PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 9.1 (1988), p. 77-93

SEQUENTIAL ESTIMATION IN RANDOM FIELDS

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Abstract. Absolute continuity of measures μ_θ, generated by a random field and a Markov stopping set τ, is considered. The analogue of Sudakov lemma is proved. Moreover, with some additional assumptions on τ, the author proves the absolute continuity of the measure μ_θ with respect to the measure μ_{θ_0} on the σ-algebra F_τ.

Results obtained in the paper make it possible to characterize efficient (in the sense of Cramer-Rao-Wolfowitz inequality) sequential plans for some random fields.

INTRODUCTION

Suppose we observe a realization $w_t \in W$ of the stochastic process X_t till a random Markov moment τ defined on the space of realizations W. The process X_t generates in the space W the measure μ_θ depending on the unknown parameter θ.

In papers [8], [11], [13], and [16] it was proved, under some additional assumptions, that the distribution m_θ of the random vector $(\tau, S(\tau))$ is absolutely continuous with respect to m_{θ_0} (S_t is a sufficient statistic for the parameter θ, defined on the space of realizations w_s, $s \in [0, t]$). It was given the form of the density function dm_θ/dm_{θ_0}. Moreover, in [3], it was proved that μ_θ is absolutely continuous with respect to μ_{θ_0} on the σ-algebra F_τ. Thus $(\tau, S(\tau))$ is sufficient for the parameter θ. These theorems made it possible to characterize efficient, in the sense of the Cramer–Rao–Wolfowitz inequality, sequential plans for some classes of stochastic processes (see [6], [9], [11]–[15], [17], and [18]).

In this paper we consider the problem of sequential estimation for random fields. We define a Markov stopping set τ (Definition 1) as a random compact set on which we observe the realization $w_s \in W$ of the random field X_s, $s \in R^2$. It is proved that the distribution m_θ of the random vector $(p(|\tau|), S(\tau))$ is absolutely continuous with respect to m_{θ_0} ($S(K)$ is a sufficient statistic defined on the set of realizations w_s, $s \in K$, where K is a compact set contained in R^2, $|\tau|$ is the Lebesgue measure of the set τ, and p is some Borel real function). The form of the density function dm_θ/dm_{θ_0} is given (Theorem 1). Moreover, it is proved (Theorem 2) that the measure μ_θ, generated by the random field X_s, is absolutely continuous with respect to μ_{θ_0} on the σ-algebra F_τ generated by the Markov stopping set τ. The statistic $(p(|\tau|), S(\tau))$ is, therefore, sufficient for the parameter θ. Although the thesis of Theorem 1 may be deduced from Theorem 2, they are formulated separately because the first theorem is proved with less restrictive assumptions on $\mathscr{K}$ and τ (without conditions (5) and (6)). Theorems 1 and 2 allow to prove the Cramer–Rao–Wolfowitz inequality. Afterwards, we consider the random field of the Ornstein–Uhlenbeck type, which is a generalization of the stochastic process examined in [11]. For this random field, using results of Stefanov [15], we have proved theorems characterizing efficient sequential plans.

The problem of efficient sequential plans for a random Wiener field and a Poisson field is reduced to the problem of efficient sequential plans for the Wiener process and the Poisson homogeneous process.

In Sections I and II we prove some auxilliary lemmas extending the well-known results from the 1-dimensional time-parameter case to the 2-dimensional case, which seems to be new in the sense of Definition 1.

I. SUDAKOV LEMMA FOR RANDOM FIELDS

Let us consider a random field X_s, $s \in R^2$. Let W be a set of realizations of a random field X_s. This random field generates the measure μ_θ defined on (W, F), where F is a σ-algebra of subsets of W, generated by the cylindrical sets. $\theta \in A \subset R^l$ is a parameter.

Let $\mathscr{K}$ be a family of compact sets K contained in R^2 satisfying the following condition:

(1) There exists a countable family of compact sets $P_i(n)$ with a diameter $\delta[P_i(n)] \to 0$ for $n \to \infty$ such that for every $K \in \mathscr{K}$ there exists a finite minimal unique covering C_n of K by the sets $P_i(n)$ for which $C_{n+1} \subset C_n$ and $\bigcap_{n=1}^{\infty} C_n = K$.

Remark 1. Condition (1) is satisfied if we take

$$P_i(n) = [x_{k_i}^n, x_{k_i+1}^n] \times [y_{l_i}^n, y_{l_i+1}^n],$$

$$(x_{k_i}^n, y_{l_i}^n) = \left(\frac{k_i}{2^n}, \frac{l_i}{2^n}\right), \quad k_i, l_i = 0, \pm 1, \pm 2, \ldots \ (i = 1, 2, \ldots).$$

By F_K we denote a σ-algebra of subsets of W generated by the cylindrical sets $\{w\colon (w(s_1), w(s_2), \ldots, w(s_n)) \in B\}$, where $B \in \mathscr{B}_{R^n}$ and $s_i \in K$ $(i = 1, 2, \ldots, n)$.

μ_θ^K is a restriction of the measure μ_θ to the σ-algebra F_K.

Assume that μ_θ^K is absolutely continuous with respect to the measure $\mu_{\theta_0}^K$ and that

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K} = g\big(p(|K|), S(K, w), \theta, \theta_0\big), \tag{2}$$

where g is a continuous function; $S(K, \cdot)$ is a mapping which is F_K-measurable and, for every nonincreasing sequence of compact sets $K_n \searrow K$, $S(K_n, w) \to S(K, w)$ μ_θ a.s. for every $\theta \in A$; p is a Borel function from R_+ into R; $|K|$ is the Lebesgue measure of the compact set K.

Definition 1 (see [4]). A *Markov stopping set* τ is the mapping $\tau\colon W \to \mathscr{K}$ such that $\{w\colon \tau(w) \subseteq K\} \in F_K$ for every $K \in \mathscr{K}$.

Remark 2. Let (Ω, F) be a measurable space. F_z, $z \in R^2$, is a family of sub-σ-algebras of F which satisfy some other conditions specified in [5] and [10]. These papers define a Markov stopping point as a mapping $Z\colon \Omega \to R_+^2 \cup \{\infty\}$ for which $\{w\colon Z(\omega) \leqslant z\} \in F_z$ ($\leqslant$ is a relation of partial ordering in R^2). If $\mathscr{K} = \{K_z,\ z \in R_+^2\} = \{[0, z],\ z \in R_+^2\}$, $F_z = F_{K_z}$ and $\tau(w) = [0, Z(w)]$, then τ is a Markov stopping set $[0, z] = \{s \in R_+^2\colon s \leqslant z\}$.

