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A REPRESENTATION THEOREM FOR RANDOM SETS

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Abstract. It is shown that every random closed set X in $\mathbf{R}^d$, taking its realizations in the extended convex ring, is (up to equivalence) the union set of a point process Y of convex particles with the same invariance properties as X.

1. Random closed sets and point processes of compact particles in $\mathbf{R}^d$ are important models for applications in image analysis and stereology (see for example [4]). In practice, quite often only the union set $X = X_Y$ of a particle process Y is observable and frequently it is essential (e.g. for projected thick sections) that the particles are compact convex sets (convex bodies). In that case, X is a random closed set with values in $\mathscr{S}^d$, the class of sets $F \subset \mathbf{R}^d$, such that, for any convex body K, $F \cap K$ is a finite union of convex bodies (*extended convex ring*). This raised the question whether every random $\mathscr{S}^d$-set X is equal (in probability) to the union set of some point process Y on the set $\mathscr{K}^d$ of convex bodies ([5], p. 398). A positive answer was given in [6] and the result was used to solve some measurability and continuity problems for geometric functionals of random $\mathscr{S}^d$-sets. We remark that in a similar situation (point processes of sets of positive reach [7]), where a corresponding result is lacking, the considerations had to be restricted to random sets which are known to be the union set of a point process.

The construction given in [6] does however not preserve any invariance property of the random set, whereas for practical applications it would be necessary to know that a stationary and/or isotropic random set X comes from a point process Y with the same invariance properties. In this note we show that a modification of the method in [6] leads to a general result of this kind.

2. First we introduce some notation. The classes $\mathscr{K}^d$ and $\mathscr{S}^d$, and the convex ring $\mathscr{R}^d$ (the class of finite unions of convex bodies) are considered as topological subspaces of the space $\mathscr{F}^d$ of all closed subsets of $\mathbf{R}^d$ (see [1] for details concerning the topology on $\mathscr{F}^d$ and its properties). A *random* $\mathscr{S}^d$*-set* is a measurable map X from some probability space $(\Omega, \mathfrak{A}, P)$ into the topological space $\mathscr{S}^d$. Similarly, a *point process* Y on $\mathscr{K}^d$ is a measurable map from $(\Omega, \mathfrak{A}, P)$ into $\mathscr{M}$, the set of locally finite (countable) collections of convex bodies with the usual σ-algebra (see, e.g. [2] and notice that a collection of sets corresponds to a simple counting measure). For a point process Y on $\mathscr{K}^d$, X_Y denotes the union set of Y. SO_d denotes the rotation group of $\mathbf{R}^d$ and G^d the group of rigid motions. Each $g \in G^d$ has a unique decomposition $g = \varrho t_x$, where $\varrho \in SO_d$ and t_x is the translation by $x \in \mathbf{R}^d$. For $F \in \mathscr{F}^d$, $\mathscr{A} \subset \mathscr{F}^d$, $x \in \mathbf{R}^d$ and $g \in G^d$, we use the notation

$$F \pm x := \{y \pm x \mid y \in F\}, \quad \mathscr{A} \pm x := \{F \pm x \mid F \in \mathscr{A}\}$$

and

$$gF := \{gy \mid y \in F\}, \quad g\mathscr{A} := \{gF \mid F \in \mathscr{A}\}.$$

The map $(F, g) \mapsto gF$ from $\mathscr{F}^d \times G^d$ into $\mathscr{F}^d$ is continuous [3] and the map $(M, g) \mapsto gM$ from $\mathscr{M} \times G^d$ into $\mathscr{M}$ is measurable [7].

Finally, a random $\mathscr{S}^d$-set X (resp. a point process Y on $\mathscr{K}^d$) is said to be *g-invariant*, for $g \in G^d$, if $X = gX$ (resp. $Y = gY$). Here and in the following the equality sign means that both random sets (resp. point processes) induce the same probability measure on $\mathscr{S}^d$ (resp. on $\mathscr{M}$). A random set or point process which is g-invariant for all translations (resp. rotations) g is called *stationary* (resp. *isotropic*).

