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REMARKS ON LÉVY MEASURES AND DOMAINS OF ATTRACTION IN BANACH SPACES

BY

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Abstract. A supplementary characterization of Banach spaces in terms of conditions on the tail behavior of Lévy measures is given. A criterion for attraction to a stable law in the operator setting is proved as well. In the case of the Banach space $C(T)$ some consequences are derived.

1. Introduction. It is well known that every infinitely divisible probability measure (p.m.) μ on a real Banach space $\boldsymbol{B}$ admits (up to translation by point measures) a unique decomposition into a convolution product $\mu = \varrho * e_s(F)$ of a Gaussian measure ϱ and a generalized Poisson measure $e_s(F)$. The last measure is characterized by its associated Lévy measure (L.m.) F. In the case of a general Banach space it is known (see [4], [10], [34], [15], [21]) that the integrability condition

$$\int \min(1, \|x\|^p) F(dx) < \infty \tag{1.1}$$

is sufficient (necessary) for the σ-finite Radon measure (R.m.) F on $\boldsymbol{B}$ with $F(\{0\}) = 0$ to be an L.m. iff $\boldsymbol{B}$ is of type p, $0 < p \leqslant 2$ (cotype p, $2 \leqslant p < \infty$).

In this note we consider the case where the measure F satisfies a condition different from (1.1). For example, the relation

$$\sup_{0 < t \leqslant 1} t^p F(\{x\colon \|x\| > t\}) < \infty, \quad 0 < p < 2, \tag{1.2}$$

can hold (note that (1.1) is equivalent to

$$\int_0^1 t^{p-1} F(\{x\colon \|x\| > t\})\, dt < \infty).$$

It is worth noticing that the L.m. F of a p-stable p.m. satisfies the condition

$$F(\{x\colon \|x\| > t\}) = t^{-p} \quad \text{for all } t > 0. \tag{1.3}$$

Therefore theorem 2.11 contains the well known result on stable p.m.'s due to Mouchtari [31] and de Acosta [2]. Namely, for any finite R.m. Γ on the unit sphere of $\boldsymbol{B}$ the σ-finite R.m. $dF(t, x) = dt \cdot d\Gamma(x)/t^{1+p}$ is the L.m. of a p-stable p.m. iff $\boldsymbol{B}$ is of stable type p.

We use an inequality due to Rosinski [36] and the characterization of Banach spaces of stable type given in [32] in order to describe those Banach spaces in which condition (1.2) implies that the σ-finite R.m. is an L.m. This allows us to extend Zinn's [42] operator approach to the central limit theorem (CLT), i.e. the domain of normal attraction (DNA) of 2-stable p.m.'s, to the case of the domain of attraction (DA) of p-stable laws, $1 \leqslant p < 2$ (see theorem 4.7). Theorem 4.6 *via* the notion of prestability seems to be the correct extension of a well known result on the CLT in Banach spaces due to Hoffmann-Jørgensen and Pisier [18].

We describe the DA *via* some type of regular variation of measures, going back to Meerschaert [30] (see section 3). In theorem 3.2 it is proved that this regular variation is equivalent to the conditions known till now. This completes proposition 3.1 in Gine's paper [12], where a unified account of the theory of DA's in Banach spaces is given. It is worth noticing that all properties of the norming sequence depend on the special structure of the σ-finite measure F defined by condition (1.3).

As an immediate consequence we get also some recent results on the almost sure continuity and the DNA of p-stable continuous processes due to Marcus and Pisier [27].

Notation. Throughout the paper the following notation will be used. Let B be the unit ball in the Banach space, $\mathscr{F}(\boldsymbol{B})$ be the family of finite-dimensional subspaces of the Banach space $\boldsymbol{B}$ and $q_F(x)$ be a seminorm defined by $q_F(x) = d(x, F)$ if $F \in \mathscr{F}(\boldsymbol{B})$.

Given a Borel measure μ on $\boldsymbol{B}$, a real number α and Borel sets A and C, $C(\mu)$ denotes the class of Borel sets with boundary of μ-measure zero, measures $\alpha \circ \mu$ and $\mu|_A$ are defined by $(\alpha \circ \mu)(A) = \mu(\alpha^{-1} A)$ and $\mu|_A(C) = \mu(A \cap C)$ respectively; $|\mu| = \mu(\boldsymbol{B})$. For a function f on the Banach space $\boldsymbol{B}$ we write $\mu(f) = \int f d\mu$ and $f(\mu) = \mu(f^{-1}(\cdot))$. We write $\mu_t \xrightarrow{w} \mu$ ($\mu_t \xrightarrow{v} \mu$) if $\mu_t(f) \to \mu(f)$, as $t \to \infty$, for all real continuous and bounded functions (vanishing on some neighbourhood of the origin in addition).

By a $\boldsymbol{B}$-*random variable* X ($\boldsymbol{B}$-r.v.) we mean a Borel measurable map on some probability space $(\Omega, \mathscr{F}, P)$, provided that $\mathscr{L}(X)$ is an R.m. on $\boldsymbol{B}$. If $X, X_1, \ldots, X_n$ are i.i.d. $\boldsymbol{B}$-r.v.'s, we write $X_\delta = XI_{\{\|X\| \leqslant \delta\}}$, $X^\delta = XI_{\{\|X\| > \delta\}}$ and $S_n(X) = X_1 + \ldots + X_n$. If $\{X_{ni}: 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ is a triangular array, then

$$S_n = \sum_{1}^{k_n} X_{ni}.$$

If f and g are functions, then $f(t) \sim g(t)$ and $f(t) \asymp g(t)$, as $t \to \infty$, means that $\lim_{t\to\infty} f(t)/g(t) = 1$ and there exist constants c_1 and c_2 such that

$$c_1 \leqslant \varliminf_{t\to\infty} \frac{f(t)}{g(t)} \leqslant \varlimsup_{t\to\infty} \frac{f(t)}{g(t)} \leqslant c_2,$$

respectively.

2. Lévy measures. For any finite R.m. F on the Banach space $\boldsymbol{B}$, there exists a Radon p.m.

$$e(F) = e^{-|F|}(\delta_0 + \sum_{k\geqslant 1} F^{*k}/k!).$$

We will study the image measure $u(F)$ using operator u with $u(F)(\{0\}) = F(\ker u)$. On the other hand, $e(G) = e\big(G - G(\{0\})\delta_0\big)$ for any finite R.m.G. Hence, for the sake of uniqueness, it is worth to suppose that two measures F and G are *equivalent* if $F|_{\{0\}^c} = G|_{\{0\}^c}$ holds. By $\mathfrak{M}(\boldsymbol{B})$ we denote the set of equivalence classes of σ-finite R.m.'s on the Banach space $\boldsymbol{B}$ which are finite outside of any neighbourhood of the origin. Let us now recall the concept of the exponent of a σ-finite measure.

Definition 2.1. Let $K(x, x^*) = e^{i\langle x, x^*\rangle} - 1 - i\langle x, x^*\rangle I_B(x)$, for $x^* \in \boldsymbol{B}^*$, $x \in \boldsymbol{B}$. A symmetric measure $F \in \mathfrak{M}(\boldsymbol{B})$ is called an L.m. if the function

$$x^* \to \exp\{\textstyle\int \operatorname{Re} K(x, x^*) F(dx)\}$$

is the characteristic function (ch.f.) of a Radon p.m. (which we also denote by $e_s(F)^{\wedge}(x^*) = \exp\{\int K(x, x^*) F(dx)\}$. $\mathscr{L}\mathfrak{M}(\boldsymbol{B})$ will denote the set of all L.m.'s symmetrized measure is an L.m.. In this case associated p.m. has ch.f. $e_s(\hat{F})(x^*) = \exp\{\int K(x, x^*) F(dx)\}$. $\mathscr{L}\mathfrak{M}(\boldsymbol{B})$ will denote the set of all L.m.'s (of equivalence classes) in $\mathfrak{M}(\boldsymbol{B})$.

Next we state and prove a supplementary characterization (statement 3, in the proposition 2.2) of a symmetric L.m. For this we define the class $\mathscr{T}(\boldsymbol{B})$ of vector Hausdorff topologies $\mathscr{T}$ on $\boldsymbol{B}$ which are weaker than the norm topology, and for which exists a countable set $\Gamma \subset (\boldsymbol{B}, \mathscr{T})^*$ separating points on $\boldsymbol{B}$, and the σ-algebra of cylinders $\hat{\mathscr{C}}(\boldsymbol{B}, \Gamma)$ coincides with the Borel σ-algebra on $\boldsymbol{B}$. An example of such a topology on a separable Banach space is the weak topology.

PROPOSITION 2.2. *For a symmetric measure $F \in \mathfrak{M}(\boldsymbol{B})$ the following statements are equivalent:*

1. $F \in \mathscr{L}\mathfrak{M}(\boldsymbol{B})$;

2. *for every sequence of finite measures $F_n \uparrow F$, the sequence $\{e(F_n)\}_n$ is weakly convergent $(to\ e(F))$;*

3. *there exist a sequence of finite measures $F_n \uparrow F$ and a $\mathscr{T}$-compact set K for some $\mathscr{T} \in \mathscr{T}(\boldsymbol{B})$ such that*

$$\lim_n e(F_n)(K) > 0.$$

Proof. $1 \Rightarrow 2$. The statement follows according to Ito-Nisio theorem from the definition of L.m.'s.