LEMMA 1. *For every Markov stopping set τ there exists a sequence of Markov stopping sets τ_n such that $\tau_{n+1}(w) \subset \tau_n(w)$ and $\tau_n(w) \searrow \tau(w)$.*

Proof. Since $\tau(w) \in \mathscr{K}$ for every $w \in W$, let $\tau_n(w) = C_n(w)$, where $C_n(w) \in \mathscr{K}$ is the unique covering $C_n \in \mathscr{K}$ of the set $\tau(w)$. Let $K \in \mathscr{K}$. We have

$$\{w\colon \tau_n(w) \subseteq K\} = \bigcup_{C_n \subseteq K} \{w\colon \tau(w) \subseteq C_n\} \in F_K, \quad \text{where } C_n \in \mathscr{K}.$$

Hence τ_n is a Markov stopping set. By (1) we infer that $\tau_{n+1}(w) \subset \tau_n(w)$ and $\tau_n(w) \searrow \tau(w)$ for every $w \in W$.

LEMMA 2. $\{w\colon \tau_n(w) = C_n^0\} \in F_{C_n^0}$ *for every* $C_n^0 \in \mathscr{K}$.

Proof. We have

$$\{w\colon \tau_n(w) = C_n^0\} = \Big(\{w\colon \tau(w) \subseteq C_n^0\} \cap \bigcap_{C_n \in \mathscr{K}} \{w\colon \tau(w) \subsetneq C_n\}\Big) \in F_{C_n^0}, \quad \text{where } C_n \subset C_n^0.$$

LEMMA 3. *The mapping $S(\tau(\cdot), \cdot)\colon W \to R^r$ is F-measurable.*

Proof. Since

$$S(\tau_n(w), w) = \sum_{C_n \in \mathscr{K}} S(C_n, w)\, 1_{\{w : \tau_n(w) = C_n\}}(w),$$

the mapping $S(\tau_n(\cdot), \cdot)$ is F-measurable. By (2) and Lemma 1 we have

$$\lim_{n\to\infty} S(\tau_n(w), w) = S(\tau(w), w)\,\mu_\theta \text{ a.s.} \quad \text{for every } \theta \in A.$$

Remark 3. The mapping $|\tau|\colon W \to R_+$ is F-measurable.

For any Markov stopping set τ we define the mapping $f = (p(|\tau|), S(\tau, \cdot))$ and the measure m_θ on $(R \times R^r, \mathscr{B}_{R\times R^r})$ by the formula

$$m_\theta(B) = \mu_\theta\Big(\{w\colon \big(p(|\tau(w)|), S(\tau(w), w)\big) \in B\Big) \tag{3}$$

for every $B \in \mathscr{B}_U$ and $u = (p(u), s(u)) \in U = R \times R^r$.

THEOREM 1. *The measure m_θ is absolutely continuous with respect to the measure m_{θ_0} and*

$$\frac{dm_\theta}{dm_{\theta_0}}(u) = g\big(p(u), s(u), \theta, \theta_0\big). \tag{4}$$

Proof. Let p_z^n, $z = 1, 2, \ldots$, denote the values of the function $p(|\tau_n|)$, and

$$Y_{C_n, p_z^n} = \{w\colon \tau_n(w) = C_n \wedge p(|\tau_n(w)|) = p_z^n\}.$$

We define the measure $m_{\theta,z}^n$ by

$$m_{\theta,z}^n(B) = m_\theta^n\big(B \cap \{(p(u), s(u))\colon p(u) = p_z^n\}\big) \quad \text{for every } B \in \mathscr{B}_U,$$

where m_θ^n is the probability measure defined by (3) for the Markov stopping set τ_n.

We have

$$m_{\theta,z}^n(B) = \sum_{C_n \in \mathscr{K}} \mu_\theta\big(\{w\colon (p_z^n, S(C_n, w)) \in B\} \cap Y_{C_n, p_z^n}\big).$$

Introducing the function $S_n\colon W \to R \times R^r$ such that, for every $w \in W$, $S_n(w) = (p_z^n, S(C_n, w))$, we get

$$\begin{aligned} m_{\theta,z}^n(B) &= \sum_{C_n \in \mathscr{K}} \mu_\theta(S_n^{-1}(B) \cap Y_{C_n, p_z^n}) = \sum_{C_n \in \mathscr{K}} \int_{I_1} g(p_z^n, S(C_n, w))\,d\mu_{\theta_0}(w) \\ &= \int_{I_2} g\big(p_z^n, S(\tau_n(w), w)\big)\,d\mu_{\theta_0}(w) = \int_B g(p_z^n, s, \theta, \theta_0)\,dm_{\theta_0,z}^n, \end{aligned}$$

where

$$I_1 = S_n^{-1}(B) \cap Y_{C_n, p_z^n}, \qquad I_2 = S_n^{-1}(B) \cap \{w\colon p(|\tau_n(w)|) = p_z^n\}.$$

We have thus proved that $m_{\theta,z}^n \ll m_{\theta_0,z}^n$. Hence

$$\begin{aligned} m_\theta^n(B) &= \sum_z m_\theta^n(B \cap \{(p, s)\colon p = p_z^n\}) = \sum_z m_{\theta,z}^n(B) \\ &= \sum_z \int_B g(p_z^n, s, \theta, \theta_0)\,dm_{\theta_0,z}^n = \int_B g(p, s, \theta, \theta_0)\,dm_{\theta_0}^n. \end{aligned}$$

The theorem is proved for a Markov stopping set τ_n.

By (2) we have

$$\lim_{n\to\infty} S(\tau_n(w), w) = S(\tau\ (w), w)\,\mu_\theta \text{ a.s.}$$

Let h be a real function defined on the set U. Assuming h to be continuous and bounded, by the Lebesgue theorem we have

$$\lim_{n\to\infty}\int_U h(u)\,m_\theta^n(du) = \lim_{n\to\infty}\int_W h\big(p(|\tau_n(w)|),\ S(\tau_n(w), w)\big)\,d\mu_\theta(w)$$
$$= \int_W h(p|\tau(w)|),\ S(\tau(w), w))\,d\mu_\theta(w) = \int_U h(u)\,m_\theta(du).$$

Hence the sequence of measures m_θ^n is weakly converging to the measure m_θ.

Now we use the following lemma (see [8]):

Let m_n be a sequence of probability measures, weakly converging to the measure m, and let $g(p, s)$ be a continuous nonnegative function. Then the sequence of measure m_n', having densities $g(p, s)$ with respect to the measure m_n, is weakly converging to the measure m' with the density $g(p, s)$ with respect to the measure m.

By this lemma the proof of Theorem 1 is completed.

II. ABSOLUTE CONTINUITY OF THE MEASURE μ_θ ON THE σ-ALGEBRA F_τ

We assume that for every K_1 and K_2 belonging to $\mathscr{K}$, $K_1 \subset K_2$, there exist an n and a finite minimal covering $C_n \in \mathscr{K}$ of K_1 by the sets $P_i(n)$ such that

$$K_1 \subseteq C_n \subset K_2 \tag{5}$$

and

$$\{w\colon \tau(w) = K\} \in F_K \quad \text{for every } K \in \mathscr{K}. \tag{6}$$

LEMMA 4. *If the family $\mathscr{K}$ satisfies* (5), *then τ is a Markov stopping set if and only if $\{w\colon \tau(w) \subset K\} \in F_K$ for every $K \in \mathscr{K}$.*

Proof. Let τ be a Markov stopping set. By (5) we have

$$\{w\colon \tau(w) \subset K\} = \bigcup_n \bigcup_{C_n \in \mathscr{K}} \{w\colon \tau(w) \subseteq C_n\},$$

where $C_n \subset K$, $\{w\colon \tau(w) \subseteq C_n\} \in F_{C_n} \subset F_K$.

