in \mathbf{R}^+,\ \varepsilon > 0.$$

Hence $\beta_n(t, \varepsilon) \underset{P}{\rightarrow} 0$ for every $t \in \mathbf{R}^+$ and $\varepsilon > 0$. By the definition of the function h, for every $\delta > 0$ there exists an $\varepsilon > 0$ such that if $1 \leqslant |x| \leqslant 1+\varepsilon$, then $|x - h(x)| < \delta$. Therefore

$$\alpha_n(t) \leqslant \delta X^n(t) + \beta_n(t, \varepsilon), \qquad n \in N,\ t \in \mathbf{R}^+,\ \varepsilon > 0. \tag{16}$$

From (11) and (C) one can easily see that

$$\lim_{\eta \to \infty} \overline{\lim_{n \to \infty}}\, P[X^n(t) > \eta] = 0.$$

So $\alpha_n(t) \underset{P}{\rightarrow} 0$, $t \in \mathbf{R}^+$. Since

$$\sup_{t \leqslant q} |\tilde{X}^n(t) - \tilde{X}^{h,n}(t)| = \tilde{\alpha}_n(q), \qquad q \in \mathbf{R}^+,$$

condition (C) is a consequence of (12).

Now we prove condition (M) from Theorem 2. It is sufficient to check that

$$\int_0^t \int_{\mathbf{R}} h^2(x)\, \nu^n(ds \times dx) \to \int_0^t \int_{\mathbf{R}} h^2(x)\, \nu^\infty(ds \times dx), \qquad t \in \mathbf{R}^+, \tag{17}$$

$$\sum_{s \leqslant t} \big(\Delta \tilde{X}^{h,n}(s)\big)^2 \underset{P}{\rightarrow} 0, \qquad t \in \mathbf{R}^+. \tag{18}$$

Since $\tilde{\mathscr{X}}^\infty$ has continuous trajectories, (18) is satisfied trivially. In order to obtain (17) we define a new sequence of functions $\{h_i\}_{i\in N} \subset C_{v(0)}$ such that $h^2(x)-h_i^2(x) \geqslant 0$, $x\in \boldsymbol{R}$, and, for $i\in N$,

$$h_i(x) \overset{\text{df}}{=} \begin{cases} 0 & \text{for } |x| \leqslant 1/i, \\ h(x) & \text{for } |x| > 2/i. \end{cases} \tag{19}$$

Due to condition (N) there exists a sufficiently slowly increasing sequence $\{i_n\}_{n\in N}$, $(i_n) \subset (n)$, for which

$$\int_0^t \int_{\boldsymbol{R}} h_{i_n}^2(x)\, v^n(ds\times dx) \underset{P}{\to} \int_0^t \int_{\boldsymbol{R}} h^2(x)\, v^\infty(ds\times dx), \qquad t\in \boldsymbol{R}^+. \tag{20}$$

Since

$$\int_0^t \int_{\boldsymbol{R}} \big(h^2(x)-h_{i_n}^2(x)\big) N^n(ds\times dx) \leqslant (2/i_n)^2\, X^n(t), \qquad n\in N,\ t\in \boldsymbol{R}^+,$$

we have, by (10),

$$\int_0^t \int_{\boldsymbol{R}} \big(h^2(x)-h_{i_n}^2(x)\big) v^n(ds\times dx) \underset{P}{\to} 0, \qquad t\in \boldsymbol{R}^+.$$

(i) $\Rightarrow$ (ii). It is a trivial consequence of Theorem 2 and Proposition 3.

Due to Theorem 3 we may deduce the necessary and sufficient conditions for the extended convergence in the case where the limit process is a simple Poisson process $(N, \mathscr{F}^N)$.

COROLLARY 2. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of processes from $\mathscr{A}_{\text{loc}}^+$ satisfying* (J^1). *If we define the new sequences of processes $\{(\delta_n, \mathscr{F}^n)\}_{n\in N}$ and $\{(\gamma_n(\varepsilon), \mathscr{F}^n)\}_{n\in N}$ by the formulae*

$$\delta_n(t) \overset{\text{df}}{=} X^n(t) - \sum_{s\leqslant t} \Delta X^n(s), \qquad \gamma_n(t, \varepsilon) = \sum_{s\leqslant t} \Delta X^n(s)\, I\big(|\Delta X^n(s)-1| > \varepsilon\big),$$

$$n\in N,\ t\in \boldsymbol{R}^+,\ \varepsilon > 0,$$

then the following two conditions are equivalent:

