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CONVERGENCE RATES IN THE STRONG LAW OF LARGE NUMBERS FOR SUMS OF RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

BY

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Abstract. We consider a set of independent random variables indexed by Z^d $(d \geqslant 1)$, the positive integer d-dimensional lattice points, and study the convergence rates in the strong law of large numbers. The results presented provide us with much deeper understanding of the tail probability of distributions.

1. Introduction. Let Z^d, $d \geqslant 1$, be the positive integer d-dimensional lattice points with coordinatewise partial ordering $\leqslant$. Points in Z^d are denoted by $\boldsymbol{m}$, $\boldsymbol{n}$ etc. Also, for $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$, we define

$$|\boldsymbol{n}| = \prod_{i=1}^{d} n_i$$

and $\boldsymbol{n} \to \infty$ is to be interpreted as $|\boldsymbol{n}| \to \infty$.

Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ be a set of independent random variables with $EX_{\boldsymbol{n}} = 0$ and $EX_{\boldsymbol{n}}^2 = \sigma_{\boldsymbol{n}}^2 < \infty$, $\boldsymbol{n} \in Z^d$. Let

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} X_{\boldsymbol{k}}, \quad B_{\boldsymbol{n}}^2 = \sum_{\boldsymbol{k} \leqslant \boldsymbol{n}} \sigma_{\boldsymbol{k}}^2 = E(S_{\boldsymbol{n}}^2).$$

It has been shown by Smythe [12] that if $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ are independent and identically distributed (i.i.d.) random variables, then the strong law of large numbers holds if and only if $E|X_1|(\log_+ |X_1|)^{d-1} < \infty$, where $\log_+ x = \max(1, \ln x)$. Moreover, Gut [7, 8] has recently proved the following

THEOREM 1 [7, 8]. *Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ be i.i.d. random variables, let $r \geqslant 1/\alpha$ and $\alpha > 1/2$. The following statements are equivalent*:

(1) $$E|X_1|^r (\log_+ |X_1|)^{d-1} < \infty \text{ and if } r \geqslant 1,\ EX_1 = 0.$$

(2) $$\sum_{n} |n|^{\alpha r-2} P(|S_n| \geqslant |n|^{\alpha} t) < \infty \quad \textit{for all } t > 0.$$

(3) $$\sum_{n} |n|^{\alpha r-2} P(\max_{k \leqslant n} |S_k| \geqslant |n|^{\alpha} t) < \infty \quad \textit{for } t > 0.$$

If $\alpha = 1/2$, *then the following are equivalent*:

(4) $$EX_1^2 (\log_+ |X_1|)^{d-1} < \infty \quad \textit{and} \quad EX_1 = 0.$$

(5) $$\sum_{n} |n|^{-1} \log_+ |n| \, P(|S_n| > t(|n| \log_+ |n|)^{1/2}) < \infty \quad \textit{for all } t > 0.$$

Let h be a finite and positive function defined on $[0, \infty)$. Furthermore, let, for every $t > 0$, f_t be an increasing and positive function. Let us put

$$A_n(h, f_t) = \sum_{k \leqslant n} h(|k|) \, P(|S_k| > B_k f_t(|k|)),$$

$$A_{\infty}(h, f_t) = \lim_{n \to \infty} A_n(h, f_t),$$

$$F(h, f_t) = 2 \sum_{n \geqslant 1} h(|n|) \, \Phi(-f_t(|n|)),$$

where Φ denotes the standard normal distribution function. This paper deals with order of magnitude of $A_{\infty}(h, f_t)$ and $F(h, f_t)$ as $t \to 0^+$ in the case $\{X_n, n \in Z^d\}$ are independent but necessarily identically distributed. By specializing the functions h and f_t, putting, for example, $h(|n|) = |n|^{\alpha} (\log_+ |n|)^{\beta}$, $f_t(|n|) = t |n|^{\gamma} (\log_+ |n|)^{\delta}$, $\alpha, \beta, \gamma, \delta \in (-\infty, +\infty)$, we will deduce generalization of some results obtained by Gut [7, 8], Klesov [9] and Łagodowski and Rychlik [10]. Our results provide us with a much better and deeper understanding of the tail probability of distributions. From the results presented we also get extensions, for $d \geqslant 2$, of some theorems given by Chen [2, 3] and Csörgő and Rychlik [4].

