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SEMI-STABLE AND SELF-DECOMPOSABLE MEASURES ON SEPARABLE BANACH SPACES

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Abstract. The aim of this paper is to characterize in terms of characteristic functionals the intersection of semi-stable and self-decomposable measures on a separable Banach space. These are subclasses of infinitely divisible measures the characterization of which on a Hilbert space has already been done in [4]. Here the proof is based on Urbanik's method which exploits Choquet's theorem on extreme points.

1. Introduction. Let Y be a topological space. $B(Y)$ denotes the class of Borel measurable subsets of Y. $M(Y)$ stands for the class of all probability measures on $B(Y)$.

Let $(Y, B(Y), \mu)$ be a probability space and Z be a topological space. Then, for a measurable function $f\colon Y \to Z$, we define the *measure* $f\mu$ on $B(Z)$ by $f\mu(A) = \mu(f^{-1}A)$ for each $A \in B(Z)$.

Let X be a separable Banach space with a norm $\|\cdot\|$ and with the dual space X^*. For $\mu \in M(X)$ the *characteristic functional* is defined as follows:

$$\hat{\mu}(y) = \int_X e^{i\langle y,x\rangle}\,\mu(dx), \qquad \text{where } y \in X^*.$$

A measure $\mu \in M(X)$ is called *infinitely divisible* if, for each n, there exists a $\mu_n \in M(X)$ such that $\mu_n^{*n} = \mu$, where the power $*n$ is taken in the sense of convolution.

Tortrat [9], p. 311 (see also [2]), proved the following analogue of the Levy-Khinchine representation of infinitely divisible laws:

PROPOSITION 1. *A probability measure μ on X is infinitely divisible iff $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure and M is a generalized Poisson exponent of $\tilde{e}(M)$. Moreover the decomposition is unique.*

The characteristic functionals of these measures are:

$$\hat{\tilde{e}}(M)(y) = \exp\{i\langle y, x_0\rangle + \int_X [e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle 1_D(x)]\,M(dx)\},$$

where D is a ball which is a set of continuity of the measure M and $x_0 \in X$ ([3], Th. 2.3);

$$\hat{\varrho}(y) = \exp(-\tfrac{1}{2}\langle y, Ry\rangle),$$

where R is a covariance operator and $y \in X^*$ ([11], p. 173).

Let us add that M is finite on the complements of neighbourhoods of zero in X and that $M(\{0\}) = 0$.

The family of all generalized Poisson exponents will be denoted by $P(X)$.

Remark. If

$$\int_{\|x\| \leqslant 1} \|x\|^2 M(dx) < +\infty,$$

then the characteristic functional of the measure $\tilde{e}(M)$ can be written in an analogous way as on the Hilbert space:

$$\hat{\tilde{e}}(M)(y) = \exp\left\{i\langle y, x_0\rangle + \int_X \left[e^{i\langle y,x\rangle} - 1 - \frac{i\langle y, x\rangle}{1+\|x\|^2}\right] M(dx)\right\}. \tag{1}$$

T_b will stand for the operator of multiplying by the scalar b. By μ^c we denote the measure having the characteristic functional $\hat{\mu}(y)^c$. It is well-known that for infinitely divisible μ such a measure exists.

Definition 1. $\mu \in M(X)$ is called *semi-stable* iff there are $b, c \in (0, 1)$ and $x_0 \in X$ such that

$$\mu^c = T_b \mu * \delta_{x_0}. \tag{2}$$

The family of semi-stable measures on X will be denoted by $S(X)$.

PROPOSITION 2. *Let μ be a non-degenerate measure on X satisfying* (2) *and α be a unique real solution of the equation $b^\alpha = c$. Then*

(a) $0 < \alpha \leqslant 2$;

(b) $\alpha = 2$ *iff μ is a Gaussian measure*;

(c) $0 < \alpha < 2$ *iff $\mu = \tilde{e}(M)$ for some $M \in P(X)$ and, moreover, $T_b M = cM$* ([6], Prop. 4.2 and 4.1).

The proof of Proposition 2 is an immediate consequence of the following statement: if μ is a semi-stable measure on X, then $y\mu$ is a semi-stable measure on the real line for all $y \in X^*$.