(i) $$(\mathscr{X}^n, \mathscr{F}^n) \to (N, \mathscr{F}^N),$$

(ii) $$\begin{cases} (21) & \tilde{X}^n(t) \underset{P}{\to} EN(t), \quad t\in \boldsymbol{R}^+, \\ (22) & \tilde{\delta}_n(t) \underset{P}{\to} 0, \quad t\in \boldsymbol{R}^+, \\ (23) & \tilde{\gamma}_n(t, \varepsilon) \underset{P}{\to} 0, \quad t\in \boldsymbol{R}^+,\ \varepsilon > 0. \end{cases}$$

It is not difficult to obtain the following characterisation of the extended convergence of local martingales, applying the arguments from the proof of Theorem 3 and Proposition 2:

THEOREM 4. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of processes from $\mathscr{M}_{\text{loc}}$*

satisfying (J^2). *Suppose also that for the limit process* $(\mathscr{X}^\infty, \mathscr{F}^\infty)$ *conditions* (H_∞) *and* (H^1_∞) *holds. Then two following conditions are equivalent*:

(i) $$(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty);$$

(ii) $$\begin{cases} \text{(D)} & \langle \mathscr{X}^n \rangle(t) \to \langle \mathscr{X}^\infty \rangle(t), \quad t \in \mathbf{R}^+, \\ \text{(N)} & \int_0^t \int_{\mathbf{R}} f(x)\, \nu^n(ds \times dx) \underset{P}{\to} \int_0^t \int_{\mathbf{R}} f(x)\, \nu^\infty(ds \times dx), \quad t \in \mathbf{R}^+, f \in C_{v(0)}. \end{cases}$$

At the end of this section we discuss the cases where the limit processes are the most interesting local martingales: $(N-EN, \mathscr{F}^N)$ and $(W, \mathscr{F}^W)$.

Corollary 3. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of processes from* $\mathscr{M}_{\mathrm{loc}}$ *satisfying* (J^2). *If we define a family of processes* $\{(\beta_n(\varepsilon), \mathscr{F}^n)\}_{n\in N}$, $\varepsilon > 0$, *by the formulae*

$$\beta_n(t, \varepsilon) = \sum_{s\leq t} (\Delta X^n(s))^2 I(|\Delta X^n(s) - 1| > \varepsilon), \quad n \in N, t \in \mathbf{R}^+, \varepsilon > 0,$$

then two following conditions are equivalent:

(i) $$(\mathscr{X}^n, \mathscr{F}^n) \to (N-EN, \mathscr{F}^N);$$

(ii) $$\begin{cases} (24) & \langle \mathscr{X}^n \rangle(t) \underset{P}{\to} EN(t), \quad t \in \mathbf{R}^+, \\ (25) & \langle \mathscr{X}^{n,c} \rangle(t) \underset{P}{\to} 0, \quad t \in \mathbf{R}^+, \\ (26) & \tilde{\beta}_n(t, \varepsilon) \underset{P}{\to} 0, \quad t \in \mathbf{R}^+, \varepsilon > 0. \end{cases}$$

Corollary 4. *Let* $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ *be a sequence of processes from* $\mathscr{M}_{\mathrm{loc}}$ *satisfying* (J^2). *If* $\{(\alpha_n(\varepsilon), \mathscr{F}^n)\}_{n\in N}$, $\varepsilon > 0$, *is a family of processes given by the formulae*

$$\alpha_n(t, \varepsilon) = \sum_{s\leq t} (\Delta X^n(s))^2 I(|\Delta X^n(s)| > \varepsilon), \quad n \in N, t \in \mathbf{R}^+, \varepsilon > 0,$$

then the three following conditions are equivalent:

(i) $$(\mathscr{X}^n, \mathscr{F}^n) \to (W, \mathscr{F}^W);$$

(ii) $$\mathscr{X}^n \underset{\mathscr{D}}{\to} W;$$

(iii) $$\begin{cases} (27) & \langle \mathscr{X}^n \rangle(t) \underset{P}{\to} EW^2(t), \quad t \in \mathbf{R}^+, \\ (28) & \tilde{\alpha}_n(t, \varepsilon) \underset{P}{\to} 0, \quad t \in \mathbf{R}^+, \varepsilon > 0. \end{cases}$$

4. PROOFS

Lemma 2. *Let* $(\mathscr{Z}, \mathscr{F})$ *be a prediction process of* $(\mathscr{X}, \mathscr{F})$. *Suppose that on a probability space* $(\bar{\Omega}, \bar{F}, \bar{P})$ *there are given the process* $\bar{\mathscr{X}}$ *and the family of processes* $\{Y_{T,\theta,m}\}$, *where* $m \in N$, $T \in \mathbf{Q}^m$, $\theta \in [\mathbf{Q} \cup -\mathbf{Q}]^m$ $(-\mathbf{Q} = \{t\colon -t \in \mathbf{Q}\})$

and Q *is a countable, dense subset of* $\mathbf{R}^+$, $Q \subset \operatorname{Cont} \mathscr{X} \cap \operatorname{Cont} \mathscr{Z}$ *such that*