$2 \Rightarrow 3$. Obvious.

$3 \Rightarrow 1$. Let $G_1 = F_1$ and $G_n = F_n - F_{n-1}$ for $n \geqslant 2$. Then $e(F_n) = e(G_1) * \ldots * e(G_n)$. By the assumptions and theorem 3.4.2 in [9] (the proof in our setting is the same), there exists a Radon p.m. ν on $\boldsymbol{B}$ such that $e(F_n) \xrightarrow{w} \nu$. Furthermore, since $F_n \uparrow F$, we get that the ch.f. $\hat{\nu}(x^*) = \exp\{\int \operatorname{Re} K(x, x^*) F(dx)\}$ for all $x^* \in \boldsymbol{B}^*$ which implies 1.

In this note we consider sets of σ-finite R.m.'s defined as follows:

$$\|F\|_{p,q} = \begin{cases} \left(\int_0^\infty [F(tB^c)]^{q/p} dt^q\right)^{1/q} & \text{for } q < \infty, \\ \left(\sup_{t>0} t^p F(tB^c)\right)^{1/p} & \text{for } q = \infty. \end{cases} \tag{2.1}$$

We write $\|X\|_{p,q}$ instead of $\|\mathscr{L}(X)\|_{p,q}$ if $F = \mathscr{L}(X)$. For $0 < p < \infty$ and $0 < q \leqslant \infty$ put

$$\mathfrak{M}_{p,q}(\boldsymbol{B}) = \{F \in \mathfrak{M}(\boldsymbol{B}) \colon \|F\|_{p,q} < \infty\}.$$

Condition (1.1) means for a σ-finite R.m. F that $F|_B \in \mathfrak{M}_{p,p}(\boldsymbol{B})$.

Now we define a class of Banach spaces in which the relation $\mathfrak{M}_{p,q}(\boldsymbol{B}) \subset \mathscr{S}\mathfrak{M}(\boldsymbol{B})$ holds (see theorem 2.11). We recall the definition of *Lorentz-Marcinkiewicz sequence spaces*

$$l_{p,q}(\boldsymbol{B}) = \{\bar{x} = (x_i)_{i \in N} \in \boldsymbol{B}^\infty \colon \|\bar{x}\|_{p,q} < \infty\},$$

where

$$\|\bar{x}\|_{p,q} = \begin{cases} \left(\frac{q}{p} \sum_{i \geqslant 1} i^{q/p-1} (\|x_i\|^*)^q\right)^{1/q} & \text{for } q < \infty, \\ \sup_{i \geqslant 1} i^{1/p} \|x_i\|^* & \text{for } q = \infty, \end{cases}$$

and an asterisk denotes a non-increasing rearrangement of the sequence.

Let $(r_i)_{i \geqslant 1}$ denote a Rademacher sequence.

Definition 2.3. Let $0 < p < 2$, $p \leqslant q \leqslant \infty$, and $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{B}$ be Banach spaces. An *operator* $u \colon \boldsymbol{E} \to \boldsymbol{F}$ *is of type* (p, q) if there exists a constant $c > 0$ such that the inequality

$$\mathrm{E}\left\|\sum r_i u(x_i)\right\| \leqslant c \|(x_i)\|_{p,q} \tag{2.2}$$

holds for any finite collection $x_1, \ldots, x_n \subset \boldsymbol{E}$. A *Banach space* $\boldsymbol{B}$ *is of type* (p, q) if the identity map on $\boldsymbol{B}$ is of type (p, q).

Remark 2.4. The notion of type (p, p) coincides with the notion of Rademacher type p. A Banach space $\boldsymbol{B}$ is of type (p, ∞) iff $\boldsymbol{B}$ is of stable type p [32]. It is easy to see that if $\boldsymbol{B}$ is of type (p, q_1) and $p \leqslant q_2 \leqslant q_1 \leqslant \infty$, then $\boldsymbol{B}$ is of type (p, q_2), but we do not know whether this inclusion is strict (except the cases $q_1 = \infty$ and $q_2 = p$). As usual, the notion of cotype (p, q) can be defined by means of the converse inequality to (2.2) and with $2 < p < \infty$, $1 \leqslant q \leqslant p$. If $\boldsymbol{B}$ is of type (p, q), then $\boldsymbol{B}^*$ is of cotype (p', q'), where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. The proof of this and some other properties of the class of type (p, q) will appear elsewhere.

The following property of the Banach spaces, introduced above, will be useful for us. For the proof see [33] (or [32] if $q = \infty$).

Proposition 2.5. *Let $1 \leqslant p < 2$, $p \leqslant q \leqslant \infty$ and $\boldsymbol{E}$, $\boldsymbol{F}$ be Banach spaces. For an operator $u\colon \boldsymbol{E} \to \boldsymbol{F}$ the following statements are equivalent:*

1. *u is of type (p, q);*
2. *there exists a constant $c > 0$ such that the inequality*

$$\|\sum u(X_i)\|_{p,q}^p \leqslant c \sum \|X_i\|_{p,q}^p$$

holds for all symmetric independent $\boldsymbol{E}$-r.v.'s $X_1, \ldots, X_n$;

3. *there exists a constant $c > 0$ such that the inequality*

$$\|S_n(u(X))\|_{p,q} \leqslant c n^{1/p} \|X\|_{p,q}$$

holds for all symmetric independent $\boldsymbol{E}$-r.v.'s $X_1, \ldots, X_n$;

The following result on integrability with respect to infinitely divisible p.m.'s on Banach spaces will serve as a tool in the proof of theorem 2.11. Proposition 2.6 is an extension of some results proved in [3], [20], [21] and [38] and has a nonvoid intersection with some of them if $p = q$. It is worth noticing that a more result is known for a class of subexponential p.m.'s (see [11]) and a more general result holds for $\boldsymbol{R}^1$-r.v.'s (see [8]).

Proposition 2.6. *Let $0 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $\mu = \delta_x * \varrho * e_s(F)$ be an infinitely divisible p.m. on the Banach space $\boldsymbol{B}$. Then $\|\mu\|_{p,q} < \infty$ iff $F|_{B^c} \in \mathfrak{M}_{p,q}(\boldsymbol{B})$.*

Lemma 2.7. *Let p, q be as above and $F, G, F_1, \ldots, F_k$ be finite R.m.'s on $\boldsymbol{B}$. Then the inequalities*

$$F^{1/p}(tB)\|G\|_{p,q} \leqslant \|F * G\|_{p,q} + t\,[F(tB)\,|G|]^{1/p}, \tag{2.3}$$

$$\|F_1 * \ldots * F_k\|_{p,q} \leqslant \Big(\sum_{i=1}^{k} \prod_{\substack{j=1\\ j\neq i}}^{k} |F_j|\Big)^{1/p} \sum_{l=1}^{k} \|F_l\|_{p,q} \tag{2.4}$$

hold for all $t > 0$ and any integer $k \geqslant 1$.

Proof. Let us rewrite (2.1) as

$$\|F\|_{p,q} = \begin{cases} \frac{q}{p}\int_0^{|F|} (s^{1/p}\,\mathcal{J}_s(F))^q\,ds/s)^{1/q} & \text{for } q<\infty, \\ \sup_{0<s\leqslant |F|} s^{1/p}\,\mathcal{J}_s(F) & \text{for } q=\infty, \end{cases} \tag{2.5}$$

where $\mathcal{J}_s(F) = \inf\{t>0\colon F(tB^c)\leqslant s\}$. From the σ-additivity it follows that $F*G(\alpha B^c)\leqslant s$. By the definition of the convolution and Fubini's theorem we have, for all $t>0$,

$$\begin{aligned} F*G(\alpha B^c) &= F\otimes G(\{(x_1, x_2)\colon \|x_1+x_2\|>\alpha\}) \\ &\geqslant \int_{\{x_2\colon \|x_2\|\leqslant t\}} G(\{x_1\colon \|x_1+x_2\|>\alpha\})\,F(dx_2) \\ &\geqslant G((\alpha+t)\,B^c)\,F(tB), \end{aligned}$$

therefore the inequality $\mathcal{J}_{s/F(tB)}(G)\leqslant \mathcal{J}_s(F*G)+t$ holds for all $s>0$. Hence, by (2.5), we get (2.3).

Let us now prove (2.4). Suppose that $\alpha_i = \mathcal{J}_s(F_i)$, $i=1,\ldots,k$. Then, by σ-additivity of measures F_i, we get

$$\bigwedge_{i=1}^{k} F_i(\alpha_i B^c)\leqslant s.$$

Therefore, from relation

$$A_k \equiv \{(x_1,\ldots,x_k)\colon \|\sum_1^k x_i\| > \sum_1^k \alpha_i\} \subset \bigcup_1^k B\times\ldots\times\alpha_i B^c\times\ldots\times B,$$

it follows the estimate

$$F_1*\ldots*F_k(\sum_1^k \alpha_i B^c) = F_1\otimes\ldots\otimes F_k(A_k)\leqslant s\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k} |F_j|.$$

Hence by (2.5) we get (2.4).