3. We now can state our result.

THEOREM. *Let X be a random $\mathscr{S}^d$-set. Then there exists a point process Y on $\mathscr{K}^d$ such that $X = X_Y$ and Y is g_0-invariant for each rigid motion g_0 for which X is g_0-invariant. Especially, if X is stationary, then Y is stationary, and if X is isotropic, then Y is isotropic.*

Proof. In [6], Lemma 2, we proved that there is measurable "dissection" of sets in $\mathscr{R}^d$ into convex bodies, i.e. a measurable map

$$\xi: \mathscr{R}^d \to \sum_{n=0}^{\infty} (\mathscr{K}^d)^n$$

(here $\sum (\mathscr{K}^d)^n$ is the topological sum of the product spaces $(\mathscr{K}^d)^n = \mathscr{K}^d \times \ldots \times \mathscr{K}^d$, $(\mathscr{K}^d)^0 = \{\emptyset\}$) which associates with each $K \in \mathscr{R}^d$ a collection $\xi(K) = (K_1, \ldots, K_n)$ of convex bodies with $K = \bigcup K_i$ $(i = 1, 2, \ldots, n)$ and $n = n(K)$ is the minimal number for which such a representation is possible. Since the bodies K_i in $\xi(K)$ are necessarily different, we may write $\xi(K) = \{K_1, \ldots, K_n\}$ in the following.

In order to apply this mapping to a set $F \in \mathscr{S}^d$, we first have to dissect F into pieces in $\mathscr{R}^d$. As in [6], we do this by a hypercube tiling C_z, $z \in \mathbf{Z}^d$, of $\mathbf{R}^d$.

Here

$$C = \{x \in R^d |\ x = (x_1, \ldots, x_d),\ -\frac{1}{2} \leqslant x_i \leqslant \frac{1}{2} \text{ for } i = 1, \ldots, d\},$$

and $C_z = C + z$. However, to ensure the invariance properties, we shift $F \cap C_z$ to the origin, apply ξ, and shift back. Thus, by

$$\Phi(F) = \bigcup_{z \in Z^d} [\xi((F \cap C_z) - z) + z],$$

a mapping Φ: $\mathscr{S}^d \to \mathscr{M}$ is defined which is measurable (see [6]) and obeys $F = \bigcup_{K \in \Phi(F)} K$.

Thus $\Phi \circ X$ is a point process on $\mathscr{K}^d$ with union set X.

So far, the construction preserves invariances with respect to lattice translations t_z, $z \in Z^d$. In order to get the corresponding result for general rigid motions we use the above construction not with a fixed but with a motion invariant random hypercube tiling which is independent of X.

For this purpose, let G_0^d be the (measurable) set of all rigid motions $g = \varrho t_x$ with $\varrho \in SO_d$ and $x \in (0, 1]^d$, let $\mathfrak{B}$ be the σ-algebra of all Borel subsets of G_0^d and let τ_0 be the Haar measure on G^d, restricted to G_0^d and normalized. For $g \in G^d$, define Φ_g: $\mathscr{S}^d \to \mathscr{M}$ by $\Phi_g(F) = g\Phi(g^{-1}F)$. Since the operation of G^d on $\mathscr{S}^d$ and $\mathscr{M}$ is measurable, $\Phi_g \circ X$ is again a point process on $\mathscr{K}^d$ with union set X, and hence the same is true for the point process Y: $(\Omega \times G_0^d, \mathfrak{A} \otimes \mathfrak{B}, P \otimes \tau_0) \to \mathscr{M}$ defined by $Y(\omega, g) = \Phi_g(X(\omega))$.

It remains to show that Y has the required invariance properties.

Thus, let X be g_0-invariant for a rigid motion g_0 and let $\mathscr{A}$ be a measurable set in $\mathscr{M}$. Since $g_0 \Phi_g(F) = g_0 g \Phi(g^{-1} g_0^{-1} g_0 F) = \Phi_{g_0 g}(g_0 F)$ whenever $g \in G^d$ and $F \in \mathscr{S}^d$, we have

$$\text{(1)} \quad \begin{aligned} P \otimes \tau_0 \{(\omega, g) \mid g_0 Y(\omega, g) \in \mathscr{A}\} &= \int_{G_0^d} P(\{\omega \mid g_0 Y(\omega, g) \in \mathscr{A}\})\, d\tau_0(g) \\ &= \int_{G_0^d} P(\{\omega \mid \Phi_{g_0 g}(g_0 X(\omega)) \in \mathscr{A}\})\, d\tau_0(g) \\ &= \int_{G_0^d} P(\Phi_{g_0 g} \circ X \in \mathscr{A})\, d\tau_0(g). \end{aligned}$$