(29) $$\mathscr{L}\big((X(t_1), X_{T,\Theta^1,m}(t_1)), \ldots, (X(t_m), X_{T,\Theta^m,m}(t_m))\big) = \mathscr{L}\big((\bar{X}(t_1), Y_{T,\Theta^1,m}(t_1)), \ldots, (\bar{X}(t_m), Y_{T,\Theta^m,m}(t_m))\big)$$

for every $m \in N$, $T \in Q^m$, $\Theta^i \in [Q \cup -Q]^m$, $i = 1, 2, \ldots, m$.

Let $\mathscr{G}$ *be a filtration on* $(\bar{\Omega}, \bar{F}, \bar{P})$,

$$\dot{G}(t) \overset{\text{df}}{=} \bigcap_{u>t} \sigma\big((\bar{X}(s), Y_{T,\Theta,m}(s)) \colon s \leqslant u,\ T \in Q^m,\ \Theta \in [Q \cup -Q]^m,\ m \in N\big)$$

and let $(\bar{\mathscr{Z}}, \mathscr{G})$ *be a prediction process of* $(\bar{\mathscr{X}}, \mathscr{G})$. *Then*

(30) $$\bar{\mathscr{X}}_{T,\Theta,m} = Y_{T,\Theta,m}, \quad m \in N,\ T \in Q^m,\ \Theta \in [Q \cup -Q]^m,$$

(31) $$\mathscr{L}\big((\mathscr{X}, \mathscr{Z})\big) = \mathscr{L}\big((\bar{\mathscr{X}}, \bar{\mathscr{Z}})\big).$$

Proof. We have to check that for every $t \in Q$, $T \in Q^m$, $\Theta \in [Q \cup -Q]^m$, $m \in N$,

$$Y_{T,\Theta,m}(t) = \bar{E}\big(h_{T,\Theta,m}(\bar{X}) \mid G(t)\big),$$

where the mapping $h_{T,\Theta,m} \colon D(\mathbf{R}) \to \mathbf{C}$ is defined by

(32) $$h_{T,\Theta,m}(x) \overset{\text{df}}{=} \exp i \sum_{k=1}^{m} \theta_k x(t_k), \quad x \in D(\mathbf{R}).$$

Since $Y_{T,\Theta,m}(t)$ is $G(t)$-measurable and, by the arguments from the proof of Proposition 1 in [12],

$$EI(A)\, Y_{T,\Theta,m}(t) = EI(A)\, h_{T,\Theta,m}(\mathscr{X}), \quad t \in Q,\ A \in G(t),$$

the first conclusion follows.

In order to finish the proof it is sufficient, due to the inclusion $Q \subset \operatorname{Cont} \mathscr{Z}$, to verify that, for every $m \in N$ and $T \in Q^m$,

$$\mathscr{L}\big((X(t_1), Z(t_1)), \ldots, (X(t_m), Z(t_m))\big) = \mathscr{L}\big((\bar{X}(t_1), \bar{Z}(t_1)), \ldots, (\bar{X}(t_m), \bar{Z}(t_m))\big).$$

But this result immediately follows from Lemma 6 [12] and the equalities

$$\mathscr{L}\big(X_{T,\Theta^1,m}(t_1), \ldots, X_{T,\Theta^m,m}(t_m)\big) = L\big(\bar{X}_{T,\Theta^1,m}(t_1), \ldots, \bar{X}_{T,\Theta^m,m}(t_m)\big)$$

for every $m \in N$, $T \in Q^m$, $\Theta^i \in [Q \cup -Q]^m$, $i = 1, 2, \ldots, m$.

