PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 7, Fasc. 2 (1986), p. 125-147

ON A RANDOM VERSION OF THE ANSCOMBE CONDITION AND ITS APPLICATIONS

BY

KRZYSZTOF S. KUBACKI AND DOMINIK SZYNAL (LUBLIN)

Abstract. The paper proves theorems of [6] which are given here in a more general form. Moreover, some applications of the introduced version of the Anscombe condition are discussed.

1. Introduction. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables (r.vs.) defined on a probability space $(\Omega, \mathscr{F}, P)$. Suppose that there exists a probability measure μ such that

(1) $$Y_n \Rightarrow \mu, \quad n \to \infty \text{ (converges weakly)};$$

for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

(A) $$\limsup_{n\to\infty} P[\max_{|i-n|\leq \delta n} |Y_i - Y_n| \geq \varepsilon] \leq \varepsilon.$$

Suppose that $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued r.vs. defined on the same probability space $(\Omega, \mathscr{F}, P)$. The well known Anscombe's theorem [2] proves that if a sequence $\{Y_n, n \geq 1\}$ of r.vs. satisfies conditions (1) and (A), then

(2) $$Y_{N_n} \Rightarrow \mu, \quad n \to \infty,$$

for every sequence $\{N_n, n \geq 1\}$ of positive integer-valued r.vs. satisfying

(3) $$N_n/a_n \xrightarrow{P} 1, \quad n \to \infty \text{ (in probability)},$$

where $\{a_n, n \geq 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$.

Condition (A), called the *Anscombe condition*, plays very important role in proofs of limit theorems for sequences of r.vs. with random indices. Aldous [1] has pointed out that condition (A) is a necessary and sufficient one for (2) when (3) holds. A more general and stronger result than that of [2] and [1] has been given by Csörgö and Rychlik [4].

THEOREM 1 [4]. *Let $\{k_n, n \geqslant 1\}$ be a non-decreasing sequence of positive numbers. The following conditions are equivalent:*

(i) *the sequence $\{Y_n, n \geqslant 1\}$ satisfies (1) and the so-called "generalized Anscombe condition" with norming sequence $\{k_n, n \geqslant 1\}$: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\limsup_{n\to\infty} P\Big[\max_{|k_i^2 - k_n^2| \leqslant \delta k_n^2} |Y_i - Y_n| \geqslant \varepsilon\Big] \leqslant \varepsilon; \tag{A°}$$

(ii) *$Y_{N_n} \Rightarrow \mu \ (n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying*

$$k_{N_n}^2 / k_{a_n}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{4}$$

where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty \ (n \to \infty)$.

These results essentially base on the assumption $Y_n \Rightarrow \mu \ (n \to \infty)$. In [6] we have considered the case where that fact is not known but we want to know whether a sequence of r.vs. with random indices weakly converges to a measure (we are interested in, e.g., Gaussian measure). The following theorem has been given without proof:

THEOREM 2 [6]. *Let $\{k_n, n \geqslant 1\}$ be a non-decreasing sequence of positive numbers and let $\{\nu_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.vs. such that $\nu_n \xrightarrow{P} \infty \ (n \to \infty)$. The following conditions are equivalent:*

(i) *$Y_{\nu_n} \Rightarrow \mu \ (n \to \infty)$, and the sequence $\{Y_n, n \geqslant 1\}$ satisfies the so-called "Anscombe random condition" with norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{\nu_n, n \geqslant 1\}$: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\limsup_{n\to\infty} P\Big[\max_{|k_i^2 - k_{\nu_n}^2| \leqslant \delta k_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geqslant \varepsilon\Big] \leqslant \varepsilon; \tag{A*}$$

(ii) *$Y_{N_n} \Rightarrow \mu \ (n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying*

$$k_{N_n}^2 / k_{\nu_{a_n}}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{5}$$

where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty \ (n \to \infty)$.

We prove here Theorem 2 in a more general version and discuss some applications of the introduced random version of the Anscombe condition.

2. Random version of the Anscombe condition. Let $\{Y_n, n \geqslant 1\}$ be a sequence of r.vs. defined on a probability space $(\Omega, \mathscr{F}, P)$ and let $\{\alpha_n, n \geqslant 1\}$ be a non-decreasing sequence of positive r.vs. ($0 < \alpha_n \leqslant \alpha_{n+1}$ a.s., $n \geqslant 1$) defined on the same probability space $(\Omega, \mathscr{F}, P)$. Furthermore, let $\{\tau_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.vs. defined on $(\Omega, \mathscr{F}, P)$ and such that $\tau_n \xrightarrow{P} \infty \ (n \to \infty)$.

Definition (Anscombe random condition). A sequence $\{Y_n, n \geqslant 1\}$ is said *to satisfy the Anscombe random condition* with the norming sequence $\{\alpha_n, n \geqslant 1\}$ of positive r.vs. and filtering sequence $\{\tau_n, n \geqslant 1\}$ of positive integer-valued r.vs. if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\mathrm{A}^{**}) \qquad \limsup_{n\to\infty} P\Big[\max_{|\alpha_i^2-\alpha_{\nu_n}^2|\leqslant \delta\alpha_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geqslant \varepsilon\Big] \leqslant \varepsilon.$$

One can easily see that in the special case, when $\alpha_n = k_n$ a.s., $\tau_n = n$ a.s., $n \geqslant 1$, (A**) reduces to (A°), and hence, when $k_n^2 = n$ ($n \geqslant 1$), (A**) reduces to (A). On the other hand, if $\alpha_n = k_n$ a.s. ($n \geqslant 1$), then (A**) reduces to (A*).

Moreover, we notice that if a sequence $\{Y_n, n \geqslant 1\}$ satisfies (A), then it satisfies (A**) with the norming sequence $\{\sqrt{n}, n \geqslant 1\}$ (i.e. $\alpha_n = \sqrt{n}$ a.s., $n \geqslant 1$) and any filtering sequence $\{a_n, n \geqslant 1\}$ of positive integers such that $a_n \to \infty$ ($n \to \infty$).

The analogous remark refers to (A°). Namely, if a sequence $\{Y_n, n \geqslant 1\}$ satisfies (A°) with norming sequence $\{k_n, n \geqslant 1\}$, then it satisfies (A**) with the same norming sequence $\{k_n, n \geqslant 1\}$ (i.e. $\alpha_n = k_n$ a.s., $n \geqslant 1$) and any filtering sequence $\{a_n, n \geqslant 1\}$ of positive integers such that $a_n \to \infty$ ($n \to \infty$).

The following lemma generalizes these remarks and proves Lemma 1 from [6]:

LEMMA 1. *If a sequence $\{Y_n, n \geqslant 1\}$ of r.vs. satisfies* (A°) *with the norming sequence $\{k_n, n \geqslant 1\}$, then it satisfies* (A**) *with the same norming sequence $\{k_n, n \geqslant 1\}$ and any filtering sequence $\{\tau_n, n \geqslant 1\}$ independent of $\{Y_n, n \geqslant 1\}$.*

Proof. Let $K \in N$ be fixed. Then

$$P\Big[\max_{|k_i^2-k_{\tau_n}^2|\leqslant \delta k_{\tau_n}^2} |Y_i - Y_{\tau_n}| \geqslant \varepsilon\Big] \leqslant P[\tau_n \leqslant K] + \sum_{r=K+1}^{\infty} P[\tau_n = r]\, P\Big[\max_{|k_i^2-k_r^2|\leqslant \delta k_r^2} |Y_i - Y_r| \geqslant \varepsilon\Big],$$

as τ_n is for every $n \geqslant 1$ independent of $\{Y_n, n \geqslant 1\}$. But

$$\lim_{n\to\infty} P[\tau_n \leqslant K] = 0,$$

since $\tau_n \xrightarrow{P} \infty$ ($n \to \infty$). Choosing then K so large that

$$P\Big[\max_{|k_i^2-k_r^2|\leqslant \delta k_r^2} |Y_i - Y_r| \geqslant \varepsilon\Big] \leqslant \varepsilon \quad \text{for every } r > K,$$

we obtain the desired result.

There arises, obviously, a question about conditions which the filtering sequence $\{\tau_n, n \geqslant 1\}$ should satisfy in the case where we reject in Lemma 1 the assumption that $\{\tau_n, n \geqslant 1\}$ is independent of $\{Y_n, n \geqslant 1\}$. The following lemma gives an answer to this question and proves Lemma 2 from [6]:

LEMMA 2. *If a sequence $\{Y_n, n \geqslant 1\}$ of r.vs. satisfies* (A°) *with the norming sequence $\{k_n, n \geqslant 1\}$, then it satisfies* (A**) *with the same norming sequence $\{k_n, n \geqslant 1\}$ and any filtering sequence $\{N_n, n \geqslant 1\}$ satisfying* (4).

Proof. By the assumption and the remark after Definition we conclude that the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with the norming sequence $\{k_n, n \geqslant 1\}$ and any filtering sequence $\{a_n, n \geqslant 1\}$ of positive integers such that $a_n \to \infty$ $(n \to \infty)$. Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued r.vs. satisfying (4), i.e.

$$\text{(a)} \qquad k_{N_n}^2/k_{b_n}^2 \xrightarrow{P} 1 \quad (n \to \infty),$$

where $\{b_n, n \geqslant 1\}$ is a sequence of positive integers with $b_n \to \infty$ $(n \to \infty)$. Put

$$B_n = [|k_{N_n}^2 - k_{b_n}^2| \leqslant \eta k_{b_n}^2] \quad (n \geqslant 1),$$

where η is a fixed positive number. Then for every $\varepsilon > 0$ and $\delta > 0$ we have

$$\text{(b)} \quad P[\max_{|k_i^2 - k_{N_n}^2| \leqslant \delta k_{N_n}^2} |Y_i - Y_{N_n}| \geqslant \varepsilon] \leqslant P(B_n^c) +$$

$$+ P[\max_{|k_i^2 - k_{N_n}^2| \leqslant \delta k_{N_n}^2} |Y_i - Y_{b_n}| \geqslant \varepsilon/2; B_n] + P[|Y_{N_n} - Y_{b_n}| \geqslant \varepsilon/2; B_n]$$

$$\leqslant P(B_n^c) + P[\max_{|k_i^2 - k_{b_n}^2| \leqslant \delta^* k_{b_n}^2} |Y_i - Y_{b_n}| \geqslant \varepsilon/2] + P[\max_{|k_i^2 - k_{b_n}^2| \leqslant \eta k_{b_n}^2} |Y_i - Y_{b_n}| \geqslant \varepsilon/2],$$

where $\delta^* = \delta(1+\eta)+\eta$. Since the sequence $\{Y_n, n \geqslant 1\}$ satisfies condition (A**) with the norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{b_n, n \geqslant 1\}$ and since, by (a), $P(B_n^c) \to 0$ as $n \to \infty$, inequality (b) proves that the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{N_n, n \geqslant 1\}$. The proof of Lemma 2 is completed.