So $\{w\colon \tau(w) \subset K\} \in F_K$ for every $K \in \mathscr{K}$.

Now let $\{w\colon \tau(w) \subset K\} \in F_K$ for every $K \in \mathscr{K}$. Then $\{w\colon \tau(w) \subset C_n\} \in F_{C_n}$ for every $C_n \in \mathscr{K}$ and $\{w\colon \tau(w) \subseteq K\} = \bigcap_n \{w\colon \tau(w) \subset C_n\}$.

Since $F_{C_n} \searrow F_K$, we have $\bigcap_n \{w\colon \tau(w) \subset C_n\} \in F_K$.

Definition 2. Let τ be a Markov stopping set. By F_τ we denote a σ-algebra of the sets $V \in F$ such that $V \cap \{w\colon \tau(w) \subseteq K\} \in F_K$ for every $K \in \mathscr{K}$.

Lemma 5. *If the family $\mathscr{K}$ satisfies* (5), *then $V \in F_\tau$ if and only if $(\{w\colon \tau(w) \subset K\} \cap V) \in F_K$ for every $K \in \mathscr{K}$.*

Proof. Let $V \in F_\tau$. We have $V \cap \{w\colon \tau(w) \subseteq K\} \in F_K$ for every $K \in \mathscr{K}$. By (5),

$$V \cap \{w\colon \tau(w) \subset K\} = V \cap \Big(\bigcup_n \bigcup_{C_n \in \mathscr{K}} (\{w\colon \tau(w) \subseteq C_n\})\Big)$$

$$= \bigcup_n \bigcup_{C_n \in \mathscr{K}} (V \cap \{w\colon \tau(w) \subseteq C_n\}) \in F_K, \quad \text{where } C_n \subset K.$$

Now assume that $V \cap \{w\colon \tau(w) \subset K\} \in F_K$ for every $K \in \mathscr{K}$. We have

$$V \cap \{w\colon \tau(w) \subseteq K\} = V \cap \Big(\bigcap_n \{w\colon \tau(w) \subset C_n\}\Big) = \bigcap_n \big(V \cap \{w\colon \tau(w) \subset C_n\}\big) \in F_{C_n} \searrow F_K,$$

hence $V \cap \{w\colon \tau(w) \subseteq K\} \in F_K$ for every $K \in \mathscr{K}$.

Lemma 6. *Let τ be a Markov stopping set and τ_n be a sequence of Markov stopping sets defined as in Lemma* 1. *If the family $\mathscr{K}$ satisfies* (5), *then $F_{\tau_{n+1}} \subset F_{\tau_n}$, $F_\tau = \bigcap_n F_{\tau_n}$.*

Proof. We have:

$$\tau(w) \subseteq \tau_{n+1}(w) \subseteq \tau_n(w) \quad \text{for every } w \in W,$$

$$\{w\colon \tau_n(w) \subseteq K\} \subseteq \{w\colon \tau_{n+1}(w) \subseteq K\} \quad \text{for every } K \in \mathscr{K},$$

$$\{w\colon \tau_n(w) \subseteq K\} \subseteq \{w\colon \tau(w) \subseteq K\} \quad \text{for every } n \text{ and } K \in \mathscr{K}.$$

Let $V \in F_{\tau_{n+1}}$. We have

$$V \cap \{w\colon \tau_n(w) \subseteq K\} = (V \cap \{w\colon \tau_{n+1}(w) \subseteq K\} \cap \{w\colon \tau_n(w) \subseteq K\}) \in F_K.$$

So $V \in F_{\tau_n}$ and $F_{\tau_{n+1}} \subset F_{\tau_n}$.

Let $V \in F_\tau$, which means that, for every $K \in \mathscr{K}$,

$$V \cap \{w\colon \tau(w) \subseteq K\} \in F_K,$$

$$V \cap \{w\colon \tau_n(w) \subseteq K\} = (V \cap \{w\colon \tau(w) \subseteq K\} \cap \{w\colon \tau_n(w) \subseteq K\}) \in F_K.$$

Hence, for every n, $V \in F_{\tau_n}$ and $F_\tau \subset \bigcap_n F_{\tau_n}$.

Now let $V \in \bigcap_n F_{\tau_n}$. We have

$$V \cap \{w\colon \tau(w) \subset K\} = V \cap \Big(\bigcup_n \{w\colon \tau_n(w) \subset K\}\Big) = \bigcup_n \big(V \cap \{w\colon \tau_n(w) \subset K\}\big) \in F_K.$$

By Lemma 5 we see that $V \in F_\tau$. So $\bigcap_n F_{\tau_n} \subset F_\tau$ and $F_\tau = \bigcap_n F_{\tau_n}$.

LEMMA 7. *If τ is a Markov stopping set for which* (6) *is satisfied, then*

$$V \in F_\tau \Rightarrow (V \cap \{w\colon \tau(w) = K\}) \in F_K \quad \textit{for every } K \in \mathscr{K}.$$

Proof. We have $V \cap \{w\colon \tau(w) = K\} = (V \cap \{w\colon \tau(w) \subseteq K\} \cap \{w\colon \tau(w) = K\}) \in F_K$.

LEMMA 8. *If τ is a Markov stopping set, then* $\{w\colon \tau_n(w) = C_n\} \in F_{\tau_n}$.

Proof. Let $K \in \mathscr{K}$. We have

$$\{w\colon \tau_n(w) = C_n\} \cap \{w\colon \tau_n(w) \subseteq K\} = \{w\colon \tau_n(w) = C_n \wedge C_n \subseteq K\}$$
$$= \begin{cases} \emptyset & \text{if } C_n \not\subseteq K, \\ \{w\colon \tau_n(w) = C_n\} \in F_{C_n} & \text{if } C_n \subseteq K. \end{cases}$$

LEMMA 9. *If the family $\mathscr{K}$ and a Markov stopping set τ satisfy* (5) *and* (6), *then the mapping $S(\tau(\cdot), \cdot)\colon W \to R^r$ is F_τ-measurable.*

Proof. We will prove that $S(\tau_n(\cdot), \cdot)$ is F_{τ_n}-measurable.