2. General theorem. Let us define $d(x) = \text{Card}\{n \in Z^d \colon |n| = [x]\}$ and $M_d(x) = \text{Card}\{n \in Z^d \colon |n| < [x]\} = M(x)$, where $[x]$ denotes the greatest integer not exceeding x, $x \in [0, \infty)$. We have (cf. [13])

(6) $$M_d(n) = n(\log_+ n)^{d-1}/(d-1)! - M_{d-1}(n), \qquad d \geqslant 2,$$

so that, as $x \to \infty$,

(7) $$M(x) = O\big(x(\log_+ x)^{d-1}\big).$$

Furthermore, for every $\delta > 0$, $d(x) = o(x^{\delta})$ as $x \to \infty$.

THEOREM 2. *Let* $\{X_n, n \in Z^d\}$ *be independent random variables such that* $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \in Z^d$. *If there exists a function* g *such that*

(8) $g(x)$ *is nondecreasing on the interval* $(0, \infty)$, *is even on* $(-\infty, \infty)$, *and* $g(x) \to \infty$ *as* $x \to \infty$,

(9) *the function $x/g(x)$ does not decrease on $(0, \infty)$,*
(10) *for every $t > 0$*

$$S(h, f_t, g) = \sum_{n \geqslant 1} b_n(g)\, h(|n|)/(1+f_t(|n|))^2,$$

$$g\big((1+f_t(|n|))B_n\big) < \infty,$$

where

$$b_n(g) = \sum_{k \leqslant n} EX_k^2 g(X_k)/B_n^2,$$

then there exists a positive constant (independent of the functions h, f_t and g) such that

(11) $$|A_\infty(h, f_t) - F(h, f_t)| \leqslant C\, S(h, f_t, g),$$

where $\infty - \infty$ is to be interpreted as 0.

Proof. Let us observe that, by (10), $F(h, f_t) < \infty$ if and only if $A_\infty(h, f_t) < \infty$. Thus suppose $F(h, f_t) < \infty$. We have

(12) $$|A_\infty(h, f_t) - F(h, f_t)| \leqslant \sum_{n \geqslant 1} h(|n|)\, \big|P(|S_n| \geqslant B_n f_t(|n|)) - 2\Phi(-f_t(|n|))\big|.$$

Let

$$\Delta_n(x) = |P(S_n < xB_n) - \Phi(x)|.$$

In view of Bikyalis result (cf. [1] or [11]) we can prove that under our assumptions

(13) $$\Delta_n(x) \leqslant C_0\, b_n(g)/(1+|x|)^2 g\big((1+|x|)B_n\big),$$

where C_0 is an absolute constant. Combining (12) and (13) with $x = f_t(|n|)$, we see that

$$|A_\infty(h, f_t) - F(h, f_t)| \leqslant 2C_0\, S(h, f_t, g),$$

which proves (11).

Now let us observe that

$$F(h, f_t) = 2\sum_{k=1}^{\infty} h(k)\, d(k)\, \Phi(-f_t(k)),$$

Thus if, for example, $h(x) = x^r(\log_+ x)^s, f_t(x) = tx^\alpha$, $\alpha > 0$, where $\log_+ x = \max(0, \ln x)$, then

$$F(h, f_t) = 2\sum_{k=1}^{\infty} k^r d(k)(\log_+ k)^s\, \Phi(-tk^\alpha)$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} k^r d(k)(\log_+ k)^s\, P\big(tk^\alpha \leqslant |N| < t(k+1)^\alpha\big),$$