Remark 1. Further it will be shown that (4) is in a sense necessary for (A°) to imply (A**). It will be proved that it is not enough to assume (5).

The following lemma generalizes Lemma 2 as well as Lemma 3 of [6]:

LEMMA 3. *If a sequence $\{Y_n, n \geqslant 1\}$ of r.vs. satisfies (A**) with the norming sequence $\{\alpha_n, n \geqslant 1\}$ and filtering sequence $\{\tau_n, n \geqslant 1\}$, then it satisfies (A**) with the same norming sequence $\{\alpha_n, n \geqslant 1\}$ and any filtering sequence $\{N_n, n \geqslant 1\}$ such that $\alpha_{N_n}^2/\alpha_{\tau_{a_n}}^2 \xrightarrow{P} 1$ $(n \to \infty)$, where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$.*

Proof. Let us put

$$B_n = [|\alpha_{N_n}^2 - \alpha_{\tau_{a_n}}^2| \leqslant \eta \alpha_{\tau_{a_n}}^2] \quad (n \geqslant 1),$$

where η is a fixed positive number. Then, for every $\varepsilon > 0$ and $\delta > 0$, we have

$$P[\max_{|\alpha_i^2 - \alpha_{N_n}^2| \leqslant \delta \alpha_{N_n}^2} |Y_i - Y_{N_n}| \geqslant \varepsilon]$$

$$\leqslant P(B_n^c) + P[\max_{|\alpha_i^2 - \alpha_{\tau_{a_n}}^2| \leqslant \delta^* \alpha_{\tau_{a_n}}^2} |Y_i - Y_{\tau_{a_n}}| \geqslant \varepsilon/2] + P[\max_{|\alpha_i^2 - \alpha_{\tau_{a_n}}^2| \leqslant \eta \alpha_{\tau_{a_n}}^2} |Y_i - Y_{\tau_{a_n}}| \geqslant \varepsilon/2],$$

where $\delta^* = \delta(1+\eta)+\eta$. The assumption and this inequality imply the desired result.

It is easy to see that if $\alpha_n = k_n$ a.s. $(n \geq 1)$, then Lemma 3 reduces to Lemma 3 of [6].

3. The main theorem. The following theorem generalizes and strengthens the statements of Theorem 1 and Theorem 2:

THEOREM 3. *Let $\{\alpha_n, n \geq 1\}$ be a non-decreasing sequence of positive r.vs. and let $\{\nu_n, n \geq 1\}$ be a sequence of positive integer-valued r.vs. such that $\nu_n \xrightarrow{P} \infty \ (n \to \infty)$. The following conditions are equivalent:*

(i) *$Y_{\nu_n} \Rightarrow \mu \ (n \to \infty)$, and the sequence $\{Y_n, n \geq 1\}$ satisfies Anscombe random condition* (A**) *with norming sequence $\{\alpha_n, n \geq 1\}$ and filtering sequence $\{\nu_n, n \geq 1\}$;*

(ii) *$Y_{N_n} \Rightarrow \mu \ (n \to \infty)$ for every sequence $\{N_n, n \geq 1\}$ of positive integer-valued r.vs. such that*

$$\alpha_{N_n}^2/\alpha_{\nu_{a_n}}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{6}$$

where $\{a_n, n \geq 1\}$ is a sequence of positive integers with $a_n \to \infty \ (n \to \infty)$.

Proof. Let $\varepsilon > 0$ and a closed set $A \subset \mathbf{R}$ be given. Then, for every $\delta > 0$, we have

$$\begin{aligned} P[Y_{N_n} \in A] &\leq P[Y_{\nu_{a_n}} \in A^\varepsilon] + P[Y_{N_n} \in A;\ Y_{\nu_{a_n}} \notin A^\varepsilon] \\ &\leq P[Y_{\nu_{a_n}} \in A^\varepsilon] + P[|Y_{N_n} - Y_{\nu_{a_n}}| \geq \varepsilon] \\ &\leq P[Y_{\nu_{a_n}} \in A^\varepsilon] + P[|\alpha_{N_n}^2 - \alpha_{\nu_{a_n}}^2| > \delta\alpha_{\nu_{a_n}}^2] + P[\max_{|\alpha_i^2 - \alpha_{\nu_{a_n}}^2| \leq \delta\alpha_{\nu_{a_n}}^2} |Y_i - Y_{\nu_{a_n}}| \geq \varepsilon], \end{aligned}$$

where $A^\varepsilon = \{x \in \mathbf{R}: \varrho(x, A) \leq \varepsilon\}$, $\varrho(x, A) = \inf\{|x-y|: y \in A\}$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we see by (6), (A**), the assumption $Y_{\nu_n} \Rightarrow \mu$ $(n \to \infty)$ and Theorem 2.1 of [3] that (i) implies (ii).

If (ii) holds, then putting $N_n = \nu_n$ a.s. $(n \geq 1)$, we conclude that $Y_{\nu_n} \Rightarrow \mu$ $(n \to \infty)$. Suppose that (A**) with the norming sequence $\{\alpha_n, n \geq 1\}$ and filtering sequence $\{\nu_n, n \geq 1\}$ fails. Then there exist an $\varepsilon > 0$ and a subsequence $n_1 < n_2 < \ldots$ of positive integers $(n_j \to \infty, j \to \infty)$ such that

$$P[\max_{i \in B_{n_j}} |Y_i - Y_{\nu_{n_j}}| \geq \varepsilon] > \varepsilon \quad \text{for all } j \geq 1, \tag{a}$$

where

$$B_{n_j} = \{i \in \mathbf{N}: \alpha_{\nu_{n_j}}^2 \leq \alpha_i^2 \leq (1+1/j)\alpha_{\nu_{n_j}}^2\} \quad (j \geq 1)$$

or

$$B_{n_j} = \{i \in \mathbf{N}: (1-1/j)\alpha_{\nu_{n_j}}^2 \leq \alpha_i^2 \leq \alpha_{\nu_{n_j}}^2\} \quad (j \geq 1).$$

We shall only consider the first case as the second one can be treated similarly.

Let $\{G_i, 1 \leqslant i \leqslant M\}$ be disjoint and open subsets such that

$$0 < \mu(G_i) = \mu(\bar{G}_i), \quad \mu\Big(\bigcup_{i=1}^{M} G_i\Big) > 1-\varepsilon/2,$$

and the diameter $G_i < \varepsilon/2$. Thus, by (a), there exist a set $G \in \{G_i, 1 \leqslant i \leqslant M\}$ and a subsequence $\{m_j, j \geqslant 1\}$ of the sequence $\{n_j, j \geqslant 1\}$ such that

$$\text{(b)} \quad P[Y_{\nu_{m_j}} \in G; \max_{\alpha^2_{\nu_{m_j}} \leqslant \alpha_i^2 \leqslant (1+1/j)\alpha^2_{\nu_{m_j}}} |Y_i - Y_{\nu_{m_j}}| \geqslant \varepsilon] > \varepsilon/(2M) \quad \text{for all } j \geqslant 1.$$

Let $N_j = \min\{C_j^1, C_j^2\}, j \geqslant 1$, where

$$C_j^1 = \max\{i \in N: \alpha_i^2 \leqslant (1+1/j)\alpha^2_{\nu_{m_j}}\}, \quad C_j^2 = \min\{i \geqslant \nu_{m_j}: Y_i \notin G\}.$$

Since the sequence $\{\alpha_n^2, n \geqslant 1\}$ is non-decreasing, we have $\alpha^2_{\nu_{m_j}} \leqslant \alpha^2_{C_j^2}$ a.s., $\alpha^2_{\nu_{m_j}} \leqslant \alpha^2_{C_j^1}$ a.s., $\alpha^2_{N_j} \leqslant (1+1/j)\alpha^2_{\nu_{m_j}}$ a.s. for all $j \geqslant 1$. Hence

$$\alpha^2_{N_j}/\alpha^2_{\nu_{m_j}} \xrightarrow{P} 1 \quad (j \to \infty),$$

which proves that the sequence $\{N_j, j \geqslant 1\}$ satisfies (6). But

$$\begin{aligned} P[Y_{N_j} \notin G] &= P[Y_{\nu_{m_j}} \notin G] + P[Y_{N_j} \notin G; Y_{\nu_{m_j}} \in G] \\ &\geqslant P[Y_{\nu_{m_j}} \notin G] + P[Y_{\nu_{m_j}} \in G; \max_{\alpha^2_{\nu_{m_j}} \leqslant \alpha_i^2 \leqslant (1+1/j)\alpha^2_{\nu_{m_j}}} |Y_i - Y_{\nu_{m_j}}| \geqslant \varepsilon] \\ &\geqslant P[Y_{\nu_{m_j}} \notin G] + \varepsilon/(2M), \end{aligned}$$

which, by Theorem 2.1 of [3], proves that $Y_{N_j} \not\Rightarrow \mu\ (j \to \infty)$. This is a contradiction to (ii), and it proves that the sequence $\{Y_n, n \geqslant 1\}$ must satisfy (A**) with the norming sequence $\{\alpha_n, n \geqslant 1\}$ and filtering sequence $\{\nu_n, n \geqslant 1\}$. The proof of Theorem 3 is completed.