LEMMA 1. *If $\mu \in S(X)$ and $\mu = \tilde{e}(M)$, then*

$$\int_{\|x\| \leqslant 1} \|x\|^2 M(dx) < +\infty.$$

Proof. By Proposition 2 (c) we have $b^2 < c$ and $T_b M = cM$. Let

$$A_n = \{x\colon b^{n+1} < \|x\| \leqslant b^n\} \quad (n = 0, 1, 2, \ldots).$$

We have

$$\int_{\|x\|\leqslant 1} \|x\|^2 M(dx)$$

$$= \int_{\bigcup_{n=0}^{\infty} A_n} \|x\|^2 M(dx) \leqslant \sum_{n=0}^{\infty} b^{2n} M(b^n A_0) = \sum_{n=0}^{\infty} b^{2n} c^{-n} M(A_0) < +\infty .$$

$M(A_0) < +\infty$ as M is finite on complements of the neighbourhoods of zero. This completes the proof of Lemma 1.

It follows from Lemma 1 and from the Remark that the characteristic functional of a semi-stable measure may be written in the form of (1).

Definition 2. A measure μ is called *self-decomposable* if it is infinitely divisible and if $M \geqslant T_c M$ for each $c \in (0, 1)$, where $\mu = \varrho * \tilde{e}(M)$ (see Proposition 1).

Original definitions of semi-stable and self-decomposable measures can be found in [7] and [8]. Our definitions 1 and 2 are, in fact, the well-known characterizations.

The family of self-decomposable measures on X will be denoted by $L(X)$.

2. Characterization of the class $L(X) \cap S(X)$.

THEOREM. $\nu \in L(X) \cap S(X)$ *iff*

(i) $$\hat{\nu}(y) = \hat{\varrho}(y) \exp(i \langle y, x_0 \rangle),$$

where $x_0 \in X$, ϱ is a symmetric Gaussian measure, $y \in X^*$, or

(ii) $$\hat{\nu}(y) = \exp\{i \langle y, x_0 \rangle +$$
$$+ \int_{U_\eta^0} \int_{R_+} \left[\frac{e^{it\langle y,x\rangle} - 1}{t} - \frac{i\langle y, x\rangle}{1+t^2\|x\|^2}\right] \sum_{k=-\infty}^{\infty} b^{\alpha k} \psi_{\|x\|} (b^k t \|x\|)\, dt \lambda(dx),$$

where $x_0 \in X$, $y \in X^*$, $\alpha \in (0, 2)$, $b \in (0, 1)$, $\eta = -\ln b$, $U_\eta^0 = \{x \in X : 0 < \|x\| < \eta\}$, λ is a finite Borel measure on the closed ball of radius η such that $\lambda(\{0\}) = 0$ and ψ_r is given by the formula

$$\psi_r(t) = \begin{cases} 1 & \text{for } t \in [1, e^r), \\ b & \text{for } t \in [e^r, b^{-1}),\ r \in (0, \eta]. \end{cases}$$

We precede the proof of this theorem by introducing some notions and notation.

U, U_η and S will denote the unit ball, the ball of radius η and the unit sphere, respectively.

Let $X_0 = X \setminus \{0\}$, $R_+ = (0, \infty)$ and $Q = U \times [0, \infty]$. Q is a separable metric space. Let $h\colon X_0 \to Q$ be the mapping given by the formula $h(x) = (x/\|x\|, \|x\|)$. h is a homeomorphism of X_0 onto $S\times R_+$. For $q=(x, r)\in Q$ let us put $\|q\| = r$ and, for $b\in R$, $bq = (x, br)$; then $\|h(x)\| = \|x\|$ and $h(bx) = bh(x)$.

The measure ν is infinitely divisible, thus $\nu = \varrho * \tilde{e}(M)$ (see Proposition 1). Let μ_0 be such a measure that $d\mu_0 = \|x\|^2/(1+\|x\|^2)\,dM$.

From Lemma 1 and from the finiteness of M on the complements of neighbourhoods of zero it easily follows that μ_0 is finite on X. Let us put $m_0 = h\mu_0$. This measure is concentrated at $S\times R_+$. We shall consider m_0 on $S\times \bar{R}_+$, where $\bar{R}_+ = [0, \infty]$. With these assumptions the following holds:

$$m_0(B) - c^2 \int_{c^{-1}B} \frac{\|q\|^2+1}{1+c^2\|q\|^2}\, m_0(dq) \geqslant 0 \tag{3}$$

for each $B\in B(Q)$ and each $c\in(0, 1)$;

$$\int_{b^{-1}B} \frac{1+\|q\|^2}{\|q\|^2}\, m_0(dq) = b^\alpha \int_B \frac{1+\|q\|^2}{\|q\|^2}\, m_0(dq) \tag{4}$$

for each $B\in B(Q)$, some $b\in(0, 1)$ and some $\alpha\in(0, 2)$.