LEMMA 3 (Aldous [1]). *Let* S *and* S_1 *be two Polish spaces. The sequence* $\{(x_n, z_n)\}_{n \in N} \subset D(S \times S_1)$ *is relatively compact iff the following conditions are satisfied:*

(33) $\{(x_n(t), z_n(t);\ t \leqslant q\}$ *is relatively compact in* $S \times S_1$;

(34) *suppose that the sequence* $\{t_n^i\}_{n \in N}$, $i = 1, 2, 3$, *are such that* $t_n^1 \leqslant t_n^2 \leqslant t_n^3$, $n \in N$, $t_n^i \to t$, $i = 1, 2, 3$, *and* $\big(x_n(t_n^i), z_n(t_n^i)\big) \to (x^i, z^i)$,

$i = 1, 2, 3$; *then*

$$(x^1, z^1) = (x^2, z^2) \quad or \quad (x^2, z^2) = (x^3, z^3);$$

(35) *suppose that the sequence* $\{t_n\}_{n \in N}$ *is such that* $t_n \downarrow 0$ *and* $(x_n(0), z_n(0)) \to (x^0, z^0)$, $(x_n(t_n), z_n(t_n)) \to (x^1, z^1)$; *then* $(x^0, z^0) = (x^1, z^1)$.

Proof of Proposition 1. First suppose that assumptions (i) and (ii) hold for every $m \in N$, every $T \in Q^m$, $\Theta \in [Q \cup -Q]^m$ for some countable and dense subset of $Q \subset R^+$ such that

$$Q \subset \bigcap_{n \in N \cup \{\infty\}} [\operatorname{Cont} \mathscr{X}^n \cap \operatorname{Cont} \mathscr{X}^n].$$

Let $h_{T,\Theta,m}$ be the mapping defined by (32) for which $T \in Q^m$ and $\Theta \in [Q \cup -Q]^m$. It follows from the definition that $P(\mathscr{X}^n \in D_{h_{T,\Theta,m}}) = 0$, $n \in N \cup \{\infty\}$, where

$$D_{h_{T,\Theta,m}} = \{x \in D(R):\ h_{T,\Theta,m} \text{ is discontinuous in } x\}.$$

Due to the maximal inequality we have, for every $\varepsilon > 0$, $n \in N$ and $q \in R^+$,

$$P[\sup_{t \leqslant q} Z^n(t)(D_{h_{T,\Theta,m}}) \geqslant \varepsilon] \leqslant \varepsilon^{-1} EZ^n(q)(D_{h_{T,\Theta,m}})$$
$$= \varepsilon^{-1} P[X^n \in D_{h_{T,\Theta,m}}] = 0.$$

Hence

(36) $P[\tilde{h}_{T,\Theta,m}$ is continuous in $Z^n(t)$ and

$$Z^n(t-):\ t \in R^+,\ n \in N \cup \{\infty\}] = 1,$$

where $\tilde{h}_{T,\Theta,m}\colon \mathscr{P}(D(R)) \to C$ is defined by the formulae

$$\tilde{h}_{T,\Theta,m}(p) = \int_{R^m} \exp i \sum_{k=1}^{m} \theta_k x_k\, p_{t_1,\ldots,t_m}(dx_1, \ldots, dx_m), \qquad p \in \mathscr{P}(D(R)).$$

By simple calculations

(37) $$P[\tilde{h}_{T,\Theta,m}(Z^n(t)) = X^n_{T,\Theta,m}(t):\ t \in R^+,\ n \in N \cup \{\infty\}] = 1$$

for every $m \in N$, $T \in Q^m$, $\Theta \in [Q \cup -Q]^m$.

Since $Q \subset \operatorname{Cont} \mathscr{X}^\infty_{T,\Theta,m}$, by (i) and (ii),

(38) $$((\mathscr{X}^n, \mathscr{X}^n_{T,\Theta^1,m}), \ldots, (\mathscr{X}^n, \mathscr{X}^n_{T,\Theta^m,m})) \underset{\mathscr{D}}{\to} ((\mathscr{X}^\infty, \mathscr{X}^\infty_{T,\Theta^1,m}), \ldots, (\mathscr{X}^\infty, \mathscr{X}^\infty_{T,\Theta^m,m}))$$

in $[D(C^2)]^m$, $m \in N$, $T \in Q^m$, $\Theta^i \in [Q \cup -Q]^m$, $i = 1. 2, \ldots, m$.

Condition (38) may be replaced by the equivalent one:

$$\mathscr{Y}^n \to \mathscr{Y}^\infty \quad \text{in } [D(C^2)]^\infty,$$

where $\mathscr{Y}^n = (\mathscr{Y}^n_1, \mathscr{Y}^n_2, \ldots)$ and $\mathscr{Y}^n_k = (\mathscr{X}^n, \mathscr{X}^n_{T_k,\Theta_k,m_k})$ for some $T_k \in Q^{m_k}$, $\Theta_k \in [Q \cup -Q]^{m_k}$, $m_k \in N$, $n \in N$.