Remark 2. It is easy to see that putting $\alpha_n = k_n$ a.s., $\nu_n = n$ a.s., $n \geqslant 1$, Theorem 3 reduces to Theorem 1. It is so since in this case (A**) reduces to (A°), whereas (6) reduces to (4). On the other hand, if $\alpha_n = k_n$ a.s. ($n \geqslant 1$), then Theorem 3 reduces to Theorem 2.

COROLLARY 1. *Let $\{k_n, n \geqslant 1\}$ be a non-decreasing sequence of positive numbers and let $\{a_n, n \geqslant 1\}$ be a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$. The following conditions are equivalent:*

(i) *$Y_{a_n} \Rightarrow \mu\ (n \to \infty)$, and the sequence $\{Y_n, n \geqslant 1\}$ satisfies* (A**) *with the norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{a_n, n \geqslant 1\}$;*

(ii) $Y_{N_n} \Rightarrow \mu$, $n \to \infty$, *for every* $\{N_n, n \geqslant 1\}$ *satisfying*

$$k_{N_n}^2 / k_{b_n}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{7}$$

where $\{b_n, n \geqslant 1\}$ *is a sequence of positive integers with* $b_n \to \infty$ $(n \to \infty)$ *and* $\{b_n, n \geqslant 1\} \subseteq \{a_n, n \geqslant 1\}$.

COROLLARY 2. *Let* $\{k_n, n \geqslant 1\}$ *be a non-decreasing sequence of positive numbers and let* $\{\nu_n, n \geqslant 1\}$ *be a sequence of positive integer-valued r.vs. such that* $\nu_n \xrightarrow{P} \infty$ $(n \to \infty)$. *The following conditions are equivalent*:

(i) $Y_{\nu_n} \Rightarrow \mu$ $(n \to \infty)$, *and the sequence* $\{Y_n, n \geqslant 1\}$ *satisfies* (A**) *with the norming sequence* $\{k_n, n \geqslant 1\}$ *and filtering sequence* $\{\nu_n, n \geqslant 1\}$;

(ii) $Y_{N_n} \Rightarrow \mu$ $(n \to \infty)$ *for every* $\{N_n, n \geqslant 1\}$ *satisfying*

$$k_{N_n}^2 / k_{\nu_{a_n}}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{8}$$

where $\{a_n, n \geqslant 1\}$ *is a sequence of positive integers with* $a_n \to \infty$ $(n \to \infty)$.

In the particular case, where $k_n^2 = n$, $\nu_n = [\lambda n]$ a.s. $(n \geqslant 1)$, where λ is an r.v. such that $P[0 < \lambda < \infty] = 1$, the following conditions are equivalent:

(i′) $Y_{[\lambda n]} \Rightarrow \mu$ $(n \to \infty)$, and the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with norming sequence $\{\sqrt{n}, n \geqslant 1\}$ and filtering sequence $\{[\lambda n], n \geqslant 1\}$;

(ii′) $Y_{N_n} \Rightarrow \mu$ $(n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying

$$N_n / a_n \xrightarrow{P} \lambda \quad (n \to \infty), \tag{8′}$$

where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$.

COROLLARY 3. *Let* $\{\alpha_n, n \geqslant 1\}$ *be a non-decreasing sequence of positive r. vs. and let* $\{a_n, n \geqslant 1\}$ *be a sequence of positive integers with* $a_n \to \infty$ $(n \to \infty)$. *The following conditions are equivalent*:

(i) $Y_{a_n} \Rightarrow \mu$ $(n \to \infty)$, *and the sequence* $\{Y_n, n \geqslant 1\}$ *satisfies* (A**) *with the norming sequence* $\{\alpha_n, n \geqslant 1\}$ *and filtering sequence* $\{a_n, n \geqslant 1\}$;

(ii) $Y_{N_n} \Rightarrow \mu$ $(n \to \infty)$ *for every* $\{N_n, n \geqslant 1\}$ *satisfying*

$$\alpha_{N_n}^2 / \alpha_{b_n}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{9}$$

where $\{b_n, n \geqslant 1\}$ *is a sequence of positive integers with* $b_n \to \infty$ $(n \to \infty)$ *and* $\{b_n, n \geqslant 1\} \subseteq \{a_n, n \geqslant 1\}$.

The following example elucidates the usefulness of these considerations.

Example 1 [12]. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent r.vs. defined by

$$P[X_{2^{2^n}} = 2^{2^{n-1}}] = P[X_{2^{2^n}} = -2^{2^{n-1}}] = \frac{1}{2} \quad (n \geqslant 1),$$

and X_k, for $k \neq 2^{2^n}$ $(n \geqslant 1, k \geqslant 1)$, has the normal distribution function $N(0, 1)$ with mean zero and variance one. Let us put

$$S_n = \sum_{k=1}^{n} X_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma^2 X_k,$$

and $Y_n := S_n/s_n$ $(n \geq 1)$. Then

$$Y_{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \ (n \to \infty), \qquad Y_{2^{2^n}} \Rightarrow X \ (n \to \infty),$$

where $\mathcal{N}_{a,b}$ denotes a normal r.v. with mean a and variance b, and the r.v. X has the characteristic function $\varphi(t) = \cos(t/\sqrt{2})e^{-t^2/4}$ (cf. [12]). Let $\{N'_n, n \geq 1\}$ be a sequence of positive integer-valued r.vs. such that

(10a) $$P[N'_n = 2^{2^n}-1] = 1-\frac{1}{n}, \quad P[N'_n = 2^{2^n}] = \frac{1}{n} \quad (n \geq 1).$$

Theorem 1 does not allow to confirm the weak convergence of the randomly indexed sequence $\{Y_{N'_n}, n \geq 1\}$. But it is easy to see that the sequence $\{Y_n, n \geq 1\}$ of r.vs. satisfies (A°) with norming sequence $\{k_n, n \geq 1\}$, where $k_n = s_n$ $(n \geq 1)$ (cf. [13], p. 11). Hence, by Lemma 1, the sequence $\{Y_n, n \geq 1\}$ satisfies also (A**) with the norming sequence $\{s_n, n \geq 1\}$ and any filtering sequence $\{a_n, n \geq 1\}$ of positive integers with $a_n \to \infty$ $(n \to \infty)$. Thus, by Corollary 1, $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$ for every $\{N_n, n \geq 1\}$ satisfying $s^2_{N_n}/s^2_{b_n} \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b_n, n \geq 1\} \subseteq \{2^{2^n}-1, n \geq 1\}$ $(b_n \to \infty, n \to \infty)$, and $Y_{N_n} \Rightarrow X$ $(n \to \infty)$ for every $\{N_n, n \geq 1\}$ satisfying $s^2_{N_n}/s^2_{b'_n} \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b'_n, n \geq 1\} \subseteq \{2^{2^n}, n \geq 1\}$ $(b'_n \to \infty, n \to \infty)$. For the sequence $\{N'_n, n \geq 1\}$, defined by (10a), we have

$$s^2_{N'_n}/s^2_{2^{2^n}-1} \xrightarrow{P} 1 \quad (n \to \infty).$$

Hence we conclude that in this case $Y_{N'_n} \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$.

Let us further notice that if A is an event independent of X_k $(k \geq 1)$ and

(10b) $$\nu_n = \begin{cases} 2^{2^n}-1 & \text{on } A, \\ 2^{2^n} & \text{on } A^c, \end{cases}$$

then $P[Y_{\nu_n} < x] \to \Phi(x)P(A) + P[X < x]P(A^c)$, $n \to \infty$, i.e.

(10c) $$Y_{\nu_n} \Rightarrow \mathcal{N}_{0,1} I(A) + XI(A^c) \quad (n \to \infty).$$

Moreover, the sequence $\{Y_n, n \geq 1\}$ satisfies (A**) with the norming sequence $\{s_n, n \geq 1\}$ and filtering sequence $\{\nu_n, n \geq 1\}$. Indeed, by the construction $\nu_n \xrightarrow{P} \infty$ $(n \to \infty)$, and ν_n is for every $n \geq 1$ independent of Y_k $(k \geq 1)$. Since the sequence $\{Y_n, n \geq 1\}$ satisfies (A°) with norming sequence $\{s_n, n \geq 1\}$, Lemma 1 confirms the desired result. Hence, and by (10c) and Corollary 2, we have

$$Y_{N_n} \Rightarrow \mathcal{N}_{0,1} I(A) + XI(A^c) \quad (n \to \infty)$$

for every $\{N_n, n \geq 1\}$ such that $s^2_{N_n}/s^2_{\nu_{a_n}} \xrightarrow{P} 1$ $(n \to \infty)$, where $\{a_n, n \geq 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$. This fact implies the statements after (10a), i.e. putting $A = \Omega$ or, equivalently, $\nu_n = 2^{2^n} - 1$ a.s. $(n \geqslant 1)$, we obtain $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $s^2_{N_n}/s^2_{b_n} \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b_n, n \geqslant 1\} \subseteq \{2^{2^n} - 1, n \geqslant 1\}$ $(b_n \to \infty, n \to \infty)$; putting, however, $A = \emptyset$ or, equivalently, $\nu_n = 2^{2^n}$ a.s. $(n \geqslant 1)$, we obtain $Y_{N_n} \Rightarrow X$ $(n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $s^2_{N_n}/s^2_{b'_n} \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b'_n, n \geqslant 1\} \subseteq \{2^{2^n}, n \geqslant 1\}$ $(b'_n \to \infty, n \to \infty)$.

4. More about usefulneses of the Anscombe random condition. Now we shall give two examples of sequences $\{Y_n, n \geqslant 1\}$ which fulfil (A**), whereas they satisfy neither (A) nor (A°). At the end of this section we shall give an example proving Remark 1.