Let $K \in \mathscr{K}$ and $x \in R$. We have

$$\begin{aligned} &\{w\colon S(\tau_n(w), w) < x\} \cap \{w\colon \tau_n(w) \subseteq K\} \\ &\quad = \bigcup_{C_n \in \mathscr{K}} (\{w\colon S(\tau_n(w), w) < x\} \cap \{w\colon \tau_n(w) = C_n\}) \cap \{w\colon \tau_n(w) \subseteq K\} \\ &\quad = \bigcup_{C_n \in \mathscr{K}} \{w\colon S(\tau_n(w), w) < x \wedge \tau_n(w) = C_n \wedge \tau_n(w) \subseteq K\} \\ &\quad = \bigcup_{C_n \in \mathscr{K}} \{w\colon S(\tau_n(w), w) < x \wedge \tau_n(w) = C_n\} \\ &\quad = (\bigcup_{C_n \in \mathscr{K}} \{w\colon S(C_n, w) < x\} \cap \{w\colon \tau_n(w) = C_n\}) \in F_K, \qquad C_n \subseteq K. \end{aligned}$$

Now we will show that $S(\tau(\cdot), \cdot)$ is F_τ-measurable. We can assume that the σ-algebras F_K are σ-complete σ-algebras for every $\theta \in A$ and every $K \in \mathscr{K}$. We have

$$\begin{aligned} &\{w\colon S(\tau(w), w) < x\} \cap \{w\colon \tau(w) \subseteq K\} \\ &\qquad = (\{w\colon S(K, w) < x\} \cap \{w\colon \tau(w) = K\}) \cup \\ &\qquad\qquad \cup (\{w\colon S(\tau(w), w) < x\} \cap \{w\colon \tau(w) \subset K\}). \end{aligned}$$

By (6) and F_K-measurability of the mapping $S(K, \cdot)$ we infer that

$$\{w\colon S(K, w) < x\} \cap \{w\colon \tau(w) = K\} \in F_K.$$

Since $S(\tau_n(w), w) \to S(\tau(w), w)\,\mu_\theta$ a.s. as $n \to \infty$, there exist sets C_θ and $D_\theta \subset W$ of the measure μ_θ equal to zero such that

$$\{w\colon S(\tau(w), w) < x\} \cup D_\theta = \bigcup_l \bigcup_N \bigcap_{n > N} \left\{w\colon S(\tau_n(w), w) < x - \frac{1}{l}\right\} \cup C_\theta.$$

Hence $\{w\colon S(\tau(w), w) < x\} \cap \{w\colon \tau(w) \subset K\} \in F_K$ if and only if

$$\bigcup_l \bigcup_N \bigcap_{n>N} \left\{w\colon S(\tau_n(w), w) < x - \frac{1}{l}\right\} \cap \{w\colon \tau(w) \subset K\} \in F_K.$$

But

$$\bigcup_l \bigcup_N \bigcap_{n>N} \left\{w\colon S(\tau_n(w), w) < x - \frac{1}{l}\right\} \cap \{w\colon \tau(w) \subset K\} \in F_K.$$

$$= \bigcup_l \bigcup_N \bigcap_{n>N} \left\{w\colon S(\tau_n(w), w) < x - \frac{1}{l}\right\} \cap \bigcup_N \bigcap_{n>N} \{w\colon \tau_n(w) \subset K\}$$

$$= \left(\bigcup_l \bigcup_N \bigcap_{n>N} \left\{w\colon S(\tau_n(w), w) < x - \frac{1}{l}\right\} \cap \{w\colon \tau_n(w) \subset K\}\right) \in F_K,$$

$S(\tau_n(\cdot), \cdot)$ being F_{τ_n}-measurable.

By μ_θ^τ we denote the restriction of the measure μ_θ to the σ-algebra F_τ.

THEOREM 2. *Assume that for every $K \in \mathscr{K}$ the measure μ_θ^K is absolutely continuous with respect to the measure $\mu_{\theta_0}^K$ and that*

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K}(w) = g(K, w, \theta, \theta_0), \tag{7}$$

where g is such that, for every nonincreasing sequence $K_n \searrow K$ $(K_n, K \in \mathscr{K})$,

$$\lim_{n\to\infty} g(K_n, \cdot, \theta, \theta_0) = g(K, \cdot, \theta, \theta_0)\, \mu_{\theta_0} \text{ a.s.}$$

If τ is a Markov stopping set satisfying (5) *and* (6), *then μ_θ^τ is absolutely continuous with respect to $\mu_{\theta_0}^\tau$ and*

$$\frac{d\mu_\theta^\tau}{d\mu_{\theta_0}^\tau}(w) = g(\tau, w, \theta, \theta_0). \tag{8}$$

Proof. First we prove that $\mu_\theta^{\tau_n} \ll \mu_{\theta_0}^{\tau_n}$. We have, for every $B \in F_{\tau_n}$,

$$\mu_\theta^{\tau_n}(B) = \sum_{C_n \in \mathscr{K}} \mu_\theta^{\tau_n}(B \cap \{w\colon \tau_n(w) = C_n\})$$

$$= \sum_{C_n \in \mathscr{K}} \int_{I_1} g(C_n, w, \theta, \theta_0)\, d\mu_{\theta_0} = \int_B g(\tau_n(w), w, \theta, \theta_0)\, d\mu_{\theta_0}^{\tau_n},$$

where $I_1 = B \cap \{w\colon \tau_n(w) = C_n\}$. Therefore

$$\frac{d\mu_\theta^{\tau_n}}{d\mu_{\theta_0}^{\tau_n}}(w) = g(\tau_n(w), w, \theta, \theta_0).$$

Further we use Döhler's idea [3]. Let us define the sequence (ξ_{-n}, F_{τ_n})

$= (d\mu_\theta^{\tau_n}/d\mu_{\theta_0}^{\tau_n}, F_{\tau_n})$. This sequence is a martingale. Since

$$\lim_{n\to\infty} \mathrm{E}_{\mu_{\theta_0}} \xi_{-n} = 1 > -\infty,$$

we see that $d\mu_\theta^{\tau_n}/d\mu_{\theta_0}^{\tau_n}$ is coverging in L_1 and with probability 1 (see [7]). By the assumptions on g we infer that

$$\lim_{n\to\infty} \frac{d\mu_\theta^{\tau_n}}{d\mu_{\theta_0}^{\tau_n}} = g(\tau, \cdot, \theta, \theta_0).$$

Looking at $g(\tau(\cdot), \cdot, \theta, \theta_0)$ as $S(\tau(\cdot), \cdot)$, we conclude from Lemma 9 that $g(\tau, \cdot, \theta, \theta_0)$ is F_τ-measurable, which gives the thesis.

Remark 4. The thesis of Theorem 1 may be obtained as a consequence of Theorem 2, but Theorem 1 is proved with less restrictive assumptions on $\mathscr{K}$ and τ (without conditions (5) and (6)). Therefore we have formulated both these theorems separately.

III. THE CRAMER-RAO-WOLFOWITZ INEQUALITY

In this section we assume that, for every $K \in \mathscr{K}$, $\mu_\theta^K \ll \mu_{\theta_0}^K$ and that the density function takes the form (2).