Proof of proposition 2.6. By virtue of Fernique's (p. 258 in [40]), Yurinskiĭ [41] results and lemma 2.7 it is enough to prove that $\|e(F|_{B^c}\|_{p,q} < \infty$ iff $F|_{B^c}\in\mathfrak{M}_{p,q}(B)$. The necessity of $F|_{B^c}\in\mathfrak{M}_{p,q}(B)$ follows immediately from the inequality $e(F)(A)\geqslant e^{-|F|}F(A)$, which holds for all Borel sets A. For the sufficiency we use lemma 2.7 (and Minkowski inequality for $q<\infty$). We get

$$\|e(F|_{B^c})\|_{p,q}^{p^*}\leqslant e^{-F(B^c)}\sum_{k\geqslant 1}\|(F|_{B^c})^{*k}\|_{p,q}^{p^*}/k!\leqslant c(p,F)\,\|F|_{B^c}\|_{p,q}^{p^*}$$

for some constant $c(p,F)$ and p^* equal to 1 or p according to either $p>q$ or $p\leqslant q$. This proves proposition 2.6.

If $q = \infty$, from proposition 2.6 it follows

COROLLARY 2.8 [1]. *Let $0 < p < 2$ and Γ be a p-stable p.m. on the Banach space $\boldsymbol{B}$. Then there exists a constant $c > 0$ such that $\mu(tB^c) \leqslant ct^{-p}$ for all $t > 0$.*

The following statement refines a result in [20]:

COROLLARY 2.9. *Let μ be an operator stable p.m. on a separable Banach space $\boldsymbol{B}$, i.e. a full p.m. for which there exists an operator $Q \in L(\boldsymbol{B})$ with $\lim_{t\to 0} t^Q = 0$ such that $\mu^{*t} = t^Q \circ \mu * \delta_{b_t}$ for all $t > 0$ and some $b_t \in \boldsymbol{B}$. Then there exists a constant $c > 0$ such that $\mu(tB^c) \leqslant ct^{-\|Q\|^{-1}}$ for all $t > 0$.*

Proof. Follows from the representation of the L.m. F_Q of the operator-stable p.m. μ (see [19]),

$$F_Q(A) = \int_{S_Q} \int_0^1 I_A(t^Q x) t^{-2} \, dt \, \Gamma(dx)$$

for all Borel sets A, where

$$S_Q = \{x \in \boldsymbol{B}\colon \int_0^1 \|t^Q x\| t^{-1} \, dt = 1\}.$$

The next statement concerns the completely self-decomposable p.m.'s, i.e. the infinitely divisible p.m.'s with associated L.m.'s F of the form

$$F(A) = \int_{\boldsymbol{B}} \int_0^\infty I_A(sx) s^{-2\|x\|-1} h(x) \, ds \, \Gamma(dx) \tag{2.6}$$

for all Borel sets A, some finite measure Γ on B vanishing at the origin and a weight function h such that

$$\int_{\boldsymbol{B}} \|x\|^{2\|x\|} h(x) \Gamma(dx) < \infty$$

(see [39]).

COROLLARY 2.10. *Let μ be a completely self-decomposable p.m. on the separable Banach space $\boldsymbol{B}$ with associated L. m. F (see (2.6)). If the measure Γ is concentrated on the set $\{x\colon 0 < \alpha \leqslant \|x\| < 1\}$, then there exists a constant $c > 0$ such that $\mu(tB^c) \leqslant ct^{-2\alpha}$ for all $t > 0$.*

Let us state now the main result of this section.

THEOREM 2.11. *Let $1 \leqslant p < 2$ and $p \leqslant q \leqslant \infty$. The following are equivalent for the Banach space $\boldsymbol{B}$:*

1. *$\boldsymbol{B}$ is of type (p, q);*
2. *for all $F \in \mathfrak{M}(\boldsymbol{B})$ such that $\|F|_B\|_{p,q} < \infty$, $F \in \mathscr{L}\mathfrak{M}(\boldsymbol{B})$;*
3. *if $F \in \mathfrak{M}_{p,q}(\boldsymbol{B})$, then $F \in \mathscr{L}\mathfrak{M}(\boldsymbol{B})$ and $\|e_s(F)\|_{p,q} < \infty$;*
4. *there exists a constant $c > 0$ such that $\|e(F)\|_{p,q} \leqslant c\|F\|_{p,q}$ for all symmetric $F \in \mathfrak{M}(\boldsymbol{B})$.*

Proof. $1 \Rightarrow 2$. There is no loss of generality if we assume that $F \in \mathfrak{M}(\boldsymbol{B})$ is concentrated on B and symmetric. Let us define symmetric independent $\boldsymbol{B}$-r.v.'s $(X_i)_{i \geq 1}$ with distributions $\mathscr{L}(X_i) = e(F|_{\{x:(1+i)^{-1} < \|x\| \leq i^{-1}\}})$, $i \geq 1$, and finite R.m.'s $F_{m,n} = F|_{\{x:m^{-1} < \|x\| \leq n^{-1}\}}$ for $m > n > 1$. Further, suppose that $q < \infty$. Then, by Minkowski's inequality and proposition 2.5, we get the estimate

$$\|\sum_{i=n}^{m} X_i\|_{p,q}^p$$

$$\leq e^{-|F_{m,n}|} \sum_{k \geq 1} \frac{1}{k!} |F_{m,n}|^k \left(\int_0^\infty \left(\frac{F_{m,n}^{*k}}{|F_{m,n}|^k} (tB^c) \right)^{q/p} dt^q \right)^{p/q}$$

$$\leq ce^{-|F_{m,n}|} \sum_{k \geq 1} |F_{m,n}|^{k-1} \frac{1}{(k-1)!} \left(\int_0^\infty (F_{m,n}(tB^c))^{q/p} dt^q \right)^{p/q}$$

$$= cm^{-p} F(m^{-1} B^c) + c\left(\int_{1/m}^{1/n} F^{q/p}(tB^c)\, dt^q \right)^{p/q}$$

for all $m > n > 1$ and, therefore, $\sum X_i$ $(i = 1, 2, \ldots)$ converges in probability. In particular,

$$e(F|_{n^{-1}B^c}) \xrightarrow{w} \mathscr{L}\left(\sum_{i=1}^{\infty} X_i\right). \tag{2.7}$$

Thus, by proposition 2.2, if $q < \infty$, then F is an L.m.

Assume now that $q = \infty$. Then, by proposition 2.5, for R.m.'s $F_n \equiv F_{n,1}$, $n \geq 1$, we have

$$\|e(F_n)\|_{p,\infty}^p \leq e^{-|F_n|} \sum_{k \geq 1} (k!)^{-1} \|F_n^{*k}\|_{p,\infty}^p$$

$$\leq ce^{-|F_n|} \sum_{k \geq 1} [(k-1)!]^{-1} |F_n|^{k-1} \|F_n\|_{p,\infty}^p \leq c\|F\|_{p,\infty}^p.$$

Thus, partial sums $\left(\sum_{i=1}^{n} X_i\right)_{n \geq 1}$ are stochastically bounded and hence (by Lévy's inequality) bounded a.s. In view of proposition 2.2 in [32] $\boldsymbol{B}$ is of stable type p. An appeal to results of Maurey and Pisier [29] assures us that $\boldsymbol{B}$ does not contain c_0. Another appeal to corollaty 1.7.2 in [17] implies that $\sum X_i$ $(i = 1, 2, \ldots)$ converges a.s. Hence relation (2.7) holds and F is an L.m. as well.

$2 \Rightarrow 3$. This follows from proposition 2.6.

$3 \Rightarrow 4$. Suppose that statement 4 is not true. Then, for all $n \in \boldsymbol{N}$, there exists a symmetric L.m. F_n such that $\|e(F_n)\|_{p,q} \geq n^3 \|F_n\|_{p,q}$ for all $n \in \boldsymbol{N}$.

Put $\tilde{F}_n = (n^2 \|F_n\|_{p,q})^{-1} \circ F_n$; then, by (2.5), proposition 6.1.5 in [22] and the symmetry of F_n, we get that $\|\tilde{F}_n\|_{p,q} = n^{-2}$ and $\|e(\tilde{F}_n)\|_{p,q} \geqslant n$ for all $n \in N$. Now we set $F = \sum \tilde{F}_n$ $(n = 1, 2, \ldots)$. By a simple argument (Minkowski's inequality for $q < \infty$) we infer that $F \in \mathfrak{M}_{p,q}(\boldsymbol{B})$. Therefore F is an L.m. and $\|e(F)\|_{p,q} < \infty$ by assumption. Let us define a sequence of symmetric independent $\boldsymbol{B}$-r.v.'s $(X_n)_{n \geqslant 1}$ with distributions $\mathscr{L}(X_n) = e(F_n)$ for all $n \in N$. Then, by propositions 5.4.9 and 5.4.14 in [22], $\sum X_i$ $(i = 1, 2, \ldots)$ is a $\boldsymbol{B}$-r.v. with distribution $e(F)$. An appeal to Lévy's inequality gives that

$$\sup_n \|e(F_n)\|_{p,q} \leqslant 8\,\|e(F)\|_{p,q} < \infty .$$

But this is a contradiction to $\|e(F_n)\|_{p,q} \geqslant n$, which implies statement 4.