Example 2. Let $\{X_k, k \geqslant 1\}$ be a sequence of r.vs. defined as follows: the r.vs. X_k, $k \neq 2^{2^n}$ $(n \geqslant 0, k \geqslant 1)$, are independent and have the normal distribution function $N(0, 1)$, and

$$X_2 = -X_1, \qquad X_{2^{2^n}} = -\sum_{j=2^{2^{n-1}}+1}^{2^{2^n}-1} X_j \quad (n \geqslant 1).$$

Let us put $S_n = \sum_{j=1}^{n} X_j$, $Y_n := S_n/\sqrt{n}$ $(n \geqslant 1)$. Then

$$Y_{2^{2^n}} = 0 \text{ a.s. } (n \geqslant 0), \qquad Y_{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \ (n \to \infty). \tag{11}$$

Indeed, for every $n \geqslant 0$ we have $S_{2^{2^n}} = 0$ a.s., which proves that $Y_{2^{2^n}} = 0$ a.s. $(n \geqslant 0)$. For the proof of the second property in (11) we put $S_n = S^*_n + S^{**}_n$ $(n \geqslant 1)$, where

$$S^*_n = \sum_{i=1}^{n}{}^* X_i = \sum_{\substack{i=1 \\ i \in N^*}}^{n} X_i, \qquad S^{**}_n = \sum_{i=1}^{n}{}^{**} X_i = \sum_{\substack{i=1 \\ i \in N^{**}}}^{n} X_i, \tag{12a}$$

and

$$N^* = \{j \in N \colon j \neq 2^{2^n}, n \geqslant 0\}, \qquad N^{**} = N \setminus N^*, \tag{12b}$$

and note that, for every $n \geqslant 1$, S^*_n is the sum of independent r.vs., $S^*_{2^{2^n}} = S^*_{2^{2^n}-1}$ a.s. for $n \geqslant 0$, while

$$S^{**}_1 = 0 \text{ a.s.}, \tag{12c}$$
$$S^{**}_{2^{2^n}-i} = -S^*_{2^{2^{n-1}}-1} \text{ a.s. for } 1 \leqslant i \leqslant 2^{2^n} - 2^{2^{n-1}} \quad (n \geqslant 1).$$

Moreover, by the Kolmogorov's inequality we have, for every $\varepsilon > 0$,

$$P[|S^{**}_{2^{2^n}-1}/\sqrt{2^{2^n}-1}| \geqslant \varepsilon] = P[|\sum_{\substack{i=1\\ i\in N^*}}^{2^{2^{n-1}}-1} X_i| \geqslant \varepsilon\sqrt{2^{2^n}-1}]$$
$$\leqslant 2^{2^{n-1}}/\varepsilon^2(2^{2^n}-1) \to 0 \quad (n\to\infty).$$

Hence, and taking into account that

$$E\exp\{it\, S^{*}_{2^{2^n}-1}/\sqrt{2^{2^n}-1}\} = E\exp\{it \sum_{\substack{j=1\\ j\in N^*}}^{2^{2^n}-1} X_j/\sqrt{2^{2^n}-1}\}$$
$$= \exp\{-t^2(2^{2^n}-n-1)/2(2^{2^n}-1)\} \to \exp\{-t^2/2\} \quad (n\to\infty),$$

we obtain

$$Y_{2^{2^n}-1} = S^{*}_{2^{2^n}-1}/\sqrt{2^{2^n}-1} + S^{**}_{2^{2^n}-1}/\sqrt{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \quad (n\to\infty),$$

which ends the proof of (11).

Now we shall prove that the sequence $\{Y_n,\, n\geqslant 1\}$ does not satisfy the Anscombe condition (A).

Indeed, if the sequence $\{Y_n,\, n\geqslant 1\}$ fulfilled (A), then in view of the remarks after Definition it would fulfil (A**) with norming sequence $\{\sqrt{n},\, n\geqslant 1\}$ and any filtering sequence $\{a_n,\, n\geqslant 1\}$ of positive integers with $a_n\to\infty$ $(n\to\infty)$, e.g. with $a_n = 2^{2^n}-1$ $(n\geqslant 1)$ for which

$$Y_{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \quad (n\to\infty).$$

Hence, by Corollary 1, $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$ for every $\{N_n,\, n\geqslant 1\}$ such that $N_n/(2^{2^n}-1) \xrightarrow{P} 1$ $(n\to\infty)$. So then, for $N_n = 2^{2^n}$ a.s. $(n\geqslant 1)$, we would have

$$Y_{2^{2^n}} \Rightarrow \mathcal{N}_{0,1} \quad (n\to\infty),$$

which is a contradiction to (11). Thus, the sequence $\{Y_n,\, n\geqslant 1\}$ does not satisfy (A**) with the norming sequence $\{\sqrt{n},\, n\geqslant 1\}$ and filtering sequence $\{2^{2^n}-1,\, n\geqslant 1\}$, and then, by the earlier considerations, we conclude that the sequence $\{Y_n,\, n\geqslant 1\}$ does not satisfy (A).

And now we shall prove that the sequence $\{Y_n,\, n\geqslant 1\}$ satisfies (A**) with the norming sequence $\{\sqrt{n},\, n\geqslant 1\}$ and filtering sequence $\{3^{2^n},\, n\geqslant 1\}$. To this end we note that for every $\varepsilon > 0$ and for every $\delta > 0$ we have

(13) $$P[\max_{|i-3^{2^n}|\leqslant\delta 3^{2^n}} |Y_i - Y_{3^{2^n}}| \geqslant \varepsilon] \leqslant P[\max_{|i-3^{2^n}|\leqslant\delta 3^{2^n}} |S_i - S_{3^{2^n}}| \geqslant \varepsilon\sqrt{3^{2^n}}/2] +$$
$$+ P[\max_{|i-3^{2^n}|\leqslant\delta 3^{2^n}} |S_i|\, |3^{2^n}-i|/\sqrt{i}\, 3^{2^n} \geqslant \varepsilon/2]$$

$$\leqslant 2P[\max_{[(1-\delta)3^{2^n}]\leqslant i\leqslant[(1+\delta)3^{2^n}]} |S_i - S_{[(1-\delta)3^{2^n}]}| \geqslant \varepsilon\sqrt{3^{2^n}}/4] +$$

$$+P[\max_{[(1-\delta)3^{2^n}]\leqslant i\leqslant[(1+\delta)3^{2^n}]} |S_i| \geqslant \varepsilon\sqrt{[(1-\delta)3^{2^n}]/2\delta}],$$

where $[x]$ denotes the integral part of the real number x. Further, by (12c) and the Kolmogorov's inequality, the first term on the right-hand side of (13) is less than or equal to

$$2P[\max_{[(1-\delta)3^{2^n}]\leqslant i\leqslant[(1+\delta)3^{2^n}]} |S_i^* - S^*_{[(1-\delta)3^{2^n}]}| \geqslant \varepsilon\sqrt{3^{2^n}}/4]$$

$$\leqslant 32\{[(1+\delta)3^{2^n}] - [(1-\delta)3^{2^n}]\}/\varepsilon^2 3^{2^n} \to 64\delta/\varepsilon^2 \quad (n\to\infty),$$

since, for $[(1-\delta)3^{2^n}] \leqslant i \leqslant [(1+\delta)3^{2^n}]$, we have

$$S_i^{**} - S^{**}_{[(1-\delta)3^{2^n}]} = \sum_{\substack{j=[(1-\delta)3^{2^n}]+1 \\ j\in N^{**}}}^{i} X_j = 0 \quad \text{a.s.},$$

while the second term on the right-hand side of (13) is less than or equal to

$$P[\max_{[(1-\delta)3^{2^n}]\leqslant i\leqslant[(1+\delta)3^{2^n}]} |S_i^*| \geqslant \varepsilon\sqrt{[(1-\delta)3^{2^n}]/4\delta}] +$$

$$+P[\max_{[(1-\delta)3^{2^n}]\leqslant i\leqslant[(1+\delta)3^{2^n}]} |S_i^{**}| \geqslant \varepsilon\sqrt{[(1-\delta)3^{2^n}]/4\delta}]$$

$$\leqslant 16\delta^2[(1+\delta)3^{2^n}]/\varepsilon^2[(1-\delta)3^{2^n}] + P[|S^*_{2^{2^n}-1}| \geqslant \varepsilon\sqrt{[(1-\delta)3^{2^n}]/4\delta}]$$

$$\leqslant 16\delta^2\{[(1+\delta)3^{2^n}] + 2^{2^n}\}/\varepsilon^2[(1-\delta)3^{2^n}] \to 16\delta^2(1+\delta)/\varepsilon^2(1-\delta) \quad (n\to\infty),$$

where $16\delta^2(1+\delta)/\varepsilon^2(1-\delta) \leqslant 64\delta/\varepsilon^2$ for $0<\delta\leqslant 1/2$.

Hence, for every $\varepsilon>0$ and $0<\delta\leqslant 1/2$, we have

$$\limsup_{n\to\infty} P[\max_{|i-3^{2^n}|\leqslant\delta 3^{2^n}} |Y_i - Y_{3^{2^n}}| \geqslant \varepsilon] \leqslant 128\delta/\varepsilon^2,$$

which proves that the sequence $\{Y_n, n\geqslant 1\}$ satisfies (A**) with the norming sequence $\{\sqrt{n}, n\geqslant 1\}$ and filtering sequence $\{3^{2^n}, n\geqslant 1\}$.

Let us further notice that $Y_{3^{2^n}} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$. Indeed, since for every $n\geqslant 1$ we have

$$Y_{3^{2^n}} = S^*_{3^{2^n}}/\sqrt{3^{2^n}} + S^{**}_{3^{2^n}}/\sqrt{3^{2^n}},$$

where, by the Kolmogorov inequality, for every $\varepsilon>0$,

$$P[|S^{**}_{3^{2^n}}/\sqrt{3^{2^n}}| \geqslant \varepsilon] = P[|S^*_{2^{2^n}-1}| \geqslant \varepsilon\sqrt{3^{2^n}}] \leqslant 2^{2^n}/\varepsilon^2 3^{2^n} \to 0 \quad (n\to\infty)$$

and

$$E\exp\{itS^*_{3^{2^n}}/\sqrt{3^{2^n}}\} = E\exp\{it\sum_{\substack{j=1\\ j\in N^*}}^{3^{2^n}} X_j/\sqrt{3^{2^n}}\}$$

$$= \exp\{-t^2(3^{2^n}-n-1)/2\cdot 3^{2^n}\} \to \exp\{-t^2/2\} \quad (n\to\infty),$$

we have $Y_{3^{2^n}} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$. Hence, by Corollary 1, $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$ for every $\{N_n, n\geqslant 1\}$ satisfying $N_n/b_n \xrightarrow{P} 1$ $(n\to\infty)$, where $\{b_n, n\geqslant 1\} \subseteq \{3^{2^n}, n\geqslant 1\}$ $(b_n\to\infty, n\to\infty)$.