Definition 3. Let $h\colon A \to R$, $h \not\equiv$ const. An *estimator* f of the parameter $h(\theta)$ is a function $f\colon U \to R$ which is $\mathscr{B}_U$-measurable.

Remark 5. Theorem 2 implies that the random vector $(p(|\tau|), S(\tau))$ is a sufficient statistic and we can restrict ourselves to estimators f being only a function of u.

Definition 4. By a *sequential plan* we call the pair $(\tau, f(p(|\tau|), S(\tau)))$, where τ is a Markov stopping set and $f(p(|\tau|), S(\tau))$ is an estimator of the parameter $h(\theta)$.

THEOREM 3. *Let $(\tau, f(p(|\tau|), S(\tau)))$ be a sequential plan, where $f(p(|\tau|), S(\tau))$ is an unbiased estimator for the function $h(\theta)$ i.e.*

$$\mathrm{E}_{\mu_\theta} f(p(|\tau|), S(\tau)) = h(\theta) \quad \textit{and} \quad \mathrm{Var}_{\mu_\theta} f(p(|\tau|), S(\tau)) < \infty.$$

We also assume that the density function $g(u, \theta, \theta_0)$ satisfies some regularity conditions which guarantee the following equalities:

$$\int_U \nabla_\theta (g(u, \theta, \theta_0)) m_{\theta_0}(du) = 0, \tag{9}$$

$$\nabla_\theta \int_U f(u) g(u, \theta, \theta_0) m_{\theta_0}(du) = \int_U f(u) \cdot \nabla_\theta (\ln g(u, \theta, \theta_0)) m_\theta(du). \tag{10}$$

Then

$$\mathrm{Var}_\theta f(p(|\tau|), S(\tau)) \geqslant (\nabla_\theta(h)) I^{-1}(\theta) (\nabla_\theta(h))^*. \tag{11}$$

Equality in (11) *holds at some* θ *if and only if*

(12) $$f(u) = (\nabla_\theta(h)) I^{-1}(\theta) V^*(u, \theta) + h(\theta) m_{\theta_0} \quad a.s.,$$

where ∇_θ *is a gradient with respect to* θ, $V(u, \theta) = \nabla_\theta(\ln g(u, \theta, \theta_0))$ *is a vector-row*, $I(\theta) = E_\theta V^*(\cdot, \theta) V(\cdot, \theta)$, *and* $*$ *denotes the transposition of a matrix.*

The proof of this theorem, being analogous to that in [1], is omitted.

Definition 5. A sequential plan $(\tau, f(p(|\tau|), S(\tau)))$ is said to be *efficient at* θ if (12) holds at this θ.

Definition 6. A sequential plan $(\tau, f(p(|\tau|), S(\tau)))$ is efficient for $\theta \in A$ if (12) holds for all $\theta \in A$. In this case the estimator f is called *efficient*, and the function $h(\theta)$ – *efficiently estimable*.

IV. EFFICIENT SEQUENTIAL PLANS FOR RANDOM FIELDS OF THE ORNSTEIN-UHLENBECK TYPE

Let $X(t_1, t_2)$, $t_1, t_2 \geqslant 0$, be a homogeneous Gaussian random field with the mean value θ and correlation function

$$R((t_1, t_2, t_1+h_1, t_2+h_2)) = \exp(-\alpha|h_1| - \beta|h_2|).$$

The measure generated by this field is denoted by μ_θ. This measure is defined on the space (W, F), where W is the set of continuous functions w defined on R_+^2.

By K_t we shall denote the rectangle $[0, t] \times [0, t]$, $t \geqslant 0$.

The measure $\mu_\theta^{K_t}$ is absolutely continuous with respect to the measure $\mu_{\theta_0}^{K_t}$ for $\theta_0 = 0$ and in [19] it is derived that

(13) $$\frac{d\mu_\theta^{K_t}}{d\mu_0^{K_t}}(w) = \exp\left(\frac{\theta}{4} S(K_t, w) - \frac{\theta^2}{8}(\alpha t + 2)(\beta t + 2)\right),$$

where

$$S(K_t, w) = w(0, 0) + w(t, 0) + w(0, t) + w(t, t) + \alpha \int_0^t w(u, 0)\, du +$$
$$+ \alpha \int_0^t w(u, t) du + \beta \int_0^t w(0, v)\, dv + \beta \int_0^t w(t, v)\, dv + \alpha\beta \int_0^t \int_0^t w(u, v)\, du dv.$$

Further, we assume that $X(, 0, 0) = 0$ with probability 1. Under this condition the random variable $X(t_1, t_2)$ has the normal distribution

$$N(\theta(1 - \exp(-\alpha|t_1| - \beta|t_2|), 1 - \exp(-2\alpha|t_1| - 2\beta|t_2|))).$$

The considered random field generates the measure μ_θ on the space

(W, F), where W is the set of continuous functions w defined on R_+^2, for which $w(0, 0) = 0$.

It is easy to see that the measure $\mu_\theta^{K_t}$ is absolutely continuous with respect to the measure $\mu_{\theta_0}^{K_t}$ for $\theta_0 = 0$ and the density function takes the form

$$\frac{d\mu_\theta^{K_t}}{d\mu_0^{K_t}}(w) = \exp\left(\frac{\theta}{4}S(K_t, w) - \frac{\theta^2}{2}p(|K_t|)\right), \tag{14}$$

where

$$S(K_t, w) = w(t, 0) + w(0, t) + w(t, t) + \alpha\int_0^t w(u, 0)\,du + \alpha\int_0^t w(u, t)\,du +$$
$$+\beta\int_0^t w(0, u)\,du + \beta\int_0^t w(t, u)\,du + \alpha\beta\int_0^t\int_0^t w(u, v)\,du\,dv,$$
$$p(|K_t|) = \frac{\alpha+\beta}{2}t + \frac{\alpha\beta}{4}t^2.$$

Remark 6. If this random field varies along one axis only, then it is identical to the process considered in [11].

Let $\mathscr{K} = \{K_t,\ t \geqslant 0\}$. By $\mathscr{T}$ we denote the class of Markov stopping sets τ with respect to the family of σ-algebras $\{F_{K_t}\}_{t\geqslant 0}$. Let us observe that the Markov stopping set τ must not be defined by $\big(S(K_t),\ p(|K_t|)\big)$ only.

Our aim is to examine the properties of efficient sequential plans $\big(\tau, f(p(|\tau|),\ S(\tau))\big)$ for $\tau \in \mathscr{T}$.