$4 \Rightarrow 1$. Let F be the R.m. defined by

$$F = \sum_{k=1}^{n} (\delta_{x_k} + \delta_{-x_k})$$

for some $x_1, \ldots, x_n \subset \boldsymbol{B}$. Then $e(F) = \mathscr{L}\big(\sum (\xi_k - \xi_k')\, x_k\big)$, $k = 1, 2, \ldots, n$, where ξ_k and ξ_k', $k = 1, \ldots, n$, are i.i.d. with $e(\delta_1)$ $\boldsymbol{R}^1$-r.v.'s and $\|F\|_{p,q} = 2\,\|(x_k)_1^n\|_{p,q}$. Hence

$$\Big\|\sum_1^n (\xi_k - \xi_k')\, x_k\Big\|_{p,q} \leqslant 2c\,\|(x_k)_1^n\|_{p,q}$$

by assumption. According to comparison principle due to Kwapień and Rychlik (see theorem 5.4.4 in [40]) we see that the inequality

$$\Big\|\sum_1^n r_k\, x_k\Big\|_{p,q} \leqslant c e^{-4}\,\|(x_k)_1^n\|_{p,q}$$

follows immediately from estimate $\mathrm{P}(|r_1| > t) \leqslant c^2\, \mathrm{P}(|\xi_1 - \xi_1'| > t)$ for all $t > 0$. This completes the proof of theorem 2.11.

It is easy to see that theorem 2.11 holds also in the operator setting excepted the case $q = \infty$. For the last case we can prove the following

THEOREM 2.12. *Let $1 \leqslant p < 2$ and $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ be Banach spaces. If an operator $u \in L(\boldsymbol{E}, \boldsymbol{F})$ can be factorized through $\boldsymbol{B}$ by means of the type (p, ∞) operator $v \in L(\boldsymbol{E}, \boldsymbol{B})$ and the weakly compact operator $w \in L(\boldsymbol{B}, \boldsymbol{F})$, and $\boldsymbol{F}$ is separable, then, for all $F \in \mathfrak{M}(\boldsymbol{E})$ such that $\|F|_{\boldsymbol{B}}\|_{p,\infty} < \infty$, $u(F) \in \mathscr{L}\mathfrak{M}(\boldsymbol{F})$.*

If, in addition, symmetric R.m. $F \in \mathfrak{M}_{p,\infty}(\boldsymbol{E})$, then $\big\|e\big(u(F)\big)\big\|_{p,\infty} \leqslant c\,\|F\|_{p,\infty}$ for a constant $c = c(u)$.

Proof. As in the proof of implication $1 \Rightarrow 2$ in theorem 2.11, we conclude that the sequence $\big\{\sum v(X_i)(\omega)\big\}$, $i = 1, 2, \ldots, n$, $n \geqslant 1$, is bounded for a.a. ω (with the same notation). As the unit ball of $\boldsymbol{F}$ is closed in the weak topology, then (according to corollary 5.5 in [16] and the assumption) we get that $\sum u(X_i)$, $i = 1, 2, \ldots$, converges a.s. Then, by the same arguments as in preceding theorem, we derive that $u(F) \in \mathscr{L}\mathfrak{M}(\boldsymbol{F})$ and the reminder of the statement, which completes the proof.

Remark 2.13.1. If in the preceding theorem the operator $w \in L(\boldsymbol{B}, \boldsymbol{F})$ is compact, then the separability of the Banach space $\boldsymbol{F}$ can be removed.

2. It is easy to see that the factorization in theorem 2.12 is superfluous if we restrict ourselves to the subclass of R.m.'s of $\mathfrak{M}_{p,\infty}(E)$ for which $\lim t^p F(tB^c)$ as $t \to 0$ exists and is finite. In this case, only property of type (p, ∞) for the operator $u \in L(E, F)$ is needed.

Theorems 2.11 and 2.12 generalize well-known results about spectral measures of p-stable p.m.'s on Banach spaces, as mentioned in the introduction. Theorem 2.12 extends also a result on completely self-decomposable p.m.'s due to Thu [39]. We give it for the Banach space setting for simplicity.

Corollary 2.14 [39]. *Let $1 \leqslant p < 2$. Assume that the separable Banach space $\boldsymbol{B}$ is of type (p, ∞). Then, for any finite measure Γ concentrated on the set $\{x: 0 < \|x\| \leqslant p/2\}$, the σ-finite measure F defined by* (2.6) *is an L.m. of a completely self-decomposable p.m.*

3. Regular variation. The classical criterion for domains of attraction in the real line is as follows: if η is a (p, c_1, c_2)-stable $\boldsymbol{R}^1$-r.v. with $p \in (0, 2)$, $c_1, c_2 \geqslant 0$ and $c_1 + c_2 > 0$, then the $\boldsymbol{R}^1$-r.v. $\xi \in DA(\eta)$ iff

$$\frac{\mathrm{P}(\xi > x)}{\mathrm{P}(|\xi| > x)} \to c_1, \quad \frac{\mathrm{P}(\xi < -x)}{\mathrm{P}(|\xi| > x)} \to c_2,$$

as $x \to \infty$, and the function $R(x) = \mathrm{P}(|\xi| > x)$ varies regularly with index $-p$ (write $R \in \mathrm{RV}(-p)$ in the sequel), i.e. $R(x)$ is measurable on $(0, \infty)$ and

$$\lim_{t \to \infty} R(tx)/R(t) = x^{-p} \quad \text{for all } x > 0.$$

The domain of attraction of Gaussian laws is characterized by slow variation of the truncated second moment. This suggests that a more concise criterion for attraction to a stable law *via* some kind of regular variation of the law $\mathscr{L}(\xi)$ is possible. Following Meerschaert [30], we provide in this section the concept of regular variation of measures (see definition 3.1) and prove that this is equivalent to conditions which characterize the domain of attraction (see Theorem 3.2).

We define

$$\mathfrak{M}^p(\boldsymbol{B}) = \{F \in \mathfrak{M}(\boldsymbol{B}): t^{1/p} \circ F = tF, \ \forall t > 0\}, \qquad p > 0,$$

the measures $\gamma_p(dt) = t^{-1-p} dt$ on $\boldsymbol{R}^+$ and $\Gamma_F(W) = pF(x: \|x\| > 1, x/\|x\| \in W)$ on Borel sets W of the unit sphere S, for any $F \in \mathfrak{M}^p(\boldsymbol{B})$. Note that the map $i: (t, x) \to tx$ is a homeomorphism of $\boldsymbol{R}^+ \times S$ onto $\boldsymbol{B}/\{0\}$. Therefore any measure $F \in \mathfrak{M}^p(\boldsymbol{B})$ is a continuous image of the product measure $\gamma_p \otimes \Gamma_F$, i.e. $F = i(\gamma_p \otimes \Gamma_F)$. Conversely, if $F = i(\gamma_p \otimes \Gamma)$ for some finite R.m. Γ on S and $p \in \boldsymbol{R}^+$, then $F \in \mathfrak{M}^p(\boldsymbol{B})$.

Let us remark that $F_t \xrightarrow{v} F$ (see notation in the introduction) iff $F_t|_{\delta B^c} \xrightarrow{W} F|_{\delta B^c}$ for all $\delta B \in C(F)$.

Definition 3.1. Let $\alpha \in \boldsymbol{R}^1$ and $\boldsymbol{B}$ be a Banach space. We say that a measure $G \in \mathfrak{M}(\boldsymbol{B})$ *varies regularly with index* α if there exists a measure $F \in \mathfrak{M}^{\alpha}(\boldsymbol{B})$ with

$$\pi_t(G)(\cdot) - G(t\,\cdot)/G(tB^c) \xrightarrow{v} F \quad \text{as } t \to \infty. \tag{3.1}$$

We denote it by $G \in \mathrm{RV}(\boldsymbol{B}, \alpha, F)$.

For any R.m. $G \in \mathfrak{M}(\boldsymbol{B})$ put

$$\dot{a}_G(t) = \begin{cases} \sup\{s\colon tG(sB^c) \geqslant 1\} & \text{for } t > 1/G(B^c), \\ 1 & \text{for } 0 \leqslant t \leqslant 1/G(B^c). \end{cases} \tag{3.2}$$

THEOREM 3.2. *Let $p \in \boldsymbol{R}^+$ and $\boldsymbol{B}$ be a Banach space. For an R.m. $G \in \mathfrak{M}(\boldsymbol{B})$ the following are equivalent*:

1. $G \in \mathrm{RV}(\boldsymbol{B}, p, F)$ *for* $F \neq 0$;
2. *the function* $G(\cdot B^c) \in \mathrm{RV}(-p)$ *and* (3.1) *holds for some R.m. F with* $B \in C(F)$;
3. $a_G \in \mathrm{RV}(1/p)$ *and*

$$nG\big(a_G(n)\,\cdot\big) \xrightarrow{v} F \quad \textit{as } n \to \infty \tag{3.3}$$

for some R.m. F with $B \in C(F)$;

4. *there exists an R.m.* $F \in \mathfrak{M}^p(\boldsymbol{B})$ *such that* (3.3) *holds*;
5. $G(\cdot B^c) \in \mathrm{RV}(-p)$ *and*

$$\frac{G(\{x\colon \|x\| > t,\ x/\|x\| \in W\})}{G(tB^c)} \to \frac{\Gamma(W)}{\Gamma(S)} \quad \textit{as } t \to \infty$$

for some R.m. Γ on S and all $W \in C(\Gamma)$.