Example 3. Let $\{X_k, k\geqslant 1\}$ be a sequence of r.vs. defined as follows: the r.vs. X_k, $k\neq 2^{2^n}$ $(n\geqslant 0)$, are independent and have the normal distribution function $N(0, 1)$ for $k\neq 3^{2^n}$ $(n\geqslant 1)$ and

$$P[X_{3^{2^n}} = 3^{2^{n-1}}] = P[X_{3^{2^n}} = -3^{2^{n-1}}] = \frac{1}{2} \quad (n\geqslant 1),$$

while

$$X_2 = -X_1, \quad X_{2^{2^n}} = -\sum_{j=2^{2^{n-1}}+1}^{2^{2^n}-1} X_j \quad (n\geqslant 1).$$

Let us put

$$k_n^2 = \sum_{j=1}^{n} \sigma^2 X_j \quad \text{for } n\neq 2^{2^m} (m\geqslant 0),$$

and

$$k^2_{2^{2^n}} = k^2_{2^{2^n}-1} \quad \text{for } n\geqslant 0, \quad S_n = \sum_{j=1}^{n} X_j \text{ and } Y_n := S_n/k_n \quad (n\geqslant 1).$$

Then

(14) $$Y_{2^{2^n}} = 0 \text{ a.s.}, \; n\geqslant 0, \quad Y_{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \quad (n\to\infty).$$

Indeed, for every $n\geqslant 0$, $S_{2^{2^n}} = 0$ a.s., which proves that $Y_{2^{2^n}} = 0$ a.s. for $n\geqslant 0$. For the proof of the second property in (14) we notice that $k^2_{2^{2^n}-1} \approx 2^{2^n}$ for sufficiently large n, and

$$Y_{2^{2^n}-1} = S^*_{2^{2^n}-1}/k_{2^{2^n}-1} + S^{**}_{2^{2^n}-1}/k_{2^{2^n}-1} \quad (n\geqslant 1),$$

where S_n^* and S_n^{**} are defined in (12). Furthermore, by (12c), we have

$$Y_{2^{2^n}-1} = (S^*_{2^{2^n}-1} - S^*_{2^{2^{n-1}}-1})/k_{2^{2^n}-1} = \sum_{j=2^{2^{n-1}}}^{2^{2^n}-1}{}^{*} X_j/k_{2^{2^n}-1}$$

$$= \sum_{j=2^{2^{n-1}}+1}^{2^{2^n}-1} X_j/k_{2^{2^n}-1} \quad \text{for every } n\geqslant 1,$$

where

$$E\exp\{it\sum_{j=2^{2^{n-1}}+1}^{2^{2^n}-1} X_j/k_{2^{2^n}-1}\}$$
$$=\cos(t\cdot 3^{2^{n-2}}/k_{2^{2^n}-1})\exp\{-t^2(2^{2^n}-2^{2^{n-1}}-1)/2k^2_{2^{2^n}-1}\}$$
$$\to \exp\{-t^2/2\} \quad (n\to\infty).$$

Hence, $Y_{2^{2^n}-1} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$, which completes the proof of (14).

From (14) and that

$$k^2_{2^{2^n}} = k^2_{2^{2^n}-1} \quad (n \geqslant 1)$$

we conclude that the sequence $\{Y_n, n\geqslant 1\}$ satisfies neither the Anscombe condition (A) nor the generalized Anscombe condition (A°) with the norming sequence $\{k_n, n\geqslant 1\}$. The proofs of these facts run similarly as in Example 2.

Now we shall prove that the sequence $\{Y_n, n\geqslant 1\}$ satisfies (A**) with the norming sequence $\{k_n, n\geqslant 1\}$ and filtering sequence $\{3^{2^n}, n\geqslant 1\}$. To this end we note that, for every $\varepsilon > 0$ and for every $0<\delta<1$, we have

$$(15)\quad P[\max_{|k_i^2-k^2_{3^{2^n}}|\leqslant \delta k^2_{3^{2^n}}} |Y_i - Y_{3^{2^n}}| \geqslant \varepsilon]$$
$$\leqslant P[\max_{|k_i^2-k^2_{3^{2^n}}|\leqslant \delta k^2_{3^{2^n}}} |S_i - S_{3^{2^n}}| \geqslant \varepsilon k_{3^{2^n}}/2] +$$
$$+ P[\max_{|k_i^2-k^2_{3^{2^n}}|\leqslant \delta k^2_{3^{2^n}}} |S_i|\,|k^2_{3^{2^n}} - k_i^2|/k_i k^2_{3^{2^n}} \geqslant \varepsilon/2]$$
$$\leqslant P[\max_{k^2_{3^{2^n}}\leqslant k_i^2\leqslant (1+\delta)k^2_{3^{2^n}}} |S_i^* - S^*_{3^{2^n}}| \geqslant \varepsilon k_{3^{2^n}}/2] +$$
$$+P[\max_{k^2_{3^{2^n}}\leqslant k_i^2\leqslant (1+\delta)k^2_{3^{2^n}}} |S_n^*| \geqslant \varepsilon k_{3^{2^n}}/4\delta] + P[\max_{k^2_{3^{2^n}}\leqslant k_i^2\leqslant (1+\delta)k^2_{3^{2^n}}} |S_i^{**}| \geqslant \varepsilon k_{3^{2^n}}/4\delta],$$

since $k^2_{3^{2^n}-1} \approx 3^{2^n}$ and $k^2_{3^{2^n}} \approx 2\cdot 3^{2^n}$ for sufficiently large n, whence

$$\{i\in N\colon |k_i^2 - k^2_{3^{2^n}}| \leqslant \delta k^2_{3^{2^n}}\} = \{i\in N\colon k^2_{3^{2^n}} \leqslant k_i^2 \leqslant (1+\delta)k^2_{3^{2^n}}\}$$

for sufficiently large n, and since for $i\in\{k^2_{3^{2^n}} \leqslant k_i^2 \leqslant (1+\delta)k^2_{3^{2^n}}\}$ we have

$$S_i^{**} - S^{**}_{3^{2^n}} = \sum_{\substack{j=3^{2^n}+1\\ j\in N^{**}}}^{i} X_j = 0 \quad \text{a.s.}$$

for sufficiently large n.

As S_n^* is the sum of independent r.vs. with finite variances, then, by the

Kolmogorov's inequality, the first term on the right-hand side of (15) is less than or equal to $4\delta/\varepsilon^2$, the second one is less than or equal to

$$16\delta^2(1+\delta)k^2_{3^{2^n}}/\varepsilon^2 k^2_{3^{2^n}} \leqslant 32\delta/\varepsilon^2,$$

and the third term on the right-hand side of (15) is equal to

$$P[|S^*_{2^{2^n}-1}| \geqslant \varepsilon k_{3^{2^n}}/4\delta] \leqslant 16\delta^2 k^2_{2^{2^n}-1}/\varepsilon^2 k^2_{3^{2^n}} \to 0 \quad (n\to\infty).$$

Hence, for every $\varepsilon > 0$ and for every $0 < \delta < 1$, we have

$$\limsup_{n\to\infty} P[\max_{|k_i^2 - k^2_{3^{2^n}}| \leqslant \delta k^2_{3^{2^n}}} |Y_i - Y_{3^{2^n}}| \geqslant \varepsilon] \leqslant 36\delta/\varepsilon^2,$$

which proves that the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with the norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{3^{2^n}, n \geqslant 1\}$.

Similarly it can be proved that the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{3^{2^n}-1, n \geqslant 1\}$.

Let us further notice that

$$Y_{3^{2^n}-1} \Rightarrow \mathcal{N}_{0,1} \ (n\to\infty), \qquad Y_{3^{2^n}} \Rightarrow X \ (n\to\infty), \tag{16}$$

and

$$k^2_{3^{2^n}}/k^2_{3^{2^n}-1} \to 2 \quad (n\to\infty),$$

where X is an r.v. with the characteristic function

$$\varphi(t) = \cos(t/\sqrt{2})\, e^{-t^2/4}.$$

Then, by Corollary 1, $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n\to\infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $k^2_{N_n}/k^2_{b_n} \xrightarrow{P} 1$ $(n\to\infty)$, where

$$\{b_n, n \geqslant 1\} \subseteq \{3^{2^n}-1, n \geqslant 1\} \quad (b_n\to\infty, n\to\infty)$$

and $Y_{N_n} \Rightarrow X$ $(n\to\infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $k^2_{N_n}/k^2_{b'_n} \xrightarrow{P} 1$ $(n\to\infty)$, where $\{b'_n, n \geqslant 1\} \subseteq \{3^{2^n}, n \geqslant 1\}$ $(b'_n\to\infty, n\to\infty)$.

Let us still notice that if A is an event independent of X_k, $k \neq 2^{2^n}$ $(n \geqslant 0, k \geqslant 1)$, and, for $n \geqslant 1$,

$$\nu_n = \begin{cases} 3^{2^n}-1 & \text{on } A, \\ 3^{2^n} & \text{on } A^c, \end{cases} \tag{17}$$

then $P[Y_{\nu_n} < x] \to \Phi(x)P(A) + P[X < x]P(A^c)$, $n\to\infty$, i.e.

$$Y_{\nu_n} \Rightarrow \mathcal{N}_{0,1} I(A) + XI(A^c) \quad (n\to\infty), \tag{17b}$$

and the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with the norming sequence $\{k_n, n \geqslant 1\}$ and filtering sequence $\{\nu_n, n \geqslant 1\}$.

