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ADMISSIBLE TRANSLATES FOR SUBGAUSSIAN MEASURES

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Abstract. Zinn [6] asks whether it is true that every stable measure with the spectral measure vanishing on finite-dimensional sets has no admissible translates. It turns out that the answer is "no". Precisely, the author shows that the distribution of $X\sqrt{\theta}$ is a measure which is stable, has non-trivial admissible translates and its spectral measure vanishes on finite-dimensional sets (X denotes a Gaussian vector and θ is a p-stable random variable concentrated on $(0, \infty)$).

Introduction. We will deal with p-stable measures on a real separable Banach space E. A symmetric measure μ is called *p-stable* $(0 < p \leqslant 2)$ if, for all independent random vectors X and Y with the distribution μ and every α, $\beta > 0$ $(\alpha^p + \beta^p = 1)$, the random vector $\alpha X + \beta Y$ has the same distribution μ. 2-stable measures are Gaussian. In the case of real line some information about stable measures is contained in [2]. The proof of the following fact (basic for our considerations) may be found in [2]:

LEMMA 1. *Let X be a symmetric q-stable random variable and let θ be a p-stable random variable, independent of X, with the Laplace transform e^{-s^p} $(s > 0,\ 0 < p < 1)$.*

Then $X\theta^{1/q}$ is a symmetric pq-stable random variable.

If μ is a symmetric p-stable measure $(0 < p < 2)$ on E, then there exists a finite symmetric Borel measure Γ on the unit sphere S_1 of E such that the characteristic functional of μ has the following form (cf. [5]):

$$\hat{\mu}(x^*) = \exp\Big(-\int_{S_1} |x^*(x)|^p \, d\Gamma(x)\Big) \quad \text{for } x^* \in E^*.$$

Γ is called the *spectral measure* of μ. To compute Γ, let $A \in \mathscr{B}(S_1)$ be such that $\Gamma(\partial A) = 0$ and let X be a random vector with the distribution μ. Then (cf. [1] and [5])

$$(1) \qquad \Gamma(A) = \lim_{t \to \infty} pt^p \, \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > t\right\}.$$

We can always assume that $\Gamma = \Gamma_f + \Gamma_\infty$, where Γ_f is concentrated on finite-dimensional sets and $\Gamma_\infty(F) = 0$ for any finite-dimensional set F. Hence $\mu = \mu_f * \mu_\infty$ (cf. [6]).

We consider a translate of measure: if $a \in E$, then $\mu_a = \mu * \delta_a$.

A translate $a \in E$ of a measure μ is called *admissible* if μ_a is absolutely continuous with respect to μ. The set of all admissible translates of a measure μ is denoted by A_μ.

It is easy to show that if $\mu_f \neq \delta_0$, then A_μ is non-trivial. Zinn [6] shows some examples of p-stable measures for which $A_\mu = (0)$ (but in these cases $\Gamma_f \equiv 0$) and asks whether A_{μ_∞} is always trivial. We show in Theorem 2 that the answer is „no".

2. Dichotomy for translates of a class of stable measures. Let X be a symmetric Gaussian random vector with values in a separable Banach space E and let θ be a p-stable random variable, independet of X, with the Laplace transform e^{-s^p} ($s > 0$, $0 < p < 1$). By Lemma 1 it is easy to see that $X\sqrt{\theta}$ is a symmetric $2p$-stable random vector. The law of $X\sqrt{\theta}$ is a particular case of the so-called "elliptically-contoured distribution" and it is easy to see that, for such vectors (cf. [3]),

$$A_X \subset A_{X\sqrt{\theta}}. \tag{2}$$

Since A_X is always non-trivial for a Gaussian vector X, we conclude that $X\sqrt{\theta}$ has always admissible translates.

Let M_μ be the intersection of all linear Borel subspaces $M \subset E$ such that $\mu(M) = 1$. If we show that $A_\mu = M_\mu$ for a symmetric p-stable measure μ, then, for every $a \in E$, either $\mu_a \perp \mu$ or $\mu_a \sim \mu$ (cf. [6], p. 248). For the Gaussian measure μ we have $A_\mu = M_\mu$ and the above-mentioned dichotomy appears [6].

Sacała [4] has shown the dichotomy theorem for the translates of elliptically contoured distributions. For the sake of completeness we formulate and prove this result, but only for p-stable measures of type $X\sqrt{\theta}$; the proof is the same in the general case[1].

Let S_μ denote a set of all singular translates of μ.