Remark. A characterization of DA's in Hilbert space *via* the validity of (3.1) and $G(\cdot B^c) \in \mathrm{RV}(-p)$ with additional restrictions on the support of the stable measure was given by Kłosowska [23]. Condition 5 appears for the same purpose also in [23]. Relation (3.3), which presupposes the knowledge of the norming sequence, is more or less standard. For a somewhat more complicated, but equivalent conditions see Kuelbs and Mandrekar ([24], (4.2)), Araujo and Gine ([5], 4.10 (i) (a) and (b)), Gine ([12], proposition 3.1.2).

We precede the proof of theorem 3.2 by several lemmas. The proofs of them follow along the arguments as in the proof of statement 1.4.5° and lemma 1.8 in [37].

LEMMA 3.3. *Let* $p \in \boldsymbol{R}^+$ and $R_1 \in \mathrm{RV}(-p)$. *Then there exists a function* $R_2 \in \mathrm{RV}(1/p)$ *such that*

$$R_1(R_2(x)) \sim x^{-1} \quad as\ x \to \infty, \tag{3.4}$$

$$R_2(R_1(x)) \sim x \quad as\ x \to \infty. \tag{3.5}$$

Conversely, if any function $\hat{R}_3(\cdot) \uparrow \infty$ *satisfies* (3.4) *instead of* R_2, *then* $R_3(x) \sim R_2(x)$ *as* $x \to \infty$ *and, therefore* R_3 *satisfies* (3.5). *If, in addition,* R_3 *varies regularly, then the same conclusion holds if* (3.5) *is valid for* R_3 *instead of* R_2.

LEMMA 3.4. *Let* $p \in \boldsymbol{R}^+$, *and* $R \in \mathrm{RV}(-p)$ *be a non-increasing on* $[A, \infty)$, $A > 0$. *Let, for* $x \geqslant R(A)$,

$$R_*(x) = \sup\{y \geqslant A \colon xR(y) \geqslant 1\}.$$

Then $R(R_*(x)) \sim x^{-1}$ *as* $x \to \infty$.

COROLLARY 3.5. *Let* c, $p \in \boldsymbol{R}^+$, *and the functions* R, R_* *be as in lemma* 3.4. *Then* $R(t) \sim ct^{-p}$ *iff* $R_*(t) \sim (ct)^{1/p}$ *and* $R(t) \overset{\cup}{\underset{\cap}{\sim}} t^{-p}$ *iff* $R_*(t) \overset{\cup}{\underset{\cap}{\sim}} t^{1/p}$ (*as* $t \to \infty$).

COROLLARY 3.6. *Let* $p \in \boldsymbol{R}^+$ *and* $G(\cdot B^c) \in \mathrm{RV}(-p)$. *If there exists a function* $b(\cdot) \uparrow \infty$ *which varies regularly and* $nG(b(n)B^c) \sim 1$ *as* $n \to \infty$, *then* $b(t) \sim a_G(t)$ *as* $t \to \infty$.

Proof. An appeal to the theorem of uniform convergence of slowly varying functions (theorem 1.1 in [37]) assures that $b(\lambda_n n) \sim b(n)$ as soon as $\lambda_n \sim 1$. Therefore, by the assumptions, we get that

$$\lim_{n \to \infty} b\big(1/G(b(n)B^c)\big)/b(n) = 1.$$

Put

$$n_t = \sup\{n \geqslant 1 \colon b(n) \leqslant t\} \quad \text{for all } t > 0. \tag{3.6}$$

Then

$$\frac{b\big(G^{-1}(b(n_t+1)B^c\big)}{b(n_t+1)} \leqslant \frac{b\big(G^{-1}(tB^c)\big)}{t} \leqslant \frac{b\big(G^{-1}(b(n_t)B^c)\big)}{b(n_t)} \quad \text{for all } t > 0.$$

We get that $b\big(G^{-1}(tB^c)\big) \sim t$, as $t \to \infty$, by taking limits in the above inequality. Therefore (3.5) holds, which implies that $b(t) \sim a_G(t)$ as $t \to \infty$, whenever $G(\cdot B^c) \in \mathrm{RV}(-p)$.

COROLLARY 3.7. *Let* $p \in \boldsymbol{R}^+$ *and* $G \in \mathfrak{M}(\boldsymbol{B})$. *If there exists an R.m.* $F \in \mathfrak{M}^p(\boldsymbol{B})$ *such that* $nG(b_n \cdot) \xrightarrow{v} F$, *as* $n \to \infty$, *for some sequence* $b_n \uparrow \infty$, *then* $b_n F^{1/p}(B^c) \sim a_G(n)$ *as* $n \to \infty$.

Proof. Let us define n_t by (3.6); then

$$\frac{n_t}{n_t+1}\frac{(n_t+1)\,G(b_{n_t+1}\,\lambda B^c)}{n_t\,G(b_{n_t}\,B^c)} \leqslant \frac{G(t\lambda B^c)}{G(tB^c)}$$

$$\leqslant \frac{n_t+1}{n_t}\frac{n_t\,G(b_{n_t}\lambda B^c)}{n_t+1}\,G(b_{n_t+1}\,B^c) \qquad \text{for all } t>0.$$

Taking limits in the previous inequalities and observing that $F \in \mathfrak{M}^p(B)$, we get $G(\cdot B^c) \in \mathrm{RV}(-p)$. An appeal to corollary 3.6 completes the proof.

Proof of theorem 3.2. $1 \Rightarrow 2$. We remark that $F(B^c) = 1$. Thus, from definition 3.1, it follows that $G(tsB^c)/G(tB^c) \to F(sB^c) = s^{-p}$, as $t \to \infty$, for all $s > 0$. Therefore 2 holds.

$2 \Rightarrow 3$. By the assumptions, lemma 3.4 and lemma 3.3, the function $a_G \in \mathrm{RV}(1/p)$ and $nG\big(a_G(n)\,B^c\big) \sim 1$ as $n \to \infty$. This implies

$$nG\big(a_G(n)\,\cdot\big) = nG\big(a_G(n)\,B^c\big)\big[G\big(a_G(n)\,\cdot\big)/G\big(a_G(n)\,B^c\big)\big] \xrightarrow{v} F$$

as $n \to \infty$. Therefore 3 holds.

$3 \Rightarrow 4$. We will prove that $F \in \mathfrak{M}^p(B)$. Fix $s > 0$. Then there exists a $\delta \in (0, s]$ such that $\delta B^c \in C(F)$ and

$$F(sB^c) = F|_{sB^c}(\delta B^c) = \lim_{n\to\infty} nG\big(a_G(n)\,sB^c\big). \tag{3.7}$$

By assumptions, $sa_G(n) \sim a_G(ns^p)$ as $n \to \infty$. Thus, for any fixed $\varepsilon > 0$ and all sufficiently large $n \in N$, we get

$$\begin{aligned}(3.8)\qquad &\big([s^p n]/([s^p n]+1)\big)([s^p n]+1)\,G\big((1+\varepsilon)\,a_G([s^p n]+1)\,B^c\big)\\ &\leqslant s^p nG\big(a_G(n)\,sB^c\big)\\ &\leqslant \big(([s^p n]+1)/[s^p n]\big)[s^p n]\,G\big((1-\varepsilon)\,a_G([s^p n])\,B^c\big).\end{aligned}$$

By virtue of the assumption $F(\partial B) = 0$ and by (3.7), taking limits as $n \to \infty$ and $\varepsilon \to 0$ $\big($along $(1\pm\varepsilon)\,B \in C(\Gamma)\big)$ on both sides of (3.8), we have $F(sB^c) = s^{-p}F(B^c)$ for arbitrary $s > 0$. Therefore $F \in \mathfrak{M}^p(B)$.

$4 \Rightarrow 5$. According to $F \in \mathfrak{M}^p(B)$ and (3.3) it follows that $G(\cdot B^c) \in \mathrm{RV}(-p)$, as in corollary 3.7. Now, if n_t is the largest n such that $a_G(n) \leqslant t$, then

$$\frac{G\big(\{x\colon \|x\| > a_G(n_t),\ x/\|x\| \in W\}\big)}{G\big(a_G(n_t+1)\,B^c\big)} \to \frac{F(\{x\colon \|x\| > 1,\ x/\|x\| \in W\})}{F(B^c)} = \frac{\Gamma_F(W)}{\Gamma_F(S)}$$

as $t \to \infty$. Similar arguments as above assure that 5 holds.