Thus, by Corollary 2,

$$Y_{N_n} \Rightarrow \mathcal{N}_{0,1} I(A) + X I(A^c) \quad (n \to \infty)$$

for every $\{N_n, n \geqslant 1\}$ satisfying $k_{N_n}^2/k_{v_{a_n}}^2 \xrightarrow{P} 1$ $(n \to \infty)$, where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$. That fact implies the statements after (16), i.e. putting $A = \Omega$ or, equivalently, $v_n = 3^{2^n} - 1$ a.s. $(n \geqslant 1)$, we obtain $Y_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $k_{N_n}^2/k_{b_n}^2 \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b_n, n \geqslant 1\} \subseteq \{3^{2^n} - 1, n \geqslant 1\}$ $(b_n \to \infty, n \to \infty)$; putting however $A = \emptyset$ or, equivalently, $v_n = 3^{2^n}$ a.s. $(n \geqslant 1)$, we obtain $Y_{N_n} \Rightarrow X$ $(n \to \infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying $k_{N_n}^2/k_{b'_n}^2 \xrightarrow{P} 1$ $(n \to \infty)$, where $\{b'_n, n \geqslant 1\} \subseteq \{3^{2^n}, n \geqslant 1\}$ $(b'_n \to \infty, n \to \infty)$.

The following example proves Remark 1.

Example 4 [5]. Let $U, Z_1, Z_2, \ldots$ be independent r.vs. such that U has a uniform distribution on (0, 1) and, for each $n \geqslant 1$, Z_n has a normal distribution with mean zero and variance one. Let

$$Y_n = n^{-1/2} \sum_{i=1}^{n} Z_i \quad (n \geqslant 1),$$

$$J(\omega) = \{[2^n U(\omega)] + 1, 1 \leqslant n < \infty\},$$

and

$$Y'_n = Y_n I[n \notin J] \quad (n \geqslant 1).$$

The sequence $\{Y'_n, n \geqslant 1\}$ satisfies (A) [5] and then, by Lemma 2, it satisfies as well (A**) with norming sequence $\{\sqrt{n}, n \geqslant 1\}$ and any filtering sequence $\{N_n, n \geqslant 1\}$ of positive integer-valued r.vs. satisfying (3) (i.e. (4) with $k_n^2 = n, n \geqslant 1$).

Let us put $N_n = [2^n U]$, $n \geqslant 1$. The sequence $\{N_n, n \geqslant 1\}$ does not satisfy (3) but satisfies (5). Indeed, putting e.g. $v_n = [2^n U] + 1$, $n \geqslant 1$, we have

(5′) $$N_n/v_n \xrightarrow{P} 1 \quad (n \to \infty),$$

i.e. condition (5) with $k_n^2 = n$ $(a_n = n, n \geqslant 1)$. Now we shall prove that the sequence $\{Y_n, n \geqslant 1\}$ does not satisfy (A**) with the norming sequence $\{\sqrt{n}, n \geqslant 1\}$ and filtering sequence $\{N_n, n \geqslant 1\}$. This fact will prove Remark 1.

It follows from Theorem 1 of [11], p. 472, that

(18) $$Y_{N_n} \Rightarrow \mathcal{N}_{0,1} \ (n \to \infty), \quad \text{where } N_n = [2^n U] \ (n \geqslant 1),$$

since $Y_n \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$, $N_n \xrightarrow{P} \infty$ $(n \to \infty)$ and, for every $n \geqslant 1$, the r.v. N_n is independent of Z_i $(i \geqslant 1)$. Furthermore, by the construction of the set $J(\omega)$

we have

$$P[Y'_{N_n} \neq Y_{N_n}] = P[N_n \in J] \leqslant P[U < (n-1)2^{-n}] +$$

$$+ \sum_{k=n}^{2^n} P[(k-1)2^{-n} \leqslant U < k2^{-n};\ [2^n U] \in \{[2^n U]+1,\ 1 \leqslant n < \infty\}]$$

$$\leqslant o_n(1) + \sum_{k=n}^{2^n} P[k-1)2^{-n} \leqslant U < k2^{-n};\ k-2 \leqslant 2^m U < k-1 \quad \text{for some } m \geqslant 1]$$

$$\leqslant o_n(1) + \sum_{k=n}^{2^n} \sum_{m=\log_2 2^n(1-2/n)}^{n} P[(k-1)2^{-n} \leqslant U < k2^{-n};\ (k-2)2^{-m} \leqslant U < (k-1)2^{-m}]$$

$$\leqslant o_n(1) + \sum_{m=\log_2 2^n(1-2/n)}^{n} P[(n-2)2^{-m} \leqslant U < (2^n-1)2^{-m}]$$

$$= o_n(1) + (2^n - n + 1) \sum_{m=\log_2 2^n(1-2/n)}^{n} 2^{-m}$$

$$= o_n(1) + \frac{2^{-n+1}(2^n-n+1)}{1-\frac{2}{n}}\frac{2}{n} \to 0 \quad (n \to \infty),$$

since $o_n(1) = P[U < (n-1)2^{-n}] \to 0$ $(n \to \infty)$. This fact and (18) imply (cf. [7], p. 278) $Y'_{N_n} \Rightarrow \mathcal{N}_{0,1}$ $(n \to \infty)$.

Since (5′) holds, and

(19) $$Y'_{\nu_n} = 0 \text{ a.s.} (n \geqslant 1), \quad \text{where } \nu_n = [2^n U]+1 \ (n \geqslant 1),$$

we conclude that the sequence $\{Y'_n, n \geqslant 1\}$ does not satisfy (A**) with the norming sequence $\{\sqrt{n}, n \geqslant 1\}$ and filtering sequence $\{N_n, n \geqslant 1\}$, where $N_n = [2^n U]$, $n \geqslant 1$.

5. Applications to the martingale random limit theorems. We shall now show that the given results allow to generalize the martingale random central limit theorem of Rao [8].

We write $Y_n \Rightarrow \mu$ (stably), $n \to \infty$, if $Y_n \Rightarrow \mu$ $(n \to \infty)$ and, for every $B \in \mathcal{F}$ with $P(B) > 0$, there exists a measure μ_B such that $Y_n \Rightarrow \mu_B/P(B)$, $n \to \infty$, under the conditional measure $P(\cdot|B)$. In the special case where $\mu_B = \mu P(B)$ for all $B \in \mathcal{F}$, we write $Y_n \Rightarrow \mu$ (mixing), $n \to \infty$ [10].

In what follows we shall need the following simple consequence of Theorem 3:

THEOREM 4. *Let $\{\alpha_n, n \geqslant 1\}$ be a non-decreasing sequence of positive r.vs.*

and let $\{\nu_n, n \geq 1\}$ be a sequence of positive integer-valued r.vs. such that $\nu_n \xrightarrow{P} \infty$ $(n \to \infty)$. The following conditions are equivalent:

(i) $Y_{\nu_n} \Rightarrow \mu$ *(stably), $n \to \infty$, and the sequence $\{Y_n, n \geq 1\}$ satisfies* (A**) *with the norming sequence $\{\alpha_n, n \geq 1\}$ and filtering sequence $\{\nu_n, n \geq 1\}$;*

(ii) $Y_{N_n} \Rightarrow \mu$ *(stably), $n \to \infty$, for every sequence $\{N_n, n \geq 1\}$ of positive integer-valued r.vs. satisfying* (6), *i.e.*

$$\alpha_{N_n}^2/\alpha_{\nu_{a_n}}^2 \xrightarrow{P} 1 \quad (n \to \infty),$$

where $\{a_n, n \geq 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n \to \infty)$.

Proof. Obviously, (ii) implies (i). To prove the reverse implication let B be a random event from $\mathscr{F}$ with $P(B) > 0$. Then, by (i), there exists a probability measure μ_B such that $Y_{\nu_n} \Rightarrow \mu_B/P(B)$ $(n \to \infty)$ under the measure $P(\cdot\,|B)$. Furthermore, the sequence $\{Y_n, n \geq 1\}$ satisfies (A**) with the norming sequence $\{\alpha_n, n \geq 1\}$ and filtering sequence $\{\nu_n, n \geq 1\}$ under the measure $P(\cdot\,|B)$.

Indeed, for every $\varepsilon > 0$ we can choose $\delta > 0$, $\delta = \delta(\varepsilon, B)$, such that

$$\limsup_{n\to\infty} P[\max_{|\alpha_i^2 - \alpha_{\nu_n}^2| \leq \delta\alpha_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geq \varepsilon P(B)] \leq \varepsilon P(B).$$

Hence

$$\limsup_{n\to\infty} P[\max_{|\alpha_i^2 - \alpha_{\nu_n}^2| \leq \delta\alpha_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geq \varepsilon\,|B]$$

$$\leq \limsup_{n\to\infty} P[\max_{|\alpha_i^2 - \alpha_{\nu_n}^2| \leq \delta\alpha_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geq \varepsilon]/P(B)$$

$$\leq \limsup_{n\to\infty} P[\max_{|\alpha_i^2 - \alpha_{\nu_n}^2| \leq \delta\alpha_{\nu_n}^2} |Y_i - Y_{\nu_n}| \geq \varepsilon P(B)]/P(B) \leq \varepsilon,$$

which states the desired result.

Thus, by Theorem 3, $Y_{N_n} \Rightarrow \mu_B/P(B)$, $n \to \infty$, under the measure $P(\cdot\,|B)$, for every sequence $\{N_n, n \geq 1\}$ of positive integer-valued r.vs. satisfying (6). Hence, $Y_{N_n} \Rightarrow \mu$ (stably) $(n \to \infty)$ for every $\{N_n, n \geq 1\}$ satisfying (6).

Remark 3. We note that if $Y_{\nu_n} \Rightarrow \mu$ (mixing) $(n \to \infty)$ not only stably, then in part (ii) of Theorem 4 we obtain $Y_{N_n} \Rightarrow \mu$ (mixing), $n \to \infty$.

Let $\{X_n, \mathscr{F}_n, n \geq 1\}$ be a martingale difference sequence (MDS) with $\sigma_n^2 = E(X_n^2\,|\mathscr{F}_{n-1}) < \infty$ a.s. $(n \geq 1)$. $\mathscr{F}_0$ need not be the trivial sigma field $\{\emptyset, \Omega\}$. We put

$$V_n^2 = \sum_{k=1}^{n} \sigma_k^2, \quad s_n^2 = EV_n^2 \quad (n \geq 1)$$

and assume that s_n^2 is finite for all $n \geq 1$, and $s_n^2 \to \infty$ $(n \to \infty)$. Furthermore, we assume that the sequence $\{X_n, n \geq 1\}$ satisfies the following φ-mixing condition:

$$\sup\{|P(B\,|\,A)-P(B)|:\ A \in \mathfrak{M}_1^k,\ B \in \mathfrak{M}_{k+n}^{\infty}\} \leq \varphi(n) \tag{20}$$

with $\varphi(n) \to 0$ $(n \to \infty)$, where $\mathfrak{M}_n^m = \sigma\{X_k,\ n \leq k \leq m\}$ $(1 \leq n \leq m \leq \infty)$.