where N denotes a standard normal random variable. But, taking into account Abel's transform and (6), we get

$$(14)\qquad \sum_{k=1}^{n} k^r d(k)(\log_+ k)^s \cong n^{r+1}(\log_+ n)^{s+d-1}/(r+1)(d-1)!$$

as $n \to \infty$, provided $s = 0, 1, 2, \ldots$, and $r \neq -1$, so that

$$(15)\qquad (r+1)(d-1)!\, t^{(r+1)/\alpha} F(h, f_t) = t^{(r+1)/\alpha} \sum_{n=1}^{\infty} n^{r+1}(\log_+ n)^{s+d-1} \times \times P\big(tn^\alpha \leqslant |N| < t(n+1)^\alpha\big)$$

$$\cong (1/\alpha)^{s+d-1}(2/\pi)^{1/2} \int_t^\infty x^{(r+1)/\alpha}(\log_+ (x/t))^{s+d-1} \exp(-x^2/2)\,dx$$

$$\cong (1/\alpha)^{s+d-1}(2/\pi)^{1/2} \sum_{i=0}^{s+d-1} \binom{s+d-1}{i} (\log_+ (1/t))^i \times \times \int_t^\infty x^{(r+1)/\alpha}(\log_+ x)^{s+d-1-i} \exp(-x^2/2)\,dx \quad \text{as } t \to 0^+.$$

On the other hand, in case of $h(x) = x^{-1}(\log_+ x)^s$, $f_t(x) = t(\log_+ x)^u$, $u > 0$, we have

$$F(h, f_t) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} k^{-1} d(k)(\log_+ k)^s P\big(t \log_+^u n \leqslant |N| < t \log_+^u (n+1)\big)$$

and

$$(16)\qquad \sum_{k=1}^{n} k^{-1} d(k) \log_+^s k \cong (\log_+ n)^{s+d}/(s+d)(d-1)! \quad \text{as } n \to \infty.$$

Thus, in this case, we get

$$(17)\qquad (s+d)(d-1)!\, t^{(s+d)/u} F(h, f_t)$$

$$\cong (2\pi)^{-1/2} \int_{-\infty}^{\infty} |x|^{(s+d)/u} \exp(-x^2/2)\,dx$$

$$= E|N|^{(s+d)/u} = C_{u/(s+d)} \quad \text{as } t \to 0^+.$$

Note that $C_\alpha = \pi^{-1/2} 2^{1/2\alpha} \Gamma(1/2 + 1/2\alpha)$.

By applying Theorem 2 and (17) we have

$$(18)\qquad \lim_{t \to 0^+} t^{(s+d)/u} \sum_{n \geqslant 1} |n|^{-1}(\log_+ |n|)^s P\big(|S_n| \geqslant tB_n(\log_+ |n|)^u\big) = C_{u/(s+d)}/(s+d)(d-1)!,$$

provided

(19) $$\lim_{t\to 0^+} t^{(s+d)/u}\sum_{n\geqslant 1}\frac{b_n(g)|n|^{-1}(\log_+|n|)^s}{(1+t(\log_+|n|)^u)^2 g(B_n)}=0.$$

But (19) holds in case of $(s+d)/u > 2$ and

$$\sum_{n\geqslant 1}\frac{b_n(g)(\log_+|n|)^{s-2u}}{|n|\,g(B_n)}<\infty.$$

Thus (18) gives extension of (5) not only to the case of nonidentically distributed random variables, but (18) also gives the order of magnitude of $A_\infty(h, f_t)$ as $t\to 0^+$, for $h(x)=x^{-1}(\log_+ x)^s$, $f_t(x)=t(\log_+ x)^u$, $u>0$.