THEOREM 1. *Let X be a symmetric Gaussian random vector in E and let θ be a p-stable random variable with the Laplace transform e^{-s^p} ($s > 0$), where $0 < p < 1$.*

Then

$$A_X = A_{X\sqrt{\theta}} \quad \textit{and} \quad A_{X\sqrt{\theta}} = M_{X\sqrt{\theta}}. \tag{3}$$

Proof. For a Gaussian vector X we have $A_X = M_X$ (cf. [6] and

(1) The author is very indebted to Mr. J. Sacała for permitting to include this result.

references therein). Let M be a Borel linear subspace of E. We have

$$\mathrm{P}\{X\sqrt{\theta}\in M\} = \mathrm{P}\{X\in M\}, \tag{4}$$

hence $M_{X\sqrt{\theta}} = M_X$.

Now we show that $A_\mu \subset S_\mu^c \subset M_\mu$ for every μ. We prove only the second inclusion (the first is trivial).

If there exists a subspace M such that $\mu(M)=1$ and $x\notin M$, then $(M-x)\cap M = \emptyset$, hence $\mu(M-x)=0$ and $x\notin S_\mu$. Finally,

$$A_{X\sqrt{\theta}} \subset S_{X\sqrt{\theta}}^c \subset M_{X\sqrt{\theta}} = M_X = A_X. \tag{5}$$

Combining (2) and (5) we get (3), which completes the proof.

3. Spectral measures of p-stable vectors of the form $X\theta^{1/q}$. To answer the question of Zinn we prove the following

THEOREM 2. *Let X be a symmetric Gaussian random vector with values in $(l_2, \|\cdot\|_2)$ and with the covariance operator of the diagonal form and with all non-zero entries on the diagonal, and let θ be a p-stable, independent of X, random variable with the Laplace transform e^{-s^p} $(s>0)$, where $0<p<1$.*

Then the spectral measure Γ of the distribution of $X\sqrt{\theta}$ vanishes on finite-dimensional sets.

To prove this we need a lemma and several corollaries which may be interesting in their own.

LEMMA 2. *Let X be a symmetric q-stable random vector with values in a separable Banach space $(E, \|\cdot\|)$ and let θ be an independent of X p-stable random variable with the Laplace transform e^{-s^p} $(s>0,\ 0<p<1)$. Let Γ be a spectral measure of the distribution of $X\theta^{1/q}$. Write*

$$c = \lim_{t\to\infty} t^p\,\mathrm{P}\{\theta > t\}$$

and, for $A\in\mathscr{B}(S_1)$, let $C(A) = \{x\in E\setminus\{0\}:\ x/\|x\|\in A\}$.

Then, for every $A\in\mathscr{B}(S_1)$ such that $\Gamma(\partial A)=0$, we have

$$\Gamma(A) = cpq\,\mathrm{E}\big(\mathbf{1}_{C(A)}(X)\,\|X\|^{pq}\big). \tag{6}$$

Proof. Let $p(y)$ denote the density of θ. Then (cf. [2]) $\lim t^{p+1}p(t) = cp$ as $t\to\infty$. Hence, for every $\varepsilon>0$, there exists an $M>0$ such that

$$\frac{cp-\varepsilon}{t^{1+p}} \leqslant p(t) \leqslant \frac{cp+\varepsilon}{t^{1+p}} \quad \text{for all } t>M. \tag{7}$$

Let $A\in\mathscr{B}(S_1)$ be such that $\Gamma(\partial A)=0$. By formula (1) we have

$$\Gamma(A) = \lim_{t\to\infty} pqt^{pq}\,\mathrm{P}\left\{\frac{X\theta^{1/q}}{\|X\theta^{1/q}\|}\in A,\ \|X\theta^{1/q}\|>t\right\}$$

$$= \lim_{t\to\infty} pqt^{pq} \int_0^\infty \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} p(y)\,dy$$

$$= \lim_{t\to\infty} pqt^{pq} \int_{D_t^1 \cup D_t^2} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} p(y)\,dy,$$

where $D_t^1 = (0, (t/\ln t)^q)$ and $D_t^2 = ((t/\ln t)^q, \infty)$.