The following theorem extends and strengthens Theorem 1 of [8]:

THEOREM 5. *Let* $\{X_n, \mathscr{F}_n, n \geq 1\}$ *be an MDS satisfying* (20). *Suppose that*

$$Y_n := S_n/s_n \Rightarrow \mu \quad (n \to \infty), \tag{21}$$

where

$$S_n = \sum_{k=1}^{n} X_k \quad (n \geq 1).$$

If $\{N_n, n \geq 1\}$ *is a sequence of positive integer-valued r.vs. such that*

$$s_{N_n}^2/s_{[\lambda a_n]}^2 \xrightarrow{P} 1 \quad (n \to \infty), \tag{22}$$

where $\{a_n, n \geq 1\}$ *is a sequence of positive integers with* $a_n \to \infty$ $(n \to \infty)$, *and* λ *is a positive r.v. having a discrete distribution, then*

$$Y_{N_n} = S_{N_n}/s_{N_n} \Rightarrow \mu(\textit{mixing}) \quad (n \to \infty).$$

The proof of Theorem 5 bases on Theorem 4 and the following lemmas:

LEMMA 4. *If* $\{X_n, \mathscr{F}_n, n \geq 1\}$ is *an MDS satisfying* (20) *and* (21), *then* $Y_n \Rightarrow \mu(\text{mixing})$, $n \to \infty$.

LEMMA 5 [8]. *Let* $\{k_n, n \geq 1\}$ *and* $\{m_n, n \geq 1\}$ *be sequences of positive integers tending to infinity, and let* $A_n \in \mathfrak{M}_{k_n}^{m_n}$ $(n \geq 1)$. *Then, for any event* A,

$$\limsup_{n\to\infty} P(A_n\,|\,A) = \limsup_{n\to\infty} P(A_n),$$

where we set $P(A_n\,|\,A) = P(A_n)$ *if* $P(A) = 0$.

LEMMA 6. *Let* $\{X_n, \mathscr{F}_n, n \geq 1\}$ *be an MDS satisfying* (20) *and* (21). *If* λ *is a positive r.v. having a discrete distribution, then*

$$Y_{[\lambda n]} \Rightarrow \mu(\text{mixing}) \quad (n \to \infty). \tag{23}$$

LEMMA 7. *Let* $\{X_n, \mathscr{F}_n, n \geq 1\}$ be *an MDS satisfying* (20) *and* (21). *If* λ *is a positive r.v. having a discrete distribution, then the sequence* $\{Y_n, n \geq 1\}$ *satisfies* (A**) *with the norming sequence* $\{s_n, n \geq 1\}$ *and filtering sequence* $\{[\lambda n], n \geq 1\}$.

Proof of Lemma 4. Let $A_0 = \Omega$, $A_n = [Y_n < x]$ $(n \geq 1)$. By Theorem 2 of [9] it is enough to prove that, for any A_k,

$$\lim_{n\to\infty} P(A_n\,|\,A_k) = \lim_{n\to\infty} P(A_n). \tag{24}$$

Let $\{r_n, n \geqslant 1\}$ be a sequence of positive integers such that $s_{r_n}^2 \to \infty$ $(n \to \infty)$ and $s_{r_n}^2/s_n^2 \to 0$ $(n \to \infty)$.

For every $k \geqslant 1$ we put $r_n^k = \max(r_n - k, 0)$, $n \geqslant 1$. Of course,

$$s_{r_n^k+k}^2/s_n^2 \to 0 \ (n \to \infty) \quad \text{for every } k \geqslant 1,$$

and then, by the Kolmogorov inequality, we have

$$P[|S_{r_n^k+k}| \geqslant \varepsilon s_n] \to 0 \ (n \to \infty) \quad \text{for every } \varepsilon > 0, \ k \geqslant 1.$$

Therefore, for every $k \geqslant 1$,

$$\lim_{n\to\infty} P(A_n | A_k) = \lim_{n\to\infty} P[(S_n - S_{r_n^k+k})/s_n < x | A_k]. \tag{25}$$

Note that

$$[(S_n - S_{r_n^k+k})/s_n < x] \in \mathfrak{M}_{r_n^k+k+1}^{\infty} \quad \text{for all } n \geqslant k,$$

and $A_k \in \mathfrak{M}_1^k$ $(k \geqslant 1)$. Hence, by the φ-mixing condition (20), for every $k \geqslant 1$,

$$\lim_{n\to\infty} P[(S_n - S_{r_n^k+k})/s_n < x | A_k] = \lim_{n\to\infty} P[(S_n - S_{r_n^k+k})/s_n < x] \tag{26}$$

since $r_n^k \to \infty$ $(n \to \infty)$ for every $k \geqslant 1$. Again, since

$$S_{r_n^k+k}/s_n \xrightarrow{P} 0 \quad (n \to \infty),$$

we get

$$\lim_{n\to\infty} P[(S_n - S_{r_n^k+k})/s_n < x] = \lim_{n\to\infty} P(A_n). \tag{27}$$

Combining (25), (26) and (27), we obtain (24). The proof of Lemma 4 is completed.

P r o o f of L e m m a 6. Let l_j $(j \geqslant 1)$ denote the values taken on by the r.v. λ with positive probabilities and let $\Omega_j = [\lambda = l_j]$ $(j \geqslant 1)$. Then

$$P[Y_{[\lambda n]} < x] = \sum_{j=1}^{\infty} P[Y_{[l_j n]} < x | \Omega_j] P(\Omega_j)$$

and, by (20), Lemma 4 and Theorem 1 of [9], we obtain

$$\lim_{n\to\infty} P[Y_{[\lambda n]} < x] = \sum_{j=1}^{\infty} P(\Omega_j) \lim_{n\to\infty} P[Y_{[l_j n]} < x | \Omega_j]$$

$$= \sum_{j=1}^{\infty} P(\Omega_j) \lim_{n\to\infty} P[Y_n < x | \Omega_j] = \sum_{j=1}^{\infty} P(\Omega_j) \lim_{n\to\infty} P[Y_n < x] = F(x)$$

for every continuity point x of F, where $F(\cdot)=\mu\{(-\infty,\cdot)\}$, which proves that $Y_{[\lambda n]}\Rightarrow\mu\ (n\to\infty)$.

Let $A_0^{(j)}=\Omega_j$, $A_n^{(j)}=[Y_{[l_j n]}<x;\ \Omega_j]$ $(n\geqslant 1)$ for every $j\geqslant 1$. It is easy to see that the sequence $\{A_n^{(j)},\ n\geqslant 0\}$ is a mixing sequence of sets in the space $(\Omega_j,\ \mathscr{A}_j,\ P_j)$, where

$$\mathscr{A}_j=\{A\in\mathscr{F}:\ A\subset\Omega_j\},\qquad P_j(A)=P(A\mid\Omega_j)\qquad \text{for } A\in\mathscr{A}_j.$$

Indeed, for every fixed $k\geqslant 1$ we have

$$\lim_{n\to\infty}P_j(A_n^{(j)}\mid A_k^{(j)})=\lim_{n\to\infty}P_j(A_n^{(j)}A_k^{(j)})/P_j(A_k^{(j)})$$
$$=\lim_{n\to\infty}P[Y_{[l_j n]}<x\mid A_k^{(j)}\Omega_j]\,P(A_k^{(j)}\Omega_j)/P(\Omega_j)\,P_j(A_k^{(j)}).$$

The last expression is, by Lemma 4, equal to

$$\lim_{n\to\infty}P[Y_{[l_j n]}<x\mid\Omega_j]\,P(A_k^{(j)}\mid\Omega_j)/P_j(A_k^{(j)})=\lim_{n\to\infty}P_j(A_n^{(j)}),$$

where

$$\lim_{n\to\infty}P_j(A_n^{(j)})=\lim_{n\to\infty}P[Y_n<x\mid\Omega_j]=\lim_{n\to\infty}P[Y_n<x]=F(x)$$

for every continuity point x of F, $F(\cdot)=\mu\{(-\infty,\cdot)\}$, which proves that $\{A_n^{(j)},\ n\geqslant 1\}$ is a mixing sequence of sets on $(\Omega_j,\ \mathscr{A}_j,\ P_j)$ with the local density $F(x)$.

Thus, for

$$A_n=[Y_{[\lambda n]}<x]=\bigcup_{j=1}^{\infty}A_n^{(j)}\qquad(n\geqslant 1)$$

the sequence $\{A_n,\ n\geqslant 1\}$ is a mixing sequence of sets in $(\Omega,\ \mathscr{F},\ P)$ [10]. Hence $Y_{[\lambda n]}\Rightarrow\mu$ (mixing), $n\to\infty$.