IV.1. The Wald identities. We assume that the function $\psi\colon U \times A \to R$, m_θ-integrable for every $\theta \in A$, satisfies some regularity conditions which guarantee that

$$\frac{d}{d\theta}\int_U \psi(p(u), s(u), \theta)\,g(u, \theta)\,m_0(du) = \int_U \frac{d}{d\theta}\big(\psi(u, \theta)\,g(u, \theta)\big)\,m_0(du). \tag{15}$$

We then have

$$\mathrm{E}_\theta\psi(p(|\tau|), S(\tau), \theta)\frac{d}{d\theta}\ln g(p(|\tau|), S(\tau), \theta)$$
$$= \frac{d}{d\theta}\mathrm{E}_\theta\psi(p(|\tau|), S(\tau), \theta) - \mathrm{E}_\theta\frac{d}{d\theta}\psi(p(|\tau|), S(\tau), \theta). \tag{16}$$

If $\psi(p(|\tau|),\ S(\tau),\ \theta) = 1$, we get the first Wald identity:

$$\frac{1}{4}\mathrm{E}_\theta S(\tau) = \theta\mathrm{E}_\theta p(|\tau|). \tag{17}$$

Let

$$\psi\big(p(|\tau|),\, S(\tau),\, \theta\big) = \frac{d}{d\theta}\ln g\big(p(|\tau|),\, S(\tau),\, \theta\big).$$

Then

(18) $$\mathrm{E}_\theta\left[\frac{1}{4}S(\tau)-\theta p(|\tau|)\right]^2 = \mathrm{E}_\theta\, p(|\tau|).$$

Let $\psi\big(p(|\tau|),\, S(\tau),\, \theta\big) = S(\tau)$. Then

(19) $$\mathrm{E}_\theta\, S(\tau)\left[\frac{1}{4}S(\tau)-\theta p(|\tau|)\right] = \frac{d}{d\theta}\mathrm{E}_\theta\, S(\tau).$$

So we have

(20) $$\mathrm{Var}_\theta\, S(\tau) = 16\mathrm{E}_\theta\, p(|\tau|)+32\theta\frac{d}{d\theta}\mathrm{E}_\theta\, p(|\tau|)+16\theta^2\,\mathrm{Var}_\theta\, p(|\tau|).$$

Putting $\psi\big(p(|\tau|),\, S(\tau),\, \theta\big) = p(|\tau|)$, we obtain

(21) $$\mathrm{E}_\theta\, p(|\tau|)\left[\frac{1}{4}S(\tau)-\theta p(|\tau|)\right] = \frac{d}{d\theta}\mathrm{E}_\theta\, p(|\tau|).$$

IV.2. Properties of efficient sequential plans. By Theorem 3 and formula (18) we have

(22) $$\mathrm{Var}_\theta\, f\big(p(|\tau|),\, S(\tau)\big) \geqslant \left[\frac{d}{d\theta}h(\theta)\right]^2\Big/\mathrm{E}_0\, p(|\tau|).$$

This inequality becomes equality at some θ if and only if

(23) $$f(u) = k(\theta)\left[\frac{1}{4}s(u)-\theta p(u)\right]+h(\theta)\quad m_0 \text{ a.s.},$$

where $k(\theta)\neq 0$.

THEOREM 4. *If a sequential plan* $\big(\tau, f\big(p(|\tau|),\, S(\tau)\big)\big)$, $\tau\in\mathcal{T}$, *is efficient at* θ_1, *then there exists a constant* k_1 *such that*

(24) $$h(\theta) = k_1(\theta-\theta_1)\,\mathrm{E}_\theta\, p(|\tau|)+h(\theta_1).$$

Proof. By (23) we have

$$f(u) = k(\theta_1)\left[\frac{1}{4}s(u)-\theta_1\, p(u)\right]+h(\theta_1)\quad m_0 \text{ a.s.}$$

Taking the expected value E_θ and using (17) we obtain the thesis.

THEOREM 5. *If a sequential plan* $(\tau, f(p(|\tau|), S(\tau)), \tau \in \mathcal{T}$, *is efficient, then there are constants* a_1, a_2, *and* a_3 *such that*

$$a_1 s(u) + a_2 p(u) + a_3 = 0 \ m_0 \ a.s., \tag{25}$$

where $a_1^2 + a_2^2 \neq 0$ *and* $a_3 \neq 0$.

Proof. If a sequential plan $(\tau, f(p(|\tau|), S(\tau))$ is efficient, then it is efficient at some θ_1 and θ_2. By (23) the following equalities hold:

$$f(u) = k(\theta_1)\left[\frac{1}{4}s(u) - \theta_1 p(u)\right] + h(\theta_1),$$

$$f(u) = k(\theta_2)\left[\frac{1}{4}s(u) - \theta_2 p(u)\right] + h(\theta_2),$$

$$k(\theta_1) \neq 0, \ k(\theta_2) \neq 0.$$

Subtracting one equality from the other, we obtain

$$\frac{1}{4}(k(\theta_1) - k(\theta_2))s(u) + (\theta_2 k(\theta_2) - \theta_1 k(\theta_1))p(u) + h(\theta_2) - h(\theta_1) = 0,$$

which gives the thesis.

Let G_{K_t} denote the σ-algebra generated by the process $(S(K_v), p(|K_v|))_{v \leqslant t}$. Evidently, $G_{K_t} \subset F_{K_t}$.

By $\mathcal{G}$ we denote the class of Markov stopping sets with respect to the family of σ-algebras $\{G_{K_t}\}_{t \geqslant 0}$. Using the same arguments as in [17] and [15], we obtain the following

COROLLARY 1. *If a sequential plan* $(\tau, f(p(|\tau|), S(\tau)))$, $\tau \in \mathcal{T}$, satisfying regularity conditions (which guarantee the Wald identities), is efficient, then $\tau \in \mathcal{G}$ *and the measure* m_θ *is accumulated on the line* (25).

Let

$$t_r = \inf\{t \colon p(|K_t|) \geqslant r\} \quad \text{and} \quad (S_r, G_r, \mu_\theta)_{r \geqslant 0} = (S(K_{t_r}), G_{K_{t_r}}, \mu_\theta).$$

The process $(S_r, G_r, \mu_\theta)_{r \geqslant 0}$ belongs to the exponential class of processes considered in [15], where a full characterization of efficient sequential plans for this class of process was given (Theorem 2). Since $p(|K_t|)$ is a strictly increasing function of t, we see that the analysis of the efficient sequential plans $(\tau, f(p(|\tau|), S(\tau)))$, $\tau \in \mathcal{G}$, is equivalent to the analysis of the efficient sequential plans for the process (S_r, G_r, μ_θ). Therefore, we can use the mentioned result of Stefanov [15].

Definition 7. We say that $(\tau, f(p(|\tau|), S(\tau)))$ is a *simple plan* if $\tau(w) = K_t$ for almost all $w \in W$.

Definition 8. We say that $(\tau_0, f(p(|\tau_0|), S(\tau_0)))$ is an *oblique plan* if

$$\tau_0(w) = \min\{K_t \in \mathscr{K}:\ S(K_t, w) = \gamma p(|K_t|) + \delta\}.$$

for almost all $w \in W$ with respect to μ_0.

Definition 9. We say that $(\tau_{x_0}, f(p(|\tau_{x_0}|), S(\tau_{x_0})))$ is an *inverse plan* if

$$\tau_{x_0}(w) = \min\{K_t \in \mathscr{K}:\ S(K_t, w) = x_0\}$$

for almost all $w \in W$.