$5 \Rightarrow 1$. It is sufficient to show that

$$i^{-1}(F)(A) \leqslant \varliminf_{t\to\infty} G\big((ti(A)\big)/G(tB^c) \tag{3.9}$$

for some $F \in \mathfrak{M}^p(B)$ and all A open and bounded away from the origin in $R^+ \times S$, because i is a homeomorphism. By the assumptions we have

$$\frac{\gamma_p \otimes \Gamma(a, b) \times W)}{\gamma_p \otimes \Gamma((1, \infty) \times S)} = \lim_{t\to\infty} \frac{i^{-1}(t^{-1} \circ G)((a, b) \times W)}{G(tB^c)} \tag{3.10}$$

for all $0 < a < b \leqslant \infty$ and $W \in C(\Gamma)$. By lemma 1.3.2 in [40] there exists a basis $\mathcal{O}$ of the topology on S which is closed under finite intersections and with $\mathcal{O} \subset C(\Gamma)$. Therefore (3.10) holds for finite unions $\bigcup_i (a_i, b_i) \times U_i$ with $U_i \in \mathcal{O}$ as well. Now, any open set A in $R^+ \times S$ can be represented by $\bigcup_\alpha U_\alpha$, where $\{U_\alpha\}$ is an increasing net of open sets and each of them is a finite union $\bigcup_i (a_i, b_i) \times U_i$. Therefore, by τ-smoothness of $\gamma_p \otimes \Gamma$ and by (3.10) we infer that (3.9) holds for the measure $F = \tilde{F}/\tilde{F}(B^c)$, where $\tilde{F} = i(\gamma_p \otimes \Gamma)$. This completes the proof of theorem 3.2.

Remark. It is easy to see that theorem 3.2 (appropriately modificated) holds also for τ-smooth measures on completely regular topological spaces.

4. Domains of attraction. Now we will apply the results given in the previous section to the domain of attraction problem in the operator setting. The openness of the interval $\{r > 0: B \text{ is of stable type } r\}$, the key property of the stable type Banach spaces (see [29]), is usually used to get "classical" conditions. But this property does not longer hold for stable type operators (in the sense of B. Maurey). We will show that type (p, ∞) operators are useful for this purpose. This is an answer to the question posed by Paulauskas and Račkauskas in [35], where they proved, among others, that above-mentioned classes of operators do not coincide in general.

We have also proposed the notion of prestability in this section (see definition 4.5). This property plays the same role as the pregaussianness in Hoffmann-Jørgensen and Pisier theorem [18] (see theorem 4.6). But it is still an open question whether this version will be useful in connection with the stable cotype spaces.

We will start with the general case of triangular arrays of B-r.v.'s. For this we put:

$$A_{ni}^{p,q}(t, F) = \begin{cases} \left(\int_0^t \mathrm{P}^{q/p}(q_F(X_{ni}) > s)\, ds^q\right)^{p/q} / t^p\, \mathrm{P}(q_F(X_{ni}) > t) & \text{for } q < \infty, \\ \sup_{s \leqslant t} s^p\, \mathrm{P}(q_F(X_{ni})) > s) / t^p\, \mathrm{P}(q_F(X_{ni}) > t) & \text{for } q = \infty; \end{cases}$$

$$a_{p,q}^{\mathscr{F}} = a_{p,q}^{\mathscr{F}}(\{X_{ni}\}) = \sup\{A_{ni}^{p,q}(t, F): t \geqslant 1, F \in \mathscr{F}(B), i = 1, \ldots, k_n, n \geqslant 1\};$$

$$a_{p,q} = a_{p,q}(\{X_{ni}\}) = \sup\{A_{ni}^{p,q}(t, \{0\}): t \geqslant 1, i = 1, \ldots, k_n, n \geqslant 1\};$$

$$\mathscr{A}_{p,q}^{\mathscr{F}}(B) = \{B\text{-r.v.'s } (X_{ni})_{i=1}^{k_n} : a_{p,q}^{\mathscr{F}}(\{X_{ni}\}) < \infty\};$$

$$\mathscr{A}_{p,q}(B) = \{B\text{-r.v.'s } (X_{ni})_{i=1}^{k_n} : a_{p,q}(\{X_{ni}\}) < \infty\}.$$

The following construction will serve as a tool in the proof of the main results. Suppose that E and F are Banach spaces and M is a closed subspace of F; then there exists an operator $u' \in L(E/u^{-1}(M), F/M)$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad u \quad} & F \\ {\scriptstyle Q_{u^{-1}(M)}}\downarrow & & \downarrow{\scriptstyle Q_M} \\ E/u^{-1}(M) & \xrightarrow{\quad u' \quad} & F/M \end{array}$$

commutes. We will need the surjectivity property (see [33] for the proof) of the type (p, ∞) operator ideal, which means that $b_{p,\infty}(u') \leqslant b_{p,\infty}(u)$ for all closed subspaces M, where $b_{p,q}(u)$ is the infimum of real numbers c for which statement 2 in proposition 2.5 holds.

PROPOSITION 4.1. *Let $p \in [1, 2)$, $q \in [p, \infty]$, and E and F be Banach spaces. Assume that the operator $u \in L(E, F)$ is of type (p, q). If the triangular array of symmetric E-r.v.'s $\{X_{ni}\}_{i=1}^{k_n} \in \mathcal{A}^{\mathcal{F}}_{p,q}(E)$ and if the sequence of R.m.'s*

$$\{\sum_1^{k_n} \mathscr{L}(X_{ni})\}_{n \geqslant 1}$$

is uniformly tight, then the sequence $\{\mathscr{L}(u(S_n))\}_{n \geqslant 1}$ is relatively compact in the topology of weak convergence of p.m.'s on F.

Proof. At first we show that the sequence of p.m.'s $\{\mathscr{L}(u(S_n))\}_{n \geqslant 1}$ is flatly concentrated. It follows by the assumptions, for fixed $\varepsilon > 0$, that there exists a subspace $F \in \mathscr{F}(E)$ such that

$$\sup_{n \geqslant 1} \{\sum_1^{k_n} \mathscr{L}(X_{ni})|_{B^c}(\{x: q_F(x) > 1\}) \leqslant \varepsilon/(1+\varepsilon^{-p} a_{p,q} b_{p,q}(u)).$$

Let $H = u(F) \in \mathscr{F}(F)$. Then, by virtue of the relation $F \subset u^{-1}(H)$ and corollary 2.3 in [33], we get

$$\mathrm{P}(q_H(u(S_n)) > \varepsilon) \leqslant \sum_1^{k_n} \mathrm{P}(q_F(X_{ni}) > 1) + \varepsilon^{-p} \|q_H(\sum_1^{k_n} u(X_{ni} I_{(q_F(X_{ni}) \leqslant 1)}))\|_{p,q}^p$$

$$\leqslant \sum_1^{k_n} \mathscr{L}(X_{ni})|_{B^c}(\{x: q_F(x) > 1\})[1+\varepsilon^{-p} a_{p,q} b_{p,q}(u)] \leqslant \varepsilon.$$

It is sufficient to show the uniform tightness of the sequence $\{y^*(\mathscr{L}(u(S_n)))\}_{n \geqslant 1}$ for all $y^* \in F^*$. For fixed $\varepsilon > 0$ there exists an $M \geqslant 1$ such that

$$\sup_{n \geqslant 1} \sum_1^{k_n} \mathrm{P}(\|X_{ni}\| > M) \leqslant \varepsilon/2.$$

Put $A = M\|y^* \circ u\| \sqrt{2a_{p,q}/(2-p)}$. Then the estimate

$$P\{|\langle y^*, u(S_n)\rangle| > A\} \leqslant \sum_1^{k_n} P(\|X_{ni}\| > M) +$$
$$+\|y^* \circ u\| A^{-2} \sum_1^{k_n} E\|X_{ni}\|^2 I(\|X_{ni}\| \leqslant M)$$
$$\leqslant \sum_1^{k_n} P(\|X_{ni}\| > M)[1+2\|y^* \circ u\|^2 a_{p,q} M^2 A^{-2}/(2-p)] \leqslant \varepsilon$$

holds for all $n \geqslant 1$, which completes the proof.

Now we prove the existence of p.m. which will be the limit of the sequence of $\{\mathscr{L}(u(S_n))\}_{n\geqslant 1}$, as will be seen later.

PROPOSITION 4.2. *Let $p \in [1, 2)$, $q \in [p, \infty]$, and E and F be Banach spaces. Assume that the operator $u \in L(E, F)$ is of type (p, q) (u satisfies the conditions of theorem 2.12 if $q = \infty$). Suppose that the triangular array of E-r.v.'s $\{X_{ni}\}_1^{k_n}$ satisfies the condition*

$$\varlimsup_{n\to\infty} \sum_1^{k_n} \Big(\int_0^1 P^{q/p}(\|X_{ni}\| > t)\, dt^q\Big)^{p/q} < \infty$$
$$\Big(\varlimsup_{n\to\infty} \sum_1^{k_n} \sup_{s\leqslant 1} s^p P(\|X_{ni}\| > s) < \infty \qquad \text{for } q = \infty\Big).$$

If there exists an R.m. $F \in \mathfrak{M}(E)$ such that

$$\sum_1^{k_n} \mathscr{L}(X_{ni}) \xrightarrow{v} F \qquad \text{as } n \to \infty,$$

then $u(F) \in \mathscr{L}\mathfrak{M}(F)$.