Proof of Lemma 7. For every $\varepsilon>0$ and for every $\delta>0$ we have

$$\limsup_{n\to\infty}P[\max_{|s_i^2-s_{[\lambda n]}^2|\leqslant\delta s_{[\lambda n]}^2}|Y_i-Y_{[\lambda n]}|\geqslant\varepsilon]$$
$$\leqslant\limsup_{n\to\infty}P[\max_{|s_i^2-s_{[\lambda n]}^2|\leqslant\delta s_{[\lambda n]}^2}|S_i-S_{[\lambda n]}|\geqslant\varepsilon\sqrt{1-\delta}\,s_{[\lambda n]}/2]+$$
$$+\limsup_{n\to\infty}P[|Y_{[\lambda n]}|\geqslant\varepsilon\sqrt{1-\delta}/2\delta],$$

where

$$\limsup_{n\to\infty}P[|Y_{[\lambda n]}|\geqslant\varepsilon\sqrt{1-\delta}/2\delta]=\mu\{x:\ |x|\geqslant\varepsilon\sqrt{1-\delta}/2\delta\}\to 0\qquad(\delta\to 0).$$

Thus, the sequence $Y_n = S_n/s_n$ $(n \geqslant 1)$ satisfies (A**) with the norming sequence $\{s_n, n \geqslant 1\}$ and filtering sequence $\{[\lambda n], n \geqslant 1\}$ if, for every $\varepsilon > 0$,

$$(28)\qquad \lim_{\delta\to\infty} \limsup_{n\to\infty} P[\max_{|s_i^2 - s_{[\lambda n]}^2| \leqslant \delta s_{[\lambda n]}^2} |S_i - S_{[\lambda n]}| \geqslant \varepsilon\sqrt{1-\delta}\, s_{[\lambda n]}/2] = 0.$$

Let $\Omega_j = [\lambda = l_j]$ $(j \geqslant 1)$. Then, by Lemma 5, the left-hand side of (28) is equal to

$$\lim_{\delta\to 0} \sum_{j=1}^{\infty} P(\Omega_j) \limsup_{n\to\infty} P[\max_{T_{n,\delta}^1 \leqslant i \leqslant T_{n,\delta}^2} |S_i - S_{[l_j n]}| \geqslant \varepsilon\sqrt{1-\delta}\, s_{[l_j n]}/2 \mid \Omega_j]$$

$$= \lim_{\delta\to 0} \sum_{j=1}^{\infty} P(\Omega_j) \limsup_{n\to\infty} P[\max_{T_{n,\delta}^1 \leqslant i \leqslant T_{n,\delta}^2} |S_i - S_{[l_j n]}| \geqslant \varepsilon\sqrt{1-\delta}\, s_{[l_j n]}/2],$$

where

$$T_{n,\delta}^1 = \min\{i\colon (1-\delta) s_{[l_j n]}^2 \leqslant s_i^2\}, \qquad T_{n,\delta}^2 = \max\{i\colon s_i^2 \leqslant (1+\delta) s_{[l_j n]}^2\}.$$

Furthermore, the last expression is less than or equal to

$$2\lim_{\delta\to 0} \sum_{j=1}^{\infty} P(\Omega_j) \limsup_{n\to\infty} P[\max_{T_{n,\delta}^1 \leqslant i \leqslant T_{n,\delta}^2} |S_i - S_{T_{n,\delta}^1}| \geqslant \varepsilon\sqrt{1-\delta}\, s_{[l_j n]}/4],$$

which, by the Kolmogorov inequality for martingales, is less than or equal to

$$32 \lim_{\delta\to 0} \sum_{j=1}^{\infty} P(\Omega_j) \limsup_{n\to\infty} E(S_{T_{n,\delta}^2} - S_{T_{n,\delta}^1})^2/\varepsilon^2(1-\delta) s_{[l_j n]}^2$$

$$\leqslant 32 \lim_{\delta\to 0} \sum_{j=1}^{\infty} P(\Omega_j) \limsup_{n\to\infty} \{(1+\delta) s_{[l_j n]}^2 - (1-\delta) s_{[l_j n]}^2\}/\varepsilon^2(1-\delta) s_{[l_j n]}^2$$

$$\leqslant \lim_{\delta\to 0} 64\delta/\varepsilon^2(1-\delta) = 0.$$

The proof of Lemma 7 is completed.

Proof of Theorem 5. By (23) and Lemma 7 we have $Y_{[\lambda n]} \Rightarrow \mu$ (mixing) $(n\to\infty)$ and the sequence $\{Y_n, n \geqslant 1\}$ satisfies (A**) with the norming sequence $\{s_n, n \geqslant 1\}$ and filtering sequence $\{[\lambda n], n \geqslant 1\}$. Thus, by Theorem 4, we obtain $Y_{N_n} \Rightarrow \mu$ (mixing) $(n\to\infty)$ for every $\{N_n, n \geqslant 1\}$ satisfying (22), which is the statement of Theorem 5.

Now we give simple consequences of Theorem 5 which extend or strengthen results given in [6] and [8].

COROLLARY 4 (cf. [8], Theorem 1). *Let $\{X_n, n \geqslant 1\}$ be a strictly stationary and ergodic sequence of r.vs. Assume that $EX_1 = 0$, $E(X_n \mid X_1, \ldots, X_{n-1}) = 0$ a.s. $(n \geqslant 1)$ and $EX_1^2 = 1$. If $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued r.vs. such that $N_n/a_n \xrightarrow{P} \lambda$ $(n\to\infty)$, where $\{a_n, n \geqslant 1\}$ is a sequence of positive integers with $a_n \to \infty$ $(n\to\infty)$, and λ is a positive r.v. having a discrete distribution, then $S_{N_n}/\sqrt{N_n} \Rightarrow \mathcal{N}_{0,1}$ (mixing) $(n\to\infty)$.*

COROLLARY 5 (cf. [6], Theorem 4 and Remark on p. 209). *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent r.vs. with* $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$ $(n \geqslant 1)$. *Suppose that* $S_n/s_n \Rightarrow \mu$ $(n \to \infty)$, *where*

$$S_n = \sum_{k=1}^{n} X_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \quad (n \geqslant 1).$$

If $\{N_n, n \geqslant 1\}$ *is a sequence of positive integer-valued r.vs. satisfying* (22), *then* $S_{N_n}/s_{N_n} \Rightarrow \mu$ *(mixing)* $(n \to \infty)$.

COROLLARY 6. *Let* $\{X_n, \mathscr{F}_n, n \geqslant 1\}$ *be an MDS satisfying* (20). *Suppose that*

$$V_n^2/s_n^2 \xrightarrow{P} 1 \quad (n \to \infty)$$

and, for every $\varepsilon > 0$,

$$s_n^{-2} \sum_{k=1}^{n} E(X_k^2 I(|X_k| \geqslant \varepsilon s_n) | \mathscr{F}_{n-1}) \xrightarrow{P} 0 \quad (n \to \infty).$$

If $\{N_n, n \geqslant 1\}$ *is a sequence of positive integer-valued r.vs. satisfying* (22), *then* $S_{N_n}/s_{N_n} \Rightarrow \mathscr{N}_{0,1}$ *(mixing)* $(n \to \infty)$.

The following result generalizes Theorem 4 of [6]:

THEOREM 6. *Let* $\{X_n, \mathscr{F}_n, n \geqslant 1\}$ *be an MDS satisfying* (20) *and* (21). *If* $\{N_n, n \geqslant 1\}$ *is a sequence of positive integer-valued r.vs. such that*

$$s_{N_n}^2/s_{a_n}^2 \xrightarrow{P} \lambda \quad (n \to \infty), \tag{29}$$

where $\{a_n, n \geqslant 1\}$ *is a sequence of positive integers with* $a_n \to \infty$ $(n \to \infty)$ *and* λ *is a positive r.v.* $(P[0 < \lambda < \infty] = 1)$ *such that, for any given* $\varepsilon > 0$,

$$\lim_{0 \leqslant c \to 0} \limsup_{n \to \infty} P\left[\left|\frac{s_{[(\lambda \pm c)n]}^2}{s_n^2} - \lambda\right| \geqslant \varepsilon\right] = 0, \tag{30}$$

then

$$S_{N_n}/s_{N_n} \Rightarrow \mu \text{ (mixing)} \quad (n \to \infty).$$

Proof. Only some modifications are necessary in the proof of Theorem 4 from [6] to make it applicable in this case.

In the particular case, from Theorem 6 we get a result stronger than that of Theorem 4 in [6]:

COROLLARY 7. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent r.vs. with* $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$ $(n \geqslant 1)$. *Suppose that* $S_n/s_n \Rightarrow \mu$ $(n \to \infty)$, *where*

$$S_n = \sum_{k=1}^{n} X_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \quad (n \geqslant 1).$$

If $\{N_n, n \geqslant 1\}$ *is a sequence of positive integer-valued r.vs. satisfying* (29) *and* (30), *then* $S_{N_n}/s_{N_n} \Rightarrow \mu$ *(mixing)* $(n \to \infty)$.

REFERENCES

[1] D. J. Aldous, *Weak convergence of randomly indexed sequences of random variables*, Math. Proc. Cambridge Phil. Soc. 83 (1978), p. 117-126.

[2] F. J. Anscombe, *Large-sample theory of sequential estimation*, Proc. Cambridge Phil. Soc. 48 (1952), p. 600-607.

[3] P. Billingsley, *Convergence of probability measures*, John Wiley, New York 1968.

[4] M. Csörgö and Z. Rychlik, *Weak convergence of sequences of random elements with random indices*, Math. Proc. Cambridge Phil. Soc. 88 (1980), p. 171-174.

[5] R. T. Durrett and S. I. Resnick, *Weak convergence with random indices*, Stochastic Processes Appl. 5 (1977), p. 213-220.

[6] K. S. Kubacki and D. Szynal, *Weak convergence of randomly indexed sequences of random variables*, Bull. Pol. Acad. Sci. 32 (1958), p. 201-210.

[7] M. Loève, *Probability Theory*, Van Nostrand, Princeton 1960.

[8] B. L. S. Prakasa Rao, *Random central limit theorems for martingales*, Acta Math. Acad. Sci. Hungar. 20 (1969), p. 217-222.

[9] A. Rényi, *On mixing sequences of sets*, ibidem 9 (1958), p. 215-228.

[10] — *On stable sequences of events*, Sankhyā, Series A, 25 (1963), p. 293-302.

[11] — *Probability theory*, Akadémiai Kiadó, Budapest 1970.

[12] Z. Rychlik, *Martingale random central limit theorems*, Acta Math. Acad. Sci. Hungar. 34 (1979), p. 129-139.

[13] — *Asymptotic distributions of randomly indexed sequences of random variables*, Dissertation, Maria Curie-Skłodowska University, Lublin 1980 (in Polish).

Institute of Applied Mathematics
Agricultural Academy in Lublin
ul. Akademicka 13, 20-934 Lublin
Poland

Institute of Mathematics, UMCS
ul. Nowotki 10, 20-031 Lublin
Poland

Received on 17. 12. 1984