LEMMA 10. *If* $\theta = 0$ *and* $\delta\gamma < 0$, *then* $\mu_0\big(\bigcup_t \{w:\ \tau_0(w) = K_t\}\big) = 1$.

Proof. We introduce the Markov stopping time $\tau_{1,0}$:

$$\tau_{1,0}(w) = \inf\{t:\ S(K_t, w) = \gamma p(|K_t|) + \delta\} \text{ for almost all } w \in W.$$

We consider only the case $\delta > 0$ and $\gamma < 0$ (the second is analogous). We have

$$\mu_0\big(\bigcup_t \{w:\ \tau_0(w) = K_t\}\big) = \mu_0\big(\{w:\ \tau_{1,0}(w) < \infty\}\big)$$

$$\geqslant \mu_0\big(\bigcup_t \{w:\ S(K_t, w) > \gamma p(|K_t|) + \delta\}\big) \geqslant \mu_0\big(\{w:\ S(K_t, w) > \gamma p(|K_t|) + \delta\}\big).$$

Simple but uphill calculations lead to the conclusion that the random variable $S(K_t, \cdot)$ has the normal distribution with $\mathrm{E}_0 S(K_t) = 0$ and $\mathrm{Var}_0 S(K_t) = O(t^2)$. Thus

$$\mu_0\big(\{w:\ S(K_t, w) > \gamma p(|K_t|) + \delta\}\big)$$

$$= \mu_0\left(\left\{w:\ \frac{S(K_t, w)}{\sqrt{\mathrm{Var}_0 S(K_t)}} > \frac{\gamma[\frac{1}{2}(\alpha+\beta)t + \frac{1}{4}\alpha\beta t^2] + \delta}{\sqrt{\mathrm{Var}_0 S(K_t)}}\right\}\right)$$

$$= 1 - \Phi\left(\frac{\gamma[\frac{1}{2}(\alpha+\beta)t + \frac{1}{4}\alpha\beta t^2] + \delta}{\sqrt{O(t^2)}}\right),$$

$$1 \geqslant \mu_0\big(\bigcup_t \{w:\ \tau_0(w) = K_t\}\big) \geqslant \mu_0\big(\bigcup_t \{w:\ S(K_t, w) > \gamma p(|K_t|) + \delta\}\big)$$

$$\geqslant \lim_{t\to\infty} \mu_0\big(\{w:\ S(K_t, w) > \gamma p(|K_t|) + \delta\}\big) = 1,$$

which completes the proof.

LEMMA 11. *If* $\theta = 0$, *then* $\mu_0\big(\bigcup_t \{w:\ \tau_{x_0}(w) = K_t\}\big) = 1$.

Proof. Let $\tau_{1,x_0}(w) = \inf\{t:\ S(K_t, w) = x_0\}$ and $x_0 > 0$. We choose the sequence $\gamma_n \searrow 0_+$ as $n \to \infty$. By $\tau_{1,0}^n$ we denote the Markov stopping time:

$$\tau_{1,0}^n(w) = \inf\{t:\ S(K_t, w) = -\gamma_n p(|K_t|) + x_0\}.$$

We have, for all $w \in W$, $\tau_{1,0}^n(w) \leqslant \tau_{1,0}^{n+1}(w)$ and $\lim_{n\to\infty} \tau_{1,0}^n(w) = \tau_{1,x_0}(w)$.

By Lemma 10 we have $\mu_0(\{w:\ \tau^n_{1,0}(w) < \infty\}) = 1$. Hence

$$\mu_0\big(\bigcup_t \{w:\ \tau_{x_0}(w) = K_t\}\big) = \mu_0(\{w:\ \tau_{1,x_0}(w) < \infty\})$$
$$= \mu_0\big(\bigcap_n \{w:\ \tau^n_{1,0}(w) < \infty\}\big) = \lim_{n\to\infty} \mu_0(\{w:\ \tau^n_{1,0}(w) < \infty\}) = 1.$$

Thus Lemma 11 is proved.

Let $\tau_{2,0}(w) = \inf\{r:\ S_r(w) = \gamma r+\delta\}$, $\tau_{2,x_0}(w) = \inf\{r:\ S_r(w) = x_0\}$.

By previous considerations and the mentioned result of Stefanov [15], we can formulate the following corollaries.

COROLLARY 2. *If a sequential plan $\big(\tau, f(p(|\tau|),\ S(\tau))\big)$, $\tau\in\mathcal{T}$, is efficient, then there exist constants a_1, a_2, b_1, and b_2 such that $h(\theta) = (a_1\theta+a_2)/(b_1\theta+b_2)$ is efficiently estimable function for this plan.*

COROLLARY 3. *If $\theta\neq 0$ and $\delta(4\theta-\gamma) > 0$, then $E_\theta|\tau_0|^2 < \infty$ and the regularity conditions, guaranteeing the Wald identities, hold.*

COROLLARY 4. *If $\theta\neq 0$ and $x_0\theta > 0$, then $E_\theta|\tau_{x_0}|^2 < \infty$ and the regularity conditions, guaranteeing the Wald identities, hold.*

COROLLARY 5. *A simple plan is efficient.*

Indeed, $f(p(|\tau|),\ S(\tau)) = aS(K_t)+b$, $h(\theta) = 4a\theta p(|K_t|)+b$, $\mathrm{Var}_\theta f(p(|K_t|),\ S(K_t)) = 16a^2 p(|K_t|)$.

COROLLARY 6. *An oblique plan is efficient.*

Indeed, $f(p(|\tau_0|), S(\tau_0)) = ap(|\tau_0|)+b$, $h(\theta) = a\delta/(4\theta-\gamma)$, $\mathrm{Var}_\alpha f(p(|\tau|), S(\tau_0)) = 16a^2\delta/(4\theta-\gamma)^3$.

COROLLARY 7. *An inverse plan is efficient.*

Indeed, $f(p(|\tau_{x_0}|), S(\tau_{x_0})) = ap(|\tau_{x_0}|)+b$, $h(\theta) = ax_0/4\theta+b$, $\mathrm{Var}_\theta f(p(|\tau_{x_0}|), S(\tau_{x_0})) = a^2x_0/4\theta^3$.

IV.3. Sequential plans efficient at a given value θ_1. By the previous theorems, the sequential plan $\big(\tau, f(p(|\tau|), S(\tau))\big)$ is efficient at a given value θ_1 if and only if

$$f\big(p(|\tau|), S(\tau)\big) = k(\theta_1)\left[\frac{1}{4}S(\tau)-\theta_1 p(|\tau|)\right]+h(\theta_1), \qquad \tau\in\mathcal{T},$$

and if the function $h(\theta)$ is efficiently estimable at θ_1, then $h(\theta) = k_1(\theta-\theta_1)E_\theta p(|\tau|)+h(\theta_1)$.