Proof. By the assumptions, if $q = \infty$, we get

$$t^p F(tB^c) \leqslant \varliminf_n \sum_1^{k_n} t^p \mathscr{L}(X_{ni})(tB^c) \leqslant \varlimsup_n \sum_1^{k_n} \sup_{s\leqslant 1} s^p P(\|X_{ni}\| > s)$$

for all $t \in (0, 1]$. Since

$$\|F|_B\|_{p,\infty} = \sup_{s\leqslant 1} s^p F(sB^c) - F(B^c),$$

it follows, by theorem 2.12, that $u(F) \in \mathscr{L}\mathfrak{M}(F)$. For $q < \infty$ we get the desired result by analogous arguments and Fatou's lemma.

PROPOSITION 4.3. *Let $p \in [1, 2)$, $q \in [p, \infty]$, E and F be Banach spaces and $\{X_{ni}\}_1^{k_n} \in \mathscr{A}_{p,q}(E)$ be an infinitesimal triangular array of E-r.v.'s. If there exists* *an R. m.* $F \in \mathfrak{M}(E)$ *such that* $u(F) \in \mathscr{L}\mathfrak{M}(F)$ *and*

$$\sum_{1}^{k_n} \mathscr{L}(X_{ni}) \xrightarrow{v} F \quad \text{as } n \to \infty,$$

then the shift compactness of the sequence $\{\mathscr{L}(u(S_n))\}_{n \geq 1}$ *implies*

$$\mathscr{L}\Big(u(S_n) - \sum_{1}^{k_n} \mathrm{E}u(X_{ni})_1\Big) \xrightarrow{w} e_s(u(F)) \quad \text{as } n \to \infty. \tag{4.1}$$

Proof. By theorem 3.5.6 in [6] and from the shift compactness it follows that the sequence

$$\mathscr{L}\Big(u(S_n) - \sum_{1}^{k_n} u(X_{ni})_1\Big)_{n \geq 1}$$

is relatively compact. We remark that also

$$\sum_{1}^{k_n} \mathscr{L}(u(X_{ni})) \xrightarrow{v} u(F) \quad \text{as } n \to \infty. \tag{4.2}$$

For any $\delta_k \in (0, 1)$ we have the decomposition

$$u(S_n) - \mathrm{E}u(S_n)_1 = [u(S_n)_{\delta_k} - \mathrm{E}u(S_n)_{\delta_k}] + [u(S_n)^{\delta_k} - \mathrm{E}u(S_n)_1^{\delta_k}]. \tag{4.3}$$

An appeal to Khinchine - Le Cam's inequality (theorem 3.4.2 in [6]), the inifinitesimality of the array $\{X_{ni}\}_1^{k_n}$, condition $u(F) \in \mathscr{L}\mathfrak{M}(F)$ and relation (4.2) permit (for details see [6]) to find a subsequence (n_k) such that $\mathscr{L}\big(u(S_{n_k})^{\delta_k}\big) \xrightarrow{w} e(u(F))$ as $k \to \infty$. Also, from (4.2), it follows that

$$\sum_{1}^{k_n} \int_B x \mathscr{L}(u(X_{ni})^{\delta})(dx) \to \int_B x u(F)|_{\delta B^c}(dx) \quad \text{as } n \to \infty,$$

whenever $\delta B \in C(u(F))$. Therefore the second term on the right-hand side of (4.3) weakly converges to $e_s(u(F))$ along the subsequence (n_k). Now we will show that our limit has no non-degenerate Gaussian components. For this aim we estimate

$$|\mathrm{E}\exp\{i\langle y^*, u(S_{n_k})_{\delta_k} - \mathrm{E}u(S_{n_k})_{\delta_k}\rangle\} - 1|$$

$$\leq \mathrm{E}\,|\exp\{i\langle y^*, u(S_{n_k})_{\delta_k} - \mathrm{E}u(S_{n_k})_{\delta_k}\rangle\} - i\langle y^*, u(S_{n_k})_{\delta_k} - \mathrm{E}u(S_{n_k})_{\delta_k}\rangle - 1|$$

$$\leq \sum_{1}^{n_k} \mathrm{E}\,\langle y^*, u(X_{ni})_{\delta_k}\rangle^2$$

$$\leq \frac{2}{2-p} a_{p,q} \|y^*\|^2 \delta_k^{2-p} \sum_{1}^{n_k} \mathrm{P}(\|X_{ni}\| > \|u\|^{-1}) \to 0 \quad \text{as } k \to \infty$$

for all $y^* \in F^*$. Because $e_s(u(F))\hat{}\,(y^*) \neq 0$ for any $y^* \in F^*$, the preceding estimate and corollary 2.2.3 in [22] imply that the left-hand side of (4.3) converges weakly to $e_s(u(F))$ along the subsequence (n'_k). By uniqueness of the decomposition of infinitely divisible p.m.'s it follows that (4.1) holds, which completes the proof.

Now we are ready to prove the general statement.

THEOREM 4.4. *Let $p \in [1, 2)$, $q \in [p, \infty]$, and E and F be Banach spaces. Assume that the operator $u \in L(E, F)$ is of type (p, q) (u satisfies the conditions of theorem 2.12 if $q = \infty$). If for the infinitesimal triangular array of E-r.v.'s $\{X_{ni}\}_{i=1}^{k_n} \in \mathscr{A}_{p,q}^{\mathscr{F}}(E)$ there exists an R. m. $F \in \mathfrak{M}(E)$ with*

$$\sum_1^{k_n} \mathscr{L}(X_{ni}) \xrightarrow{v} F \qquad \text{as } n \to \infty,$$

then (4.1) *holds.*

Proof. Let $\{X'_{ni}\}$ be a triangular array which is an independent copy of $\{X_{ni}\}$. Then, by proposition 4.1, the sequence of p.m.'s $\{\mathscr{L}(u(S_n) - u(S_n))\}_{n \geq 1}$ is relatively compact. Therefore the sequence of p.m.'s $\{\mathscr{L}(u(S_n))\}_{n \geq 1}$ is shift relatively compact. By proposition 4.2, $u(F) \in \mathscr{L}\mathfrak{M}(F)$, So we are in a position to apply proposition 4.3, which completes the proof.

Let X be a B-r.v. and an R.m. $F \in \mathfrak{M}^p(B)$, $p \in (0, 2)$, such that $\mathscr{L}(X)(t\cdot)/\mathscr{L}(X)(tB^c) \xrightarrow{v} F$ as $t \to \infty$ (according to definition 3.1 on p. 133, $\mathscr{L}(X) \in \mathrm{RV}(B, p, F)$). As in the case of the central limit theorem (see [18]), there are two main questions in connection with the domain of attraction problem. The first one is whether the R.m. F is an L.m. of some p.m. ϱ on B (if the answer is positive, then ϱ is necessarily p-stable p.m.).

Definition 4.5. Let $p \in (0, 2)$ and $F \in \mathfrak{M}^p(B)$. We say that the B-r.v. X is *prestable* (with L.m. F) if $F \in \mathscr{L}\mathfrak{M}(B)$ and

$$x^*\big(\mathscr{L}(X)(t\cdot)/\mathscr{L}(X)(tB^c)\big) \xrightarrow{v} x^*(F), \qquad \text{as } t \to \infty,$$

for all $x^* \in B$. We denote it by $\mathscr{L}(X) \in \mathrm{PS}_p(B, F)$.

Now suppose that X is prestable and $\varrho = e_s(F)$ is a p-stable p.m. on B. Put $a_X(n) = a_{\mathscr{L}(X)}(n)$ (see (3.2)) and

$$Z_n(X) = a_X^{-1}(n)\Big(\sum_1^n X_i - n\mathrm{E}XI_{(\|X\| \leq a_X(n))}\Big)$$

for i.i.d. B-r.v.'s $X, X_1, \ldots, X_n$. Does then $\mathscr{L}(Z_n(X))$ tend weakly to $e_s(F)$?

We say that the B-r.v. X *is in the domain attraction* of a p-stable r.v. with L.m. F (write $X \in \mathrm{DA}_p(B, F)$) if X is prestable with L.m. F and $\mathscr{L}(Z_n(X))$ converges weakly to $e_s(F)$. If, for $X \in \mathrm{DA}_p(B, F)$, $a_x(n) \sim cn^{1/p}$ as

$n \to \infty$, then we write $X \in \mathrm{DNA}_p(\boldsymbol{B}, c^{-p} F)$, because

$$n^{-1/p}\Big(\sum_1^n X_i - n\mathrm{E}XI_{(\|X\| \leqslant n^{1/p})}\Big) \xrightarrow{w} e_s(c^{-p} F) \quad \text{as } n \to \infty.$$

If it is known that for the $\boldsymbol{B}$-r.v. X there exist an R.m. $\tilde{F} \in \mathfrak{M}^p(\boldsymbol{B}_{\frac{1}{2}})$ and a sequence $a_n \uparrow \infty$ such that $n\mathscr{L}(X/a_n) \xrightarrow{v} \tilde{F}$ as $n \to \infty$, then, by corollary 3.7, $a_X(n) \sim \tilde{F}^{1/p}(B^c)\, a_n$ as $n \to \infty$. Moreover, it can be proved that statement 5 of theorem 3.2 holds with $\Gamma = \Gamma_{\tilde{F}}$ as an implication $4 \Rightarrow 5$. From the proof of implication $5 \Rightarrow 1$ it follows that $\mathscr{L}(X) \in \mathrm{RV}\big(\boldsymbol{B}, p, \tilde{F}/\tilde{F}(B^c)\big)$. This fact can be proved directly with the help of a generalized uniform convergence theorem for regular varying measures. The proof of it we do not enclose in this paper. Now, if $X \in \mathrm{DA}\big(\boldsymbol{B}, \tilde{F}/\tilde{F}(B^c)\big)$, then some computation with the help of proposition 6.1.5 in [22] shows that

$$\mathscr{L}\Big(a_n^{-1}\Big[\sum_1^n X_i - n\mathrm{E}XI_{(\|X\| \leqslant a_n)}\Big]\Big) \xrightarrow{w} e_s(F) \quad \text{as } n \to \infty.$$

Therefore our version of the definition of domains of attraction is completely general and, on the other hand, is completely described by random variables.