This follows the rendering of the conditions of Lemma 1 and Definitions 1 and 2 with the usage of m_0. The justification of (3) and (4) is in fact the same as in the case of the Hilbert space [4].

LEMMA 2. *For the above-mentioned measure m_0 there exists a pairwise disjoint family of compact subsets K_n of the sphere S such that $m_0 = \sum_{n=1}^{\infty} m_n$ for $m_n = m_0|_{K_n\times \bar{R}_+}$.*

The proof of Lemma 2 is similar to that of Lemma 5.4 in [10]. One uses in the proof the tightness of the measure m_0 on Q.

In the sequel by K_n we shall understand sets as those given in Lemma 2. Let us denote by $N(K_n\times\bar{R}_+)$ the set of all finite Borel measures of supports included in $K_n\times\bar{R}_+$. Similarly, $M(K_n\times\bar{R}_+)$ denotes the set of probabilistic measures of supports in $K_n\times\bar{R}_+$. This is a compact metric space with the topology of weak convergence. Let $N_{b,\alpha}(K_n\times\bar{R}_+)$ be the set of all measures $m\in N(K_n\times\bar{R}_+)$ such that (3) and (4) hold if m_0 is substituted by m therein. Finally we put

$$K_{b,\alpha}(K_n\times\bar{R}_+) = N_{b,\alpha}(K_n\times\bar{R}_+) \cap M(K_n\times\bar{R}_+).$$

On $K_{b,\alpha}(K_n\times\bar{R}_+)$ we consider the topology induced from $M(K_n\times\bar{R}_+)$.

LEMMA 3. *The set $K_{b,\alpha}(K_n\times\bar{R}_+)$ is compact and convex.*

The proof is analogous to that of Lemma 1 in [5]. To apply in the sequel Choquet's theorem we have to find the set $e(K_{b,\alpha}(K_n\times\bar{R}_+))$ of extreme points of the set $K_{b,\alpha}(K_n\times\bar{R}_+)$.

For $E \in B(K_n)$ the sets $E \times \{0\}$, $E \times R_+$, $E \times \{0, \infty\}$ are invariant under multiplication by elements of R_+. Hence, for $m \in N_{b,\alpha}(K_n \times \bar{R}_+)$, the restriction of m to any of these sets is again in $N_{b,\alpha}(K_n \times \bar{R}_+)$. Thus we see that each extreme point of $K_{b,\alpha}(K_n \times \bar{R}_+)$ has to be a measure concentrated either at a point $z \in K_n \times \{0, \infty\}$ or at $\{x\} \times R_+$ for $x \in K_n$. Moreover, the measure m concentrated at $\{x\} \times R_+$ is an extreme point of $K_{b,\alpha}(K_n \times \bar{R}_+)$ iff, being considered in a natural way as a measure on R, it is an extreme point of $K_{b,\alpha}(R)$ concentrated at R_+.

PROPOSITION 3 [5]. *The extreme points of the set $K_{b,\alpha}(R)$ are of the form*

$$p_r(F) = V_r \int_F |t|(1+t^2)^{-1} \sum_{k=-\infty}^{\infty} b^{k\alpha} \psi_r(b^k|t|)\,dt,$$

where

$$F \in B(R), \qquad V_r = \{\int_R |t|(1+t^2)^{-1} \sum_{k=-\infty}^{\infty} b^{k\alpha} \psi_r(b^k|t|)\,dt\}^{-1},$$

$r \in (0, \eta]$ and ψ_r is defined in the statement of the Theorem.

From the reasoning above we get a description of $e(K_{b,\alpha}(K_n \times \bar{R}_+))$.

Let $Z_n = K_n \times ([0, \eta] \cup \{\infty\})$, $\eta = -\ln b$. For $z = (x, r) \in K_n \times \{0, \infty\}$ let $m_z = \delta_z$ and, for $z = (x, r) \in K_n \times (0, \eta]$, let

$$m_z(B) = V_r \int_B \|q\|(1+\|q\|^2)^{-1} \sum_{k=-\infty}^{\infty} b^{k\alpha} \psi_r(b^k\|q\|)\,\lambda_x(dq), \tag{5}$$

where $B \in B(K_n \times \bar{R}_+)$.