We estimate both integrals. It is well-known that there exists a constant $K > 0$ such that, for $t > 0$, $\mathrm{P}\{\|X\| > t\} \leqslant Kt^{-q}$, whence

$$t^{pq} \int_{D_t^1} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} p(y)\,dy$$

$$\leqslant t^{pq} \int_{D_t^1} \mathrm{P}\{\|X\| > ty^{-1/q}\}\, p(y)\,dy \leqslant t^{pq} \int_{D_t^1} Kyt^{-q} K_1 y^{-1-p}\,dy$$

$$\leqslant \frac{KK_1}{1-p} (\ln t)^{pq-q} \to 0 \quad \text{if } t \to \infty,$$

as, by virtue of (7), $p(y) \leqslant K_1 y^{-1-p}$ for a constant K_1. Now, by (7), we get

$$\limsup_{t\to\infty} t^{pq} \int_{D_t^2} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} p(y)\,dy$$

$$\leqslant \limsup_{t\to\infty} t^{pq} \int_{D_t^2} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} \frac{cp+\varepsilon}{y^{1+p}}\,dy$$

$$= \limsup_{t\to\infty} (cp+\varepsilon)\, qt^{pq} \int_0^{\ln t} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > u\right\} \frac{u^{pq-1}}{t^{pq}}\,du$$

$$= (cp+\varepsilon)\, q \int_0^\infty u^{pq-1}\, \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > u\right\} du.$$

In the same way we show that

$$\liminf_{t\to\infty} t^{pq} \int_{D_2^t} \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > ty^{-1/q}\right\} p(y)\,dy$$

$$\geqslant (cp-\varepsilon)\, q \int_0^\infty u^{pq-1}\, \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > u\right\} du.$$

Finally, in view of the formula $E|\xi|^\alpha = \int_0^\infty \alpha t^{\alpha-1}\, \mathrm{P}\{|\xi| > t\}\,dt$, we get

$$\Gamma(A) = c(pq)^2 \int_0^\infty u^{pq-1}\, \mathrm{P}\left\{\frac{X}{\|X\|} \in A,\ \|X\| > u\right\} du = cpq\mathrm{E}(\mathbb{1}_{C(A)}(X)\|X\|^{pq}),$$

which completes the proof.

We derive several corollaries. By m_n we denote the normalized surface measure on the unit sphere S_1 in R^n. Let X and θ be as in Lemma 2.

COROLLARY 1. *If X is not concentrated on one-dimensional subspace, then Γ has no atoms.*

Proof. Let $A_\varepsilon(x_0) = \{x \in S_1 \colon \|x - x_0\| < \varepsilon\}$ and

$$C(A_\varepsilon(x_0)) = \left\{x \in E \colon \frac{x}{\|x\|} \in A_\varepsilon(x_0)\right\}.$$

We have $\mathbf{1}_{C(A_\varepsilon(x_0))}^{(X)} \|X\|^{pq} \to 0$ a.e. if $\varepsilon \to 0$, hence, from the Lebesgue Dominated Convergence Theorem, $\Gamma(A_\varepsilon(x_0)) \to 0$ if $\varepsilon \to 0$.

In the same way we prove the following corollaries 2 and 3.

COROLLARY 2. *If E is finite-dimensional (i.e. $E = R^n$) and* $\operatorname{supp} X = E$, *then Γ is absolutely continuous with respect to m_n.*

COROLLARY 3. *If μ is a p-stable measure on R^n with the spectral measure Γ which is not absolutely continuous with respect to m_n, then μ is not the distribution of any vector of the form $X\theta^{1/q}$.*

In the case where $E = R^n$ and X is a Gaussian vector, we can get an estimate of the density of Γ with respect to m_n.

COROLLARY 4. *Let X be a symmetric Gaussian vector in R^n. The density $h(s) = d\Gamma(s)/dm_n$ of the spectral measure of $X\sqrt{\theta}$ is continuous (hence bounded).*

Proof. We may assume that the covariance matrix of X is of the diagonal form with all non-zero $\sigma_1^2, \ldots, \sigma_n^2$ on the main diagonal. Then the density f of the distribution of X can be estimated as follows:

$$f(x_1, \ldots, x_n) \leqslant (2\pi)^{-n/2} \frac{1}{\sigma_1 \ldots \sigma_n} \exp\left(-\frac{x_1^2 + \ldots + x_n^2}{2 \max_i \sigma_i^2}\right).$$

In the polar coordinates (r, s), where $r = \|x\|$ and $s = x/\|x\|$, we have

$$f(r, s) \leqslant (2\pi)^{-n/2} \frac{1}{\sigma_1 \ldots \sigma_n} \exp\left(-\frac{r^2}{2 \max_i \sigma_i^2}\right) \leqslant a_1 \exp(-a_2 r^2). \tag{8}$$