The following theorem contains results due to Marcus and Woyczyński [28], Mandrekar and Zinn [25], Araujo and Gine [5], Gine [12]. We note that the proof of implication $1 \Rightarrow 2$ in [12] (theorem 3.2) contains some lack, because slow variation of $t^p\,\mathrm{P}\big(q_{F_m}(X) > t\big)$, for all $m \geqslant 1$, demands non-degeneracy of the L.m. of a p-stable p.m. The proof given below is free from this. We think that the proof of implication $4 \Rightarrow 1$ is new and more direct.

Let $\{\eta_i\}_{i \geqslant 1}$ be a sequence of i.i.d. standard p-stable $\boldsymbol{R}^1$-r.v.'s and

$$L^c_{p,\infty}(\boldsymbol{B}) = \big\{\mathscr{L}(X)\colon \lim_{t\to\infty} t^p\,\mathrm{P}(\|X\| > t) = c \in (0, \infty)\big\}.$$

THEOREM 4.6. *Let* $p \in (0, 2)$. For the Banach space $\boldsymbol{B}$ *the following are equivalent*:

1. $\boldsymbol{B}$ *is of stable type* p, *i.e. there exists a constant* $c < \infty$ *such that*

$$\Big(\mathrm{E}\,\Big\|\sum_1^n \eta_i x_i\Big\|^{\alpha}\Big)^{1/\alpha} \leqslant c\Big(\sum_1^n \|x_i\|^p\Big)^{1/p}$$

for all $x_1, \ldots, x_n \subset \boldsymbol{B}$ *and some (all)* $\alpha \in (0, p)$;

2. $\mathrm{RV}(\boldsymbol{B}, p, F) \subset \mathrm{DA}_p(\boldsymbol{B}, F)$;

3. $\mathrm{RV}(\boldsymbol{B}, p, F) \cap L^c_{p,\infty}(\boldsymbol{B}) \subset \mathrm{DNA}_p(\boldsymbol{B}, cF)$;

4. $\mathrm{RV}(\boldsymbol{B}, p, F) \cap L^c_{p,\infty}(\boldsymbol{B}) \subset \mathrm{PS}_p(\boldsymbol{B}, F)$.

Proof. $1 \Rightarrow 2$. It will be shown that the sequence $\{\mathscr{L}(S_n(X)/a_X(n)\}_{n \geqslant 1}$ is shift tight. At this point note that there is no loss of generality in assuming that X is symmetric. Further, note that by results of Maurey and Pisier [29] there exists a $p' \in (p, 2)$ such that B is of Rademacher type p'. By the assumptions the relation

$$n\mathscr{L}(X/a_X(n))(\{x: q_M(x) > 1\}) \to F|_{B^c}(\{x: q_M(x) > 1\}) \quad \text{as } n \to \infty$$

holds for any subspace $M \subset B$. By the definition of R.m.'s we can conclude that for fixed $\varepsilon > 0$ there exists an $M \in \mathscr{F}(B)$ such that

$$\sup_{t>0} t\mathrm{P}(q_M(X) > a_X(t)) \leqslant \varepsilon/(1 + 2b_{p',p'}(B)\varepsilon^{-p'} p'(p'-p)^{-1}). \tag{4.4}$$

This implies the estimate

$$\mathrm{P}(q_M(S_n(X)) > \varepsilon a_X(n)) \leqslant n\mathrm{P}(q_M(X) > a_X(n)) + \\ + \varepsilon^{-p'} \bar{b}_{p',p'}(B) n a_X^{-p'}(n) \mathrm{E}\, q_M^{p'}(X) I_{(q_M(X) \leqslant a_X(n))}. \tag{4.5}$$

By theorem 3.2, $a_X \in \mathrm{RV}(1/p)$. Hence from lemma 1.8 and statement 1.4.5° in [37] it follows that the function

$$b(t) = \inf(s > 0: a_X(s) \geqslant t) \tag{4.6}$$

belongs to $\mathrm{RV}(p)$ and $b(a_X(t)) \sim t$ as $t \to \infty$. An appeal to theorem 2.1 in [37] assures us that

$$\mathrm{E} q_F^{p'}(X) I_{(q_F(X) \leqslant a_X(n))} \tag{4.7}$$

$$\leqslant p' \sup_{t>0} b(t)\,\mathrm{P}(q_F(X) > t) \int_0^{a_X(n)} s^{p'-1} b^{-1}(s)\, ds$$

$$\sim (p'/(p'-p)) \sup_{s>0} s\mathrm{P}(q_F(X) > a(s)) a_X^{p'}(n)/n \quad \text{as } n \to \infty.$$

From (4.4), (4.5) and (4.7) it follows that $\mathrm{P}(q_F(S_n(X)) > \varepsilon a_X(n)) \leqslant \varepsilon$ for all sufficiently large $n \geqslant 1$. Taking into account the one-dimensional result, we see that the sequence of p.m.'s $\{\mathscr{L}(S_n(X)/a_X(n))\}_{n\geqslant 1}$ is shift compact. By theorem 2.1 in [37] we have $\{X_i\}_{i=1}^n \in \mathscr{A}_{p',p'}(B)$, therefore, by proposition 4.2 and 4.3 (for $p = q = p'$), $X \in \mathrm{DA}_p(B, F)$.

$2 \Rightarrow 3$. By virtue of corollary 3.5, $a_X(n) \sim c^{1/p} n^{1/p}$ as $n \to \infty$, where

$$c = \lim_{t\to\infty} t^p\, \mathrm{P}(\|X\| > t).$$

Now, in view of proposition 6.1.5 in [22] and the condition $F \in \mathfrak{M}^p(B)$, 2 implies 3 by a straightforward computation.

$3 \Rightarrow 4$. Obvious.

$4 \Rightarrow 1$. Let $(x_i)_{i\geqslant 1} \subset B$, $\sum_1^\infty \|x_i\|^p = 1$ and

$$\Gamma = 2^{-1} \sum_1^\infty \|x_i\|^p (\delta_{-x_i/\|x_i\|} + \delta_{x_i/\|x_i\|}).$$

We define a $\boldsymbol{B}$-r.v. X and a positive $\boldsymbol{R}^1$-r.v. ξ independent of X with distributions $\mathscr{L}(X) = \Gamma$ and $\mathrm{P}(\xi > t) = t^{-p}$ for all $t \geqslant 1$, respectively. Then $\xi X \in \mathrm{RV}(\boldsymbol{B}, p, i(\gamma_p \otimes \Gamma)) \cap L^1_{p,\infty}(\boldsymbol{B})$. By assumption, there exists a symmetric p.m. $e(i(\gamma_p \otimes \Gamma))$ with Fourier transform

$$e(i(\gamma_p \otimes \Gamma))^\wedge(x^*) = \exp\{-\int_S |\langle x^*, x\rangle|^p \Gamma(dx)\}$$

$$= \exp\{-\sum_1^\infty |\langle x^*, x_i\rangle|^p\} = \lim_{n\to\infty} \mathscr{L}(\sum_1^n \eta_i x_i)^\wedge(x^*)$$

and because of the Ito-Nisio theorem the sum $\sum \eta_i x_i$ $(i = 1, 2, \ldots)$ exists a.e. This is true for every choice of $(x_i)_{i\geqslant 1} \subset \boldsymbol{B}$ with $\sum \|x_i\|^p < \infty$ $(i = 1, 2, \ldots)$. Hence the Banach space $\boldsymbol{B}$ is of stable type p, which completes the proof of theorem 4.6.

Now we prove the main statement. Put

$$L^0_{p,\infty}(\boldsymbol{B}) = \{X - \boldsymbol{B}\text{-r.v.: } \lim_{t\to\infty} t^p \mathrm{P}(\|X\| > t) = 0 \text{ and}$$

$$(\mathrm{A})\,\mathrm{E}X \equiv \lim_{t\to\infty} \int X I_{(\|X\| \leqslant t)}\, dP = 0\};$$

$$\mathrm{WLLN}_p(\boldsymbol{B}) = \{X - \boldsymbol{B}\text{-r.v.: } n^{-1/p} S_n(X) \to 0 \text{ in probability}\}.$$

THEOREM 4.7. *Let $p \in [1, 2)$, and E and F be Banach spaces. For