If $g = \varrho t_x$ and $g_0 = \varrho_0 t_{x_0}$, then $g_0 g = \varrho_0 t_{x_0} \varrho t_x = \varrho_0 \varrho t_{\varrho^{-1} x_0} t_x = \varrho_0 \varrho t_{x + \varrho^{-1} x_0}$ and the last integral in (1) equals

$$\text{(2)} \qquad \int_{SO_d} \int_{(0,1]^d} P(\Phi_{\varrho_0 \varrho t_{x + \varrho^{-1} x_0}} \circ X \in \mathscr{A})\, d\lambda^d(x)\, d\nu(\varrho),$$

where λ^d is Lebesgue measure and ν is the normalized Haar measure on SO_d. We consider now a fixed $\varrho \in SO_d$. Then, for any $x \in (0, 1]^d$, there is a unique decomposition $x + \varrho^{-1} x_0 = y(x) + z(x)$ with $y(x) \in (0, 1]^d$ and $z(x) \in Z^d$.

Since the norm of $z(x)$ is bounded, the mapping $\varphi: x \mapsto z(x)$, $x \in (0, 1]^d$, can assume only finitely many values, $z_1, \ldots, z_k$ say. Defining $C_i := \{x \in (0, 1]^d \mid z(x) = z_i\}$, we have

$$(0, 1]^d = \bigcup_{i=1}^{k} C_i$$

and the C_i are disjoint. On each C_i, the mapping $\psi: x \mapsto y(x)$ is a translation. Since, for $x, x' \in (0, 1]^d$, $y(x) = y(x')$ implies $x - x' = z(x) - z(x') \in \boldsymbol{Z}^d$ and hence $x = x'$, ψ is injective. Obviously, it is also surjective on $(0, 1]^d$. Thus, ψ is a bijection of $(0, 1]^d$ leaving the Lebesgue measure λ^d invariant. We therefore get

$$\begin{aligned}
&\int_{(0,1]^d} P(\Phi_{\varrho_0 \varrho t_{x+\varrho^{-1} x_0}} \circ X \in \mathscr{A})\, d\lambda^d(x) \\
&\qquad = \int_{(0,1]^d} P(\Phi_{\varrho_0 \varrho t_{y(x)+z(x)}} \circ X \in \mathscr{A})\, d\lambda^d(x) \\
&\qquad = \int_{(0,1]^d} P(\varrho_0 \varrho \Phi_{t_{y(x)+z(x)}} \circ \varrho^{-1} \varrho_0^{-1} X \in \mathscr{A})\, d\lambda^d(x) \\
&\qquad = \int_{(0,1]^d} P(\varrho_0 \varrho \Phi_{t_{y(x)}} \circ \varrho^{-1} \varrho_0^{-1} X \in \mathscr{A})\, d\lambda^d(x) \\
&\qquad = \int_{(0,1]^d} P(\varrho_0 \varrho \Phi_{t_x} \circ \varrho^{-1} \varrho_0^{-1} X \in \mathscr{A})\, d\lambda^d(x) \\
&\qquad = \int_{(0,1]^d} P(\Phi_{\varrho_0 \varrho t_x} \circ X \in \mathscr{A})\, d\lambda^d(x).
\end{aligned}$$

Using (1), (2) and the invariance of ν under ϱ_0, we conclude that

$$P \otimes \tau_0\big(\{(\omega, g) \mid g_0 Y(\omega, g) \in \mathscr{A}\}\big) = P \otimes \tau_0\big(\{(\omega, g) \mid Y(\omega, g) \in \mathscr{A}\}\big).$$

Hence, Y is g_0-invariant and the proof is complete.

Finally, it should be mentioned that the same argumentation can be used to prove that a random closed set X in $\boldsymbol{R}^d$ is the union set of a point process Y on the space of all nonvoid compact subsets of $\boldsymbol{R}^d$ (considered as a topological subspace of $\mathscr{F}^d$) with the same invariance properties as X. However, for this result, a simpler proof is possible. The difficulties in our construction stem from the fact that the particles of Y are required to be convex.

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