Furthermore, taking into account (15) and Theorem 2, we get

(20) $$\lim_{t\to 0^+}\Big\{t^{(r+1)/\alpha}\sum_{n\geqslant 1}|n|^r(\log_+|n|)^s P(|S_n|\geqslant t B_n|n|^\alpha)-\sum_{i=1}^{s+d-1}D_i(\log_+(1/t))^i\Big\}$$

$$=\frac{(2/\pi)^{1/2}(1/\alpha)^{s+d-1}\int_0^\infty x^{(r+1)/\alpha}(\log_+ x)^{s+d-1}\exp(-x^2/2)\,dx}{(r+1)(d-1)!}$$

$$=(2/\pi)^{1/2}(1/\alpha)^{s+d-1}\,2^{(r+1)/2\alpha+1/2-s-d}\sum_{i=0}^{s+d-1}(\ln 2)^{s+d-1-i}\times$$

$$\times\Gamma^{(i)}\big((r+1)/2\alpha+3/2\big)/(r+1)(d-1)!,$$

where $\Gamma^{(i)}(x)$, $i=1, 2, \ldots$, denotes the i-th derivative of the function $\Gamma(x)$, and $\Gamma^{(0)}(x)$ is to be interpreted as $\Gamma(x)$; D_i, $1\leqslant i\leqslant s+d-2$, are some constants which depend on d, α, r, and s, and

$$D_{s+d-1}=\frac{(1/\alpha)^{s+d-1}(2/\pi)^{1/2}\int_0^\infty x^{(r+1)/\alpha}\exp(-x^2/2)\,dx}{(r+1)(d-1)!}$$

$$=2^{(r+1)/2\alpha-1/2}(1/\alpha)^{s+d-1}(2/\pi)^{1/2}\,\Gamma\big((r+1)/2\alpha+3/2\big)/(r+1)(d-1)!.$$

Of course (20) holds in case of $r\geqslant 2\alpha-1$ and

(21) $$\sum_{n\geqslant 1}b_n(g)|n|^r(\log|n|)^s/|n|^{2\alpha}g(B_n)<\infty.$$

Note that (20) gives generalization of (2). At the same time we would like to mention that in order to get (20) or (18) we need some moment restrictions which imply (19) and (21), respectively. Let us observe that if $\{X_n, n\in Z^d\}$ are independent and identically distributed random variables with $EX_n = 0$, $EX_n^2 = 1$, we get $b_n(g) = EX_1^2 g(X_1)$ and $g(B_n) = g(|n|)^{1/2}$, so in the case $s = 1$, $u = 1/2$, (19) requires

$$\sum_{n \geq 1} |n|^{-1}/g(|n|)^{1/2} < \infty$$

which, in consequence, implies a little bit stronger assumption than (4). This fact is obvious because we get (18) from general Theorem 2. The exact order of magnitude of (2) and (5), as $t \to 0^+$, is presented in the next section.

3. Convergence rates in the strong law of large numbers. At first we state and prove the following

LEMMA. *Let $\{X_n, n \in Z^d\}$ be independent and normally distributed random variables such that $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \in Z^d$. Then, for every $\alpha > 0$,*

$$\lim_{t \to 0^+} (\ln(1/t))^{-s-d} \sum_{n \geq 1} |n|^{-1} (\log_+ |n|)^s P(|S_n| \geq tB_n |n|^\alpha) = 1/\alpha^{s+d}(s+d)(d-1)!, \tag{22}$$

$$\lim_{t \to 0^+} t^{(r+1)/\alpha} (\ln(1/t))^{1-s-d} \sum_{n \geq 1} |n|^r (\log_+ |n|)^s P(|S_n| \geq tB_n |n|^\alpha) = (1/\alpha)^{s+d-1} (r+1)^{-1} ((d-1)!)^{-1} C_{(r+1)/\alpha}, \quad r, s = 0, 1, 2, \ldots, \tag{23}$$

and, for every $u = 1, 2, \ldots$,

$$\lim_{t \to 0^+} t^{(s+d)/u} \sum_{n \geq 1} |n|^{-1} (\log_+ |n|)^s P(|S_n| \geq t\, B_n (\log_+ |n|)^u) = C_{u/(s+d)}/(s+d)(d-1)!. \tag{24}$$