V_r is chosen to make m_z probabilistic. λ_x is a Lebesgue measure on $\{x\} \times R_+$. Hence we obtain

LEMMA 4. *The set $e(K_{b,\alpha}(K_n \times \bar{R}_+))$ coincides with the set $\{m_z: z \in Z_n\}$. Moreover, the mapping $w: Z_n \to e(K_{b,\alpha}(K_n \times \bar{R}_+))$, defined by $w(z) = m_z$, is a homeomorphism of these sets.*

LEMMA 5. *With the above-formulated assumptions about the measure m_0, for any bounded and continuous function $f: S \times \bar{R}_+ \to R$ there exists a finite Borel measure ζ, concentrated at $S \times [0, \eta]$, such that*

$$\int_{S \times (0,\eta]} \int_{S \times \bar{R}_+} f(q)\, m_z(dq)\, \zeta(dz) = \int_{S \times \bar{R}_+} f(q)\, m_0(dq) \text{ (1)}. \tag{6}$$

Proof. Choquet's theorem [1] says that "if X is a linear locally convex space and K is a compact and convex subset of X, then for each $x_0 \in K$ there

(1) Obviously, the function $\int_{S \times [0,\eta]} f(q) m_z(dq)$ is defined only on $\bigcup_{n=1}^{\infty} K_n \times [0, \eta]$ but, as we shall see in the proof, the measure ζ vanishes outside this set, which justifies the notation.

exists a probabilistic measure ν on $e(K)$ such that $\langle y, x_0\rangle = \int\limits_{e(K)} \langle y, x\rangle \nu(dx)$ for each $y \in X^*$".

By Lemma 3 the set $K_{b,\alpha}(K_n \times \bar{R}_+)$ is compact and convex in the space of all signed measures which is locally convex. The restriction of m_0, i.e.

$$\bar{m}_n = m_0(K_n \times R_+)^{-1} \cdot m_o|_{K_n \times \bar{R}_+},$$

belongs to $K_{b,\alpha}(K_n \times \bar{R}_+)$ since the set $K_n \times \bar{R}_+$ is invariant under multiplication by scalars.

By Choquet's theorem, applied to the functional

$$\langle m^*, \cdot\rangle = \int\limits_{K_n \times \bar{R}_n} f(q)(\cdot)(dq),$$

there exists a probabilistic measure τ_n, concentrated at $e(K_{b,\alpha}(K_n \times \bar{R}_+))$, such that

$$(*) \qquad \int\limits_{K_n \times \bar{R}_n} f(q)\bar{m}_n(dq) = \langle m^*, \bar{m}_n\rangle = \int\limits_{e(K_{b,\alpha}(K_n \times \bar{R}_+))} \langle m^*, m\rangle \tau_n(dm)$$

$$= \int\limits_{w(Z_n)} \langle m^*, m\rangle w\tilde{\tau}_n(dm) = \int\limits_{Z_n} \langle m^*, m_z\rangle \tau_n(dz) = \int\limits_{Z} \int\limits_{K_n \times \bar{R}_n} f(q)\, m_z(dq)\, \tilde{\tau}_n(dz)$$

(for w see Lemma 4), where $\tilde{\tau}_n = w^{-1}\tau_n$ is a finite measure concentrated at $Z_n = K_n \times (0, \eta]$. Multiplying by $m_0(K_n \times R_+)$ the first and the last integrals in $(*)$, we obtain

$$(7) \qquad \int\limits_{K_n \times \bar{R}_n} f(q)\, m_n(dq) = \int\limits_{Z_n} \int\limits_{K_n \times \bar{R}_n} f(q)\, m_z(dq)\, \tau'_n(dz),$$

where $\tau'_n = m_0(K_n \times \bar{R}_+)\tilde{\tau}_n$.

Taking the sum over n on both sides of (7) and taking Lemma 2 into account, we obtain (6), where $\zeta = \sum_{n=1}^{\infty} \tau'_n$.