According to the Skorokhod representation theorem (see e.g. [7]) there exists a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ and a sequence of processes $\{\bar{\mathscr{Y}}^n\}_{n\in N\cup\{\infty\}}$, defined on that space, such that $\mathscr{L}(\mathscr{Y}^n) = \mathscr{L}(\bar{\mathscr{Y}}^n)$ and

$$\bar{\mathscr{Y}}^n(\bar{\omega}) \to \bar{\mathscr{Y}}^\infty(\bar{\omega}) \quad \text{in } [D(C^2)]^\infty$$

for almost all $\bar{\omega}\in\bar{\Omega}$. By Lemma 2 we may assume that

$$\mathscr{Y}^n(\omega) \to \mathscr{Y}^\infty(\omega) \quad \text{in } [D(C^2)]^\infty \tag{39}$$

for almost all $\omega\in\Omega$.

Now observe that, by Proposition 43.6 from [1], $\{Z^n(t)\colon t \leqslant q\}$ is relatively compact in $\mathscr{P}(D(\boldsymbol{R}))$ for every $q\in\boldsymbol{R}^+$. Hence, passing to a subsequence if necessary, we infer that $\{Z^n(t, \omega)\colon t \leqslant q\}$ is relatively compact in $\mathscr{P}(D(\boldsymbol{R}))$ for every $q\in\boldsymbol{R}^+$ and almost all $\omega\in\Omega$.

Let us fix an $\omega\in\Omega$. We apply Lemma 3 to the sequence $\{(\mathscr{X}^n(\omega), \mathscr{Z}^n(\omega))\}_{n\in N} \subset D\big(\boldsymbol{R}\times\mathscr{P}(D(\boldsymbol{R}))\big)$. By (39) and the considerations above it is obvious that condition (33) holds.

Let us note that for a fixed function $\tilde{h}_{T,\Theta,m}$ there holds by (39) at least one of the following pairs of conditions:

$$\begin{cases}\big(X^n(t_n^1, \omega), \tilde{h}_{T,\Theta,m}(Z^n(t_n^1, \omega))\big) \to \big(X^\infty(t-, \omega), \tilde{h}_{T,\theta,m}(Z^\infty(t-, \omega))\big),\\ \big(X^n(t_n^2, \omega), \tilde{h}_{T,\Theta,m}(Z^n(t_n^2, \omega))\big) \to \big(X^\infty(t-, \omega), \tilde{h}_{T,\theta,m}(Z^\infty(t-, \omega))\big);\end{cases} \tag{40}$$

$$\begin{cases}\big(X^n(t_n^2, \omega), \tilde{h}_{T,\Theta,m}(Z^n(t_n^2, \omega))\big) \to \big(X^\infty(t, \omega), \tilde{h}_{T,\theta,m}(Z^\infty(t, \omega))\big),\\ \big(X^n(t_n^3, \omega), \tilde{h}_{T,\Theta,m}(Z^n(t_n^3, \omega))\big) \to \big(X^\infty(t, \omega), \tilde{h}_{T,\theta,m}(Z^\infty(t, \omega))\big)\end{cases} \tag{41}$$

for every sequence $\{t_n^i\}_{n\in N}$, $i = 1, 2, 3$, from (34).

In the case where (40) holds, (34) is satisfied with $x^2 = x^1 = X^\infty(t-, \omega)$, $z^2 = z^1 = Z^\infty(t-, \omega)$. Similarly, (41) implies (34) with $x^2 = x^3 = X^\infty(t, \omega)$, $z^2 = z^3 = Z^\infty(t, \omega)$. In the same way (35) can be obtained. Therefore condition (39) implies

$$(\mathscr{X}^n(\omega), \mathscr{Z}^n(\omega)) \to (\mathscr{X}^\infty(\omega), \mathscr{Z}^\infty(\omega)) \quad \text{in } D\big(\boldsymbol{R}\times\mathscr{P}(D(\boldsymbol{R}))\big) \tag{42}$$

for almost all $\omega\in\Omega$ and, as a consequence, $(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty)$.

Now we prove the converse implications. Due to the Skorokhod representation theorem we may assume (42), so conditions (36), (37) and Lemma 3 complete the proof of Proposition 1.