Proof. We note that (23) and (24) follow from (20) and (18), respectively. To see (22) we use the following asymptotic expansions:

$$\begin{aligned}
&(\ln(1/t))^{-s-d} \sum_{n \geq 1} |n|^{-1} (\log_+ |n|)^s P(|S_n| \geq tB_n |n|^\alpha) \\
&\cong (\ln(1/t))^{-s-d} \sum_{n=1}^{\infty} P(t(n-1)^\alpha \leq |N| < tn^\alpha) \sum_{k=1}^{n} d(k) k^{-1} (\log_+ k)^s \\
&\cong (\ln(1/t))^{-s-d} \sum_{n=1}^{\infty} (\log_+ n)^{s+d} P(t(n-1)^\alpha \leq |N| < tn^\alpha)/(s+d)(d-1)! \\
&\cong \frac{(2/\pi)^{1/2} (1/\alpha)^{s+d} (\ln(1/t))^{-s-d} \int_0^\infty (\log_+ (x/t))^{s+d} \exp(-x^2/2)\, dx}{(s+d)(d-1)!} \\
&= 1/\alpha^{s+d}(s+d)(d-1)! \quad \text{as } t \to 0^+.
\end{aligned}$$

Thus Lemma is proved.

THEOREM 3. *Let* $\{X_n, n \in Z^d\}$ *be independent random variables such that* $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$, $n \in Z^d$, *and* $\Delta_n = \sup_x \Delta_n(x) \to 0$ *as* $n \to \infty$,

$$\Delta_n(x) = |P(S_n < x B_n) - \Phi(x)|.$$

If

$$(25) \quad \lim_{K\to\infty} \limsup_{t\to 0} (\ln(1/t))^{-s-d} \sum_{|n| > Kt^{-1/\alpha}} |n|^{-1} (\log_+ |n|)^s \times \\ \times \sum_{1 \leqslant k \leqslant n} P(|X_k| \geqslant t B_n |n|^\alpha) = 0,$$

then (22) *holds.*

If

$$(26) \quad \lim_{K\to\infty} \limsup_{t\to 0^+} t^{(r+1)/\alpha} (\ln(1/t))^{1-s-d} \sum_{|n| > Kt^{-1/\alpha}} |n|^r (\log_+ |n|)^s \times \\ \times \sum_{1 \leqslant k \leqslant n} P(|X_k| \geqslant t B_n |n|^\alpha) = 0,$$

then (23) *holds.*

If

$$(27) \quad \lim_{K\to\infty} \limsup_{t\to 0^+} t^{(s+d)/u} \sum_{n \in C_{tk}^c} |n|^{-1} (\log_+ |n|)^s \sum_{1 \leqslant k \leqslant n} P(|X_k| \\ \geqslant t B_n (\log_+ |n|)^u) = 0,$$

where $C_{tk}^c = \{n \in Z^d\colon |n| > \exp(K t^{-1/u})\}$, *then* (24) *holds.*

Proof. At first we prove (23). By Lemma, it suffices to show that

$$(28) \quad \lim_{t\to 0^+} t^{(r+1)/\alpha} (\ln(1/t))^{1-s-d} \sum_{n \geqslant 1} |n|^r (\log_+ |n|)^s \times \\ \times |P(|S_n| \geqslant t B_n |n|^\alpha) - 2\Phi(-t|n|^\alpha)| = 0.$$

Now we prove (28) in the following two steps.