If $A \in \mathscr{B}(S_1)$, then

$$\Gamma(A) = \int_A h(s)\, dm_n(s) = 4cp^2 \int_0^\infty u^{2p-1} \iint_{uA} r^{n-1} f(r, s)\, dm_n(s)\, dr\, du$$

$$\leqslant 4cp^2 \int_A \Big[\int_0^\infty u^{2p-1} \int_u^\infty r^{n-1} a_1 \exp(-a_2 r^2)\, dr\, du\Big]\, dm_n(s),$$

hence

$$h(s) = 4cp^2 \int_0^\infty u^{2p-1} \int_u^\infty r^{n-1} f(r, s)\, dr\, du$$

and

$$h(s) \leqslant 4cp^2 a_1 \int_0^\infty u^{2p-1} \int_u^\infty r^{n-1} \exp(-a_2 r^2)\, dr\, du.$$

The density $h(s)$ is bounded and continuous on S_1 because, by virtue of (8), we can use the Lebesgue Dominated Convergence Theorem.

Recall that $A_\varepsilon(s) = \{x \in S_1\colon \|x-s\| < \varepsilon\}$ if $s \in S_1$.

COROLLARY 5. *Assume that X, θ and Γ are as in Corollary 4. Then there exists a constant $c_n > 0$ such that $\Gamma(A_\varepsilon(s)) \leqslant c_n \varepsilon^{n-1}$ for every $s \in S_1$ and $\varepsilon > 0$.*

Proof. It is easy to see that there exists a $d_n > 0$ such that $m_n(A_\varepsilon(s)) \leqslant d_n \varepsilon^{n-1}$ for $\varepsilon > 0$. Hence Corollary 4 implies the existence of a constant $c_n > 0$ such that $\Gamma(A_\varepsilon(s)) \leqslant c_n \varepsilon^{n-1}$ for $\varepsilon > 0$.

Now we can prove our Theorem 2.

Proof. Let F be any finite-dimensional Borel set in S_1. Assume the F is contained in an n-dimensional hyperplane. Then F is totally bounded and let $s^1, \ldots, s^{N(\varepsilon)}$ be an ε-net in F. The family $(A_\varepsilon(s^i))_{i=1}^{N(\varepsilon)}$ is an open (in S_1) cover of F. It is easy to see that $\operatorname{card}\{s^1, \ldots, s^{N(\varepsilon)}\} = N(\varepsilon) \sim \varepsilon^{-n}$ (because $\dim(\operatorname{span} F) \leqslant n$).

We estimate $\Gamma(A_\varepsilon(s^i))$. If $x = (x_1, x_2, \ldots) \in l_2$, then, for every $m \in N$,

$$A_\varepsilon(s^i) = \{x \in S_1\colon \|x-s^i\|_2 < \varepsilon\} \subset \{x \in S_1\colon |x_k - s_k^i| < \varepsilon \text{ for } k = 1, 2, \ldots, m\}.$$

Write $A_\varepsilon^m(s^i) = \{x \in S_1\colon |x_k - s_k^i| < \varepsilon,\ k = 1, \ldots, m\}$. Then, taking into account (1), we have

$$\begin{aligned}\Gamma(A_\varepsilon(s^i)) &\leqslant \limsup_{t\to\infty} 4cp^2 t^{2p}\, \mathrm{P}\left\{\frac{X\sqrt{\theta}}{\|X\sqrt{\theta}\|_2} \in A_\varepsilon(s^i),\ \|X\sqrt{\theta}\|_2 > t\right\} \\ &\leqslant \limsup_{t\to\infty} 4cp^2 t^{2p}\, \mathrm{P}\left\{\frac{X\sqrt{\theta}}{\|X\sqrt{\theta}\|_2} \in A_\varepsilon^{n+2}(s^i),\ \|X\sqrt{\theta}\|_2 > t\right\} \\ &\leqslant c_{n+2}\varepsilon^{n+1} \quad \text{(by Corollary 5).}\end{aligned}$$

Hence

$$\Gamma\Big(\bigcup_{i=1}^{N(\varepsilon)} A_\varepsilon(s^i)\Big) \sim c_{n+2}\varepsilon^{n+1}\varepsilon^{-n} \to 0 \quad \text{if } \varepsilon \to 0,$$

which completes the proof of Theorem 2.

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