COROLLARY 5. *Let $h\colon D(\boldsymbol{R}) \to D(\boldsymbol{R})$ be a measurable mapping such that $P(\mathscr{X}^\infty\in D_h) = 0$. Assume that $(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty)$. If the sequence of processes $\{h(\mathscr{X}^n)\}_{n\in N}$ is adapted to the sequence of filtrations $\{\mathscr{F}^n\}_{n\in N}$, then*

$$\big((\mathscr{X}^n, h(\mathscr{X}^n)), \mathscr{F}^n\big) \to \big((\mathscr{X}^\infty, h(\mathscr{X}^\infty)), \mathscr{F}^\infty\big).$$

COROLLARY 6. *Let $\{(\mathscr{X}_i^n, \mathscr{F}^n)\}_{n\in N\cup\{\infty\}, i\in N}$ and $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N\cup\{\infty\}}$ be families of processes with filtrations. Suppose that*

$$(\mathscr{X}_i^n, \mathscr{F}^n) \to (\mathscr{X}_i^\infty, \mathscr{F}^\infty), \qquad i\in N, \tag{43}$$

(44) $$(\mathscr{X}_i^\infty, \mathscr{F}^\infty) \to (\mathscr{X}^\infty, \mathscr{F}^\infty),$$

(45) $$\lim_{i\to\infty} \overline{\lim_{n\to\infty}} P[\sup_{t\leqslant q} |X_i^n(t) - X^n(t)| \geqslant \varepsilon] = 0, \quad \varepsilon > 0.$$

Then

$$(\mathscr{X}^n, \mathscr{F}^n) \to (\mathscr{X}^\infty, \mathscr{F}^\infty).$$

Proof of Proposition 2. We use the technics of Jacod [8]. We start from some elementary remarks.

Let $\{\delta_i\}_{i\in N}$ be a sequence of positive constants. For $x \in D(\boldsymbol{R})$ we define a sequence of partitions of $\boldsymbol{R}^+$, $\{t_{ik}^x\}$, such that

(46) $$t_{i0}^x = 0, \quad t_{i,k+1}^x = (t_{ik}^x + \delta_{ik}) \wedge \inf[t\colon\ t > t_{ik}^x, |\Delta x(t)| > \delta_i],$$

where $\delta_i/2 \leqslant \delta_{ik} \leqslant \delta_i$, $i, k \in N$.

Suppose that $\Delta x_\infty(t) \neq \delta_i$, $t \in \boldsymbol{R}^+$, and $\Delta x_\infty(t_{ik}^{x_\infty} + \delta_{ik}) = 0$. In this case the mappings $h^i\colon D(\boldsymbol{R}) \to D(\boldsymbol{R})$, $i \in N$, defined by

(47) $$h^i(x)(t) = \sum_{k=0}^{r_i^x(t)} \left(t_{i,k+1}^x \wedge t) - x(t_{ik}^x)\right)^2, \quad x \in D(\boldsymbol{R}),$$

are continuous in x_∞, $i \in N$.

Now let $\{\delta_i\}_{i\in N}$, $\{\delta_{ik}\}_{i\in N, k\in N\cup\{\infty\}}$ be two families of constants such that $\delta_i \downarrow 0$, $\delta_i/2 \leqslant \delta_{ik} \leqslant \delta_i$ and $P(\mathscr{X}^\infty \in D_{h^i}) = 0$, $i \in N$. Therefore, by Corollary 5,

(48) $$\left((X^n, h^i(\mathscr{X}^n)), \mathscr{F}^n\right) \to \left((\mathscr{X}^\infty, h^i(\mathscr{X}^\infty)), \mathscr{F}^\infty\right), \quad i \in N.$$

Using the arguments of Jacod [8] we obtain

LEMMA 4. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of processes from $\mathscr{M}_{\mathrm{loc}}$ and satisfy* ($\mathrm{J}_{\mathrm{E}}^1$). *If $\mathscr{X}^n \underset{\mathscr{D}}{\to} \mathscr{X}^\infty$, then, for every $\varepsilon > 0$,*

(49) $$\lim_{i\to\infty} \overline{\lim_{n\to\infty}} P[\sup_{t\leqslant q} |h^i(\mathscr{X}^n)(t) - [\mathscr{X}^n](t)| \geqslant \varepsilon] = 0, \quad q \in \boldsymbol{R}^+,$$

(50) $$\sup_{t\leqslant q} |h^i(\mathscr{X}^\infty)(t) - [\mathscr{X}^\infty](t)| \underset{P}{\to} 0, \quad q \in \boldsymbol{R}^+.$$

In order to complete the proof we use the two-dimensional version of Corollary 6. We get $\mathscr{X}_i^n = (\mathscr{X}^n, h^i(\mathscr{X}^n))$, $n \in N \cup \{\infty\}$, $i \in N$, $\mathscr{X}^n = (\mathscr{X}^n, [\mathscr{X}^n])$, $n \in N \cup \{\infty\}$, $\mathscr{F}^n = \mathscr{F}^n$, $n \in N \cup \{\infty\}$.

Proof of Proposition 3.