Step 1. Let $n_0(t)$ be such a positive integer that $n_0(t) \to \infty$ as $t \to 0^+$ and

$$t^{(r+1)/\alpha} (\log_+ n_0(t))^{s+d-1} (n_0(t))^{r+1} / (\ln(1/t))^{s+d-1} \to 0 \quad \text{as } t \to 0^+.$$

For every positive number K define

$$A_t = \{n \in Z^d, |n| \leqslant n_0(t)\},$$

$$B_{tk} = \{n \in Z^d, |n| \leqslant t^{-1/\alpha} K\}, \quad B_{tk}^c = Z^d - B_{tk}.$$

Then, by (14),

$$(29)\quad \lim_{t\to 0} t^{(r+1)/\alpha}(\ln(1/t))^{1-s-d} \sum_{n\in B_{tk}} |n|^r(\log_+|n|)^s \Delta_n(t|n|^\alpha)$$

$$\leqslant \lim_{t\to 0^+}\big(\max_{k\in B_{tK}-A_t} \Delta_k\big) t^{(r+1)/\alpha}(\ln(1/t))^{1-s-d}\times$$

$$\times \sum_{k\in B_{tK}-A_t} |k|^r(\log_+|k|)^s + 2\lim_{t\to 0^+} t^{(r+1)/\alpha}(\ln(1/t))^{1-s-d}\sum_{k\in A_t}|k|^r(\log_+|k|)^s = 0.$$

Step 2. By the result of Fuk [6] (Corollary 3 with $t = 2$, $y_1 = \ldots = y_n = atB_n|n|^\alpha/2$, where a is a positive constant such that $a < 2\alpha/(r+1)$), we get

$$(30)\quad P(|S_n| \geqslant tB_n|n|^\alpha) \leqslant \sum_{1\leqslant k\leqslant n} P(|X_k| \geqslant atB_n|n|^\alpha/2) +$$

$$+2^{1+2/a}(4+\alpha t^2|n|^{2\alpha})^{-1/a} + 2\exp(-t^2|n|^{2\alpha}/8e^2).$$

Let us observe that by Abel's transform

$$(31)\quad \lim_{K\to\infty}\lim_{t\to 0} t^{(r+1)/\alpha-2/a}(\ln(1/t))^{1-s-d}\sum_{n\in B^c_{tK}}|n|^{r-2\alpha/a}(\log_+|n|)^s$$

$$= \lim_{K\to\infty}\lim_{t\to 0} t^{(r+1)/\alpha-2/a}(\ln(1/t))^{1-s-d}\sum_{k=t^{-1/\alpha}K}^{\infty} k^{r-2\alpha/a}\log^s_+ k\, d(k)$$

$$= \lim_{K\to\infty}\lim_{t\to 0} t^{(r+1)/\alpha-2/a}(\ln(1/t))^{1-s-d}(2\alpha/a-r)\int_{t^{-1/\alpha}K}^{\infty} x^{r-2\alpha/a}(\log_+ x)^{s+d-1}\,dx/(d-1)!$$

$$= \frac{1}{(d-1)!\,(2\alpha/a-r-1)}\lim_{K\to\infty}\lim_{t\to 0}(2\alpha/a-r)\,t^{(r+1)/\alpha-2/a}\times$$

$$\times(\ln(1/t))^{1-s-d}\,t^{-(r+1)/\alpha+2/a}K^{r-2\alpha/a+1}(\log_+(1/t)^{1/\alpha}K)^{s+d-1}.$$

On the other hand,

$$x^{1/a}\exp(-\gamma x) \leqslant (a\gamma e)^{-1/a},\qquad a,\gamma,x>0,$$

so that

$$(32)\qquad \exp(-t^2|n|^{2\alpha}/8e^2) \leqslant t^{-2/a}|n|^{-2\alpha/a}(8e/a)^{1/a}.$$

Thus, similarly as in (31), we get

$$(33)\quad \lim_{K\to\infty}\limsup_{t\to 0^+} t^{(r+1)/\alpha}(\ln(1/t))^{1-s-d}\sum_{n\in B^c_{tK}}|n|^r(\log_+|n|)^s\times$$

$$\times\exp(-t^2|n|^{2\alpha}/8e^2) = 0.$$

Now let us observe that, by Lemma 2 ([5], p. 166),

$$\Phi(-x) \leqslant (2\pi)^{-1/2}x^{-1}\exp(-x^2/2)\qquad \text{as } x\to\infty.$$