To complete the proof it is enough to show the finiteness of measure ζ. Substituting $f(q) \equiv 1$ in (7), we obtain $m_n(K_n \times \bar{R}_+) = \tau'_n(K_n \times (0, \eta])$. Taking on both sides the sum over n, we get $+\infty > m_0(S \times R_+) = \zeta(X \times (0, \eta])$.

3. Proof of the Theorem. Since m_0 is concentrated at $S \times R_+$, we obtain (6) in the form

$$(8) \qquad \int\limits_{X_0} \tilde{f}(x)\mu_0(dx) = \int\limits_{U^0_\eta} \int\limits_{X} \tilde{f}(y)(h^{-1} m_{h(x)})(dy)\tilde{\zeta}(dx),$$

where $h\colon X_0 \to S \times R_+$, $h(x) = z$, $\tilde{f}(y) = f(y/\|y\|, \|y\|)$ and $\tilde{\zeta} = h^{-1}\zeta$.

Let us denote the measure $h^{-1}m_{h(x)}$ by μ^x. It is concentrated at $R_+ x$. If we consider the mapping $tx \to t$, we infer that the inner integral on the right-hand side of (8) has the form

$$(9) \qquad \int_{X_0} \tilde{f}(y)\,\mu^x(dy) = \int_{R_+} \tilde{f}(tx)(x^{-1}\mu^x)(dt),$$

where $x^{-1}\mu^x(F) = \mu^x(\{tx\colon t\in F\})$ for $F\in B(R_+)$.

By (5) we get

$$(10) \qquad (x^{-1}\mu^x)(F) = x^{-1}(h^{-1}m_{h(x)})(F) = m_{(x/\|x\|,\,\|x\|)}\big(h(Fx)\big)$$

$$= V_{\|x\|}\int_{h(Fx)} \|q\|(1+\|q\|^2)^{-1}\sum_{k=-\infty}^{\infty} b^{k\alpha}\psi_{\|x\|}(b^k\|q\|)\,\lambda_{x/\|x\|}(dq)$$

$$= V_{\|x\|}\int_F t\|x\|(1+t^2\|x\|^2)^{-1}\sum_{k=-\infty}^{\infty} b^{k\alpha}\psi_{\|x\|}(b^k t\|x\|)\,\|x\|\,dt.$$

The last equality follows from the observation that if $q = h(tx)\in h(Fx)$, then $\|q\| = \|h(tx)\| = \|(tx/t\|x\|,\, t\|x\|)\| = t\|x\|$ and that $x^{-1}h^{-1}\lambda_{x/\|x\|}$ is the Lebesgue measure multiplied by $\|x\|$. Indeed,

$$x^{-1}h^{-1}\lambda_{x/\|x\|}([0,\beta])$$
$$= h^{-1}\lambda_{x/\|x\|}([0,\beta x]) = \lambda_{x/\|x\|}(\{x/\|x\|\}\times[0,\beta\|x\|]) = \beta\|x\|.$$

Now taking (9) and (10) into account we write (8) in the following form:

$$(11) \qquad \int_X \tilde{f}(x)\,\mu_0(dx)$$

$$= \int_{U_\eta^0}\int_{R_+} \frac{\tilde{f}(tx)}{t}\,\frac{(t\|x\|)^2}{1+(t\|x\|)^2}\,V_{\|x\|}\sum_{k=-\infty}^{\infty} b^{k\alpha}\psi_{\|x\|}(b^k t\|x\|)\,dt\,\tilde{\zeta}(dx).$$

In order to calculate the characteristic functional of the measure ν let us put

$$\tilde{f}(x) = \left(e^{i\langle y,x\rangle} - 1 - \frac{i\langle y,x\rangle}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}.$$

Now (11) assumes the form

$$(12) \qquad \int_{X_0} \tilde{f}(x)\,\mu_0(dx)$$

$$= \int_{U_\eta^0}\int_{R_+} \left(\frac{e^{it\langle y,x\rangle}-1}{t} - \frac{i\langle y,x\rangle}{1+t^2\|x\|^2}\right)V_{\|x\|}\sum_{k=-\infty}^{\infty} b^{k\alpha}\psi_{\|x\|}(b^k t\|x\|)\,dt\,\tilde{\zeta}(dx).$$

In (12) the function $V_{\|x\|}$ can be omitted, for it is Borel measurable and bounded. Finally, multiplying both sides by $e^{i\langle y,x_0\rangle}$ we obtain the desired representation.

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