LEMMA 5. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of processes from $\mathscr{A}_{\mathrm{loc}}^+$ and $(\mathscr{X}^\infty, \mathscr{F}^\infty) \in \mathscr{A}^+$. Suppose that $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ satisfies* (J^1) *and $\mathscr{X}^n \underset{\mathscr{D}_f}{\to} \mathscr{X}^\infty$ ($\underset{\mathscr{D}_f}{\to}$ means convergence of the finite-dimensional distribution). Then there exists a* *sequence* $\{(\mathscr{Y}^n, \mathscr{F}^n)\}_{n\in N}$ *of processes from* $\mathscr{A}^+_{\text{loc}}$ *such that*

(51) $$\sup_{t\leqslant q}|X^n(t)-Y^n(t)| \underset{P}{\to} 0, \quad q\in \boldsymbol{R}^+,$$

(52) $$\sup_{t\leqslant q}|\tilde{X}^n(t)-\tilde{Y}(t)| \underset{P}{\to} 0, \quad q\in \boldsymbol{R}^+,$$

(53) $$\{Y^n(t)\}_{n\in N} \text{ is uniformly integrable}, \quad t\in \boldsymbol{R}^+.$$

Proof. Let $\{\varepsilon_m\}_{m\in N}$, $\{t_m\}_{m\in N}$ be two sequences of nonnegative numbers for which $\varepsilon_m \downarrow 0$, $0 = t_0 < t_1 < \ldots$,

$$\lim_{k\to\infty} t_{mk} = +\infty$$

and $\{t_m\}_{m\in N} \subset \text{Cont}(\mathscr{X}^\infty)$. The convergence $\mathscr{X}^n \underset{D_f}{\to} \mathscr{X}^\infty$ implies that, for every $m\in N$ and every $i<m$, there exists a constant $N(m)$ such that for each $n>N(m)$ we have

(54) $$\left|E(X^n(t_i)\wedge \varepsilon_m^{-1}) - E\left(X^\infty(t_i)\wedge \varepsilon_m^{-1}\right)\right| < \varepsilon_m.$$

Now define a sequence $\{x_n\}_{n\in N}$ of elements from $D(\boldsymbol{R})$ and a sequence of processes $\{\mathscr{Y}^n\}_{n\in N}$ as follows:

(55) $$x_n(t) \overset{\text{df}}{=} \begin{cases} \varepsilon_m^{-1}, & t<t_m, \\ \varepsilon_{i+1}^{-1}, & t_i \leqslant t < t_{i+1},\ i\geqslant m, \end{cases}$$

(56) $$Y^n(t) \overset{\text{df}}{=} X^n(t)\wedge x_n(t), \quad t\in \boldsymbol{R}^+,$$

for $m\in N$, $N(m)\leqslant n < N(m+1)$ and $\mathscr{Y}^n \overset{\text{df}}{=} x_n$ for $n<N(1)$.

Let us fix an $m\in N$, $\varepsilon>0$. Since $x_n(0)\to +\infty$ as $n\to +\infty$ and, for every $c>0$, $\varepsilon>0$,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\, P[\sup_{t\leqslant t_m}|X^n(t)-Y^n(t)| \geqslant \varepsilon] &\leqslant \overline{\lim_{n\to\infty}}\, P[X^n(t_m) \geqslant x_n(0)] \\ &\leqslant \overline{\lim_{n\to\infty}}\, P[X^n(t_m)\geqslant c] \leqslant P[X^\infty(t_m)\geqslant c],\end{aligned}$$

condition (51) holds.

In order to prove (52) let us observe that for $n\geqslant N(m+1)$ the process $(X^n-Y^n)(\cdot\wedge t_m)$ is increasing. Since, for $n\geqslant N(m+1)$, also $(\mathscr{X}^n-\mathscr{Y}^n)(\cdot\wedge t_m) \prec (\mathscr{X}^n-\mathscr{Y}^n)(\cdot\wedge t_m)$, conditions (51) and (10) imply (52).

Finally, let us observe that, by (54),

$$\lim_{n\to\infty} EX^n(t_m) = EX^\infty(t_m)$$

and, by Theorem 5.4 of [2], the sequence $\{Y^n(t_m)\}_{n\in N}$ is uniformly integrable, $m\in N$. Hence (53) is satisfied.