Consider the class of all sequential plans, efficient at θ_1, for the function $h(\theta)$ for which $E_\theta p(|\tau|)$ is the same. We denote this class by $\mathcal{E}_0$. So, if the sequential plan $\big(\tau, f(p(|\tau|), S(\tau))\big)$ belongs to $\mathcal{E}_0$, then

$$\mathrm{Var}_\theta f(p(|\tau|), S(\tau)) = \mathrm{Var}_\theta\left[k(\theta_1)\left(\frac{1}{4}S(\tau)-\theta_1 p(|\tau|)\right)+h(\theta_1)\right]$$
$$= (k(\theta_1))^2\left[\frac{1}{16}\mathrm{Var}_\theta S(\tau)+\theta_1^2\,\mathrm{Var}_\theta p(|\tau|)-\frac{1}{2}\theta_1 E_\theta p(|\tau|)S(\tau)+\frac{1}{2}\theta_1 E_\theta p(|\tau|)E_\theta S(\tau)\right].$$

By (17)-(19) we have

$$\operatorname{Var}_\theta f\big(p(|\tau|),\, S(\tau)\big)$$
$$= \big(k(\theta_1)\big)^2 \left[\mathrm{E}_\theta\, p(|\tau|) + 2(\theta-\theta_1)\frac{d}{d\theta}\mathrm{E}_\theta\, p(|\tau|) + (\theta-\theta_1)^2 \operatorname{Var}_\theta p(|\tau|) \right]$$
$$= A + B \operatorname{Var}_\theta p(|\tau|).$$

For all plans $\big(\tau, f\big(p(|\tau|),\, S(\tau)\big)\big)$, belonging to $\mathscr{E}_0$, the constants A and B are the same. So we can say that the smaller at θ is $\operatorname{Var}_\theta p(|\tau|)$ the better is the plan $\big(\tau, f\big(p(|\tau|),\, S(\tau)\big)\big)$ belonging to $\mathscr{E}_0$, because the variance at θ of the estimator $f\big(p(|\tau|),\, S(\tau)\big)$ is then getting smaller.

V. EFFICIENT SEQUENTIAL PLANS FOR POISSON AND WIENER FIELDS

Definition 10. Let $\mathscr{B}^b_{R^2}$ be a family of bounded Borel subsets of R^2. Assume that the family $\{N(A)_{A\in\mathscr{B}^b_{R^2}}\}$ of random variables has the following properties:

1° for an arbitrary set of disjoint bounded Borel subsets $A_1, A_2, \ldots, A_n$ of R^2 the random variables $N(A_1), N(A_2), \ldots, N(A_n)$ are independent;

2° $P\big(N(A_i) = k\big) = (\lambda |A_i|)^k \exp(-\lambda |A_i|)/k!$.

$N_z = N(K_z)$ is called the *Poisson random field* if $K_z = [0, t_1] \times [0, t_2]$ for any $z = (t_1, t_2) \in R^2_+$.

Definition 11. Assume that the collection of random variables $\{W(A)_{A\in\mathscr{B}^b_{R^2}}\}$ has the following properties:

1° for an arbitrary set of disjoint bounded Borel subsets $A_1, A_2, \ldots, A_n$ of R^2 the random variables $W(A_1), W(A_2), \ldots, W(A_n)$ are independent;

2° the random variable $W(A)$ is normally distributed with $\mathrm{E}W(A) = 0$ and $\mathrm{E}W^2(A) = |A|$.

Then $W_z = W(K_z)$, $z \in R^2_+$, is called the *Wiener random field*.

Let $\mathscr{K}$ be the family of the squares $K_v \subset R^2_+$ with $|K_v| = v$. In this case $N_v = N(K_v)$ is the homogeneous Poisson process and $W_v = W(K_v)$ is the Wiener process. With the same arguments as previously, the problem of characterization of efficient sequential plans for the random Poisson field and the Wiener field can be reduced to the problem of efficient sequential plans for the homogeneous Poisson process and the Wiener process.

REFERENCES

[1] J. R. Barra, *Notions fondamentales de statistique mathématique*, Dunod, Paris 1971.

[2] Do Sun Bai, *Efficient estimation of transition probabilities in a Markov chain*, Ann. Math. Statist. 6.3 (1975), p. 1305–1317.

[3] R. Döhler, *Dominierbarkeit und Suffizienz in der Sequentialanalyse*, Math. Operationsforsch. Statist. 12 (1981), p. 101-134.
[4] I. V. Evstigneev, *Optional times for random fields*, Theory Probab. Appl. 22.3 (1977), p. 575-581.
[5] J. Fourqué, *The past of a stopping point and stopping for two-parameter processes*, J. Multiv. Anal. 13.4 (1983), p. 561-578.
[6] J. Franz and W. Winkler, *Sequential estimation problem for the exponential class of processes with independent increments*, Scand. J. Statist. 6 (1979), p. 129-139.
[7] I. I. Gikhman and A. W. Skorokhod, *Theory of random processes*, v. 3, Moscow 1975 (in Russian).
[8] A. M. Kagan, Yu. V. Linnik and C. R. Rao, *Characterization problems of the mathematical statistics*, Moscow 1972 (in Russian).
[9] R. Magiera, *Sequential estimation for the spectral density parameter of a stationary Gaussian process*, Probab. Math. Statist. 4.1 (1984), p. 33-45.
[10] R. Meyer, *Théorie élémentaire des processus à deux indices*, in *Processus aléatoires à deux indices*, Lecture Notes in Math. 863 (1981), p. 1-39.
[11] R. Różański, *A modification of Sudakov lemma and efficient sequential plans for the Ornstein–Uhlenbeck process*, Zastos. Matem. 17.1 (1980), p. 73-86.
[12] — *Własności efektywnych planów sekwencyjnych dla procesu urodzin i śmierci*, Matem. Stos. 17 (1981), p. 83-93.
[13] V. T. Stefanov, *Sequential estimation for compound Poisson process, II*, Serdica 8 (1982), p. 255-261.
[14] — *On efficient stopping time*, Stochast. Proc. Appl. 19.2 (1985), p. 305-315.
[15] — *On the moments of some first-passage times*, J. Appl. Probab. 22.2 (1985), p. 461-466.
[16] V. N. Sudakov, *On measures defined by Markov time*, in *Investigations on the theory of random processes*, Memoirs of Scientific Seminars of the Leningrad Section of the Steklov Math. Inst. 12 (1969), p. 157-164 (in Russian).
[17] S. Trybuła, *Sequential estimation in processes with independent increments*, Diss. Math. 60 (1968).
[18] — *Sequential estimation in finite-state Markov processes*, Zastos. Matem. 17 (1982), p. 227-248.
[19] M. I. Yadrenko, *The absolute continuity of probability measures generated by Gaussian homogeneous random fields*, Probab. Theory Math. Statist. 7 (1972) (in Russian).

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Received on 23. 10. 1985;
revised version on 24. 3. 1986