Thus if K is sufficiently large, then, for every $n \in B_{tK}^c$,

$$\Phi(-t|n|^{\alpha}) \leqslant (2\pi)^{-1/2} t^{-1} |n|^{-\alpha} \exp(-t^2 |n|^{2\alpha}/2).$$

Hence, taking into account inequality (32), we get

$$\Phi(-t|n|^{\alpha}) \leqslant (2\pi)^{-1/2} (ae/2)^{-1/a} t^{-1-2/a} |n|^{-2\alpha/a-\alpha},$$

so that

$$(34)\quad \lim_{K\to\infty} \limsup_{t\to 0^+} t^{(r+1)/\alpha} (\ln(1/t))^{1-s-d} \sum_{n\in B_{tK}^c} |n|^r (\log_+ |n|)^s \Phi(-t|n|^{\alpha})$$

$$\leqslant \lim_{K\to\infty} \limsup_{t\to 0^+} t^{(r+1)/\alpha-2/a-1} (\ln(1/t))^{1-s-d} \sum_{n\in B_{tk}^c} |n|^{r-2\alpha/a-\alpha} \times$$

$$\times (\log_+ |n|)^s/(2\pi)^{1/2} (ae/2)^{1/a} = 0.$$

Now (28) follows from (29), (30), (31), (33), (34) and (26).

The proofs of (22) and (24) can be get by some modifications of the proof of (23). Namely, in the proof of (24) we use the set C_{tK}^c instead of the set B_{tK}^c and apply Abel's transform as well as Fuk's inequality with the functions from (22) and (24), respectively. Thus the details are omitted.

Let us observe that if, for example, $\{X_n, n \in Z^d\}$ are independent and identically distributed random variables such that $EX_n = 0$, $EX_n^2 = 1$, then

$$\Delta_n = \sup_x |P(S_n < x|n|^{1/2}) - \Phi(x)| \to 0 \quad \text{as } n \to \infty,$$

and

$$\lim_{K\to\infty} \limsup_{t\to 0} t^{(r+1)/\alpha} \sum_{|n| > Kt^{-1/\alpha}} |n|^r (\log_+ |n|)^s \sum_{1\leqslant k\leqslant n} P(|X_k| \geqslant t B_n |n|^{\alpha})$$

$$\leqslant \lim_{K\to\infty} \limsup_{t\to 0} (2/(1+2\alpha))^{s+d-1} t^{(r+1-2\alpha)/\alpha(1+2\alpha)} \times$$

$$\times E\{|X_1|^{2(r+1)/(1+2\alpha)} (\log_+ (|X_1|/t))^{s+d-1} I(|X_1| > t^{-1/2\alpha} K^{1/2+\alpha})\},$$

so that if $(r+1)/2 > \alpha > 0$ and

$$E\{|X_1|^{2(r+2)/(1+2\alpha)} (\log_+ |X_1|)^{s+d-1}\} < \infty,$$

then (26) holds. Furthermore, it is not difficult to show that if X_n, $n \in Z^d$, are independent and identically distributed such that $EX_n = 0$, $EX_n^2 = 1$ and $EX_1^2 \log_+^{d+s-1-2u} |X_1| < \infty$, $s+d > 2u$, then (27) also holds. Similarly, one can check that if

$$\lim_{K\to\infty} \limsup_{t\to 0} (\ln(1/t))^{-s-d} t^{-2/(1+2\alpha)} E|X_1|^{2/(1+2\alpha)} \times$$

$$\times (\log_+ |X_1|)^{s+d-1} I(|X_1| > t^{-1/2\alpha} K^{1/2+\alpha}) = 0,$$

then (25) also holds. Thus Theorem 3 and the remarks given above not only give some assertions obtained in [2]-[4] and [7]-[10], but extend them to nonidentically distributed random variables and provide us with a much better and deeper understanding of the tail probability of distributions.

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