To prove Proposition 3 we will also use the following modification of Theorem 16.3 from Aldous [1]:

LEMMA 6. *Let $\{(\mathscr{X}^n, \mathscr{F}^n)\}_{n\in N}$ be a sequence of submartingales such that the families $\{X^n(\tau_n)\colon \tau_n$ is $\mathscr{F}^n$-stopping time, $\tau_n \leqslant q\}_{n\in N}$, $\{X^\infty(\tau_\infty)\colon \tau_\infty$ is $\mathscr{F}^\infty$-stopping time, $\tau_\infty \leqslant q\}$ are uniformly integrable, $q\in \boldsymbol{R}^+$. Suppose that $v^\infty(\{t\} \times \boldsymbol{R}^+ \setminus \{0\}) = 0$, $t\in \boldsymbol{R}^+$, and $\{(\mathscr{Y}^n, \mathscr{F}^n)\}_{n\in N}$ is a sequence of processes with filtrations for which $((\mathscr{Y}^n, \mathscr{X}^n), \mathscr{F}^n) \to ((\mathscr{Y}^\infty, \mathscr{X}^\infty), \mathscr{F}^\infty)$.*

Then $(\mathscr{Y}^n, \mathscr{X}^n, \tilde{\mathscr{X}}^n) \underset{\mathscr{D}}{\to} (\mathscr{Y}^\infty, \mathscr{X}^\infty, \tilde{\mathscr{X}}^\infty)$ in $D(\boldsymbol{R}^3)$.

Now we are ready to prove Proposition 3.

It is clear, by Corollary 5, that

$$((\mathscr{Y}^n, \mathscr{X}^n \wedge i), \mathscr{F}^n) \to (\mathscr{Y}^\infty, \mathscr{X}^\infty \wedge i), \quad i\in N.$$

Since $(\mathscr{X}^\infty \wedge i, \mathscr{F}^\infty) \in \mathscr{A}^+$, by Lemmas 5 and 6 we have

$$(\mathscr{Y}^n, \mathscr{X}^n \wedge i, \widetilde{\mathscr{X}^n \wedge i}) \underset{\mathscr{D}}{\to} (\mathscr{Y}^\infty, \mathscr{X}^\infty \wedge i, \widetilde{\mathscr{X}^\infty \wedge i}) \quad \text{in } D(\boldsymbol{R}^3).$$

Now observe that

$$\lim_{i\to\infty} \overline{\lim_{n\to\infty}} P[\sup_{t\leqslant q} |X^n(t) - X^n(t) \wedge i| \geqslant \varepsilon] = 0, \qquad \varepsilon > 0,\, q\in \boldsymbol{R}^+,$$

$$\sup_{t\leqslant q} |X^\infty(t) - X^\infty(t) \wedge i| \underset{P}{\to} 0, \qquad q\in \boldsymbol{R}^+.$$

Therefore, by (10), for $\varepsilon > 0$ and $q\in \boldsymbol{R}^+$,

$$\lim_{i\to\infty} \overline{\lim_{n\to\infty}} P[\sup_{t\leqslant q} |\tilde{X}^n(t) - \widetilde{X^n \wedge i}(t)| \geqslant \varepsilon] = 0$$

and

$$\sup_{t\leqslant q} |\tilde{X}^\infty(t) - \widetilde{X^\infty \wedge i}(t)| \underset{P}{\to} 0.$$

Hence, due to the classical Theorem 4.2 from [2], the proof of Proposition 3 is completed.

Proof of Theorem 1. It is easy to see that, by Corollary 5, we have

$$((\mathscr{X}^{h,n}, B^{h,n}), \mathscr{F}^n) \to ((\mathscr{X}^{h,\infty}, B^\infty), \mathscr{F}^\infty), \tag{57}$$

$$\Big(\int_0^{\cdot}\int_{\boldsymbol{R}} f(x) N^n(ds\times dx), \mathscr{F}^n\Big) \to \Big(\int_0^{\cdot}\int_{\boldsymbol{R}} f(x) N^\infty(ds\times dx), \mathscr{F}^\infty\Big), \qquad f\in C_{v(0)}. \tag{58}$$

By (57) and the equalities $M^n = \mathscr{X}^{h,n} - B^{h,n}$, $M^\infty = \mathscr{X}^{h,\infty} - B^\infty$ $(n\in N)$ we get $(M^n, \mathscr{F}^n) \to (M^\infty, \mathscr{F}^\infty)$. Since

$$\sup_t |\Delta M^n(t)| \leqslant 4,$$

by Proposition 2 we obtain

$$((M^n, [M^n]), \mathscr{F}^n) \to ((M^\infty, [M^\infty]), \mathscr{F}^\infty). \tag{59}$$

Finally, we apply Proposition 3 to the process from (58) and (59), and conclusions (M) and (N) follow.

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Institute of Mathematics
Nicholas Copernicus University
87-100 Toruń
ul. Chopina 12/18
Poland

Received on 4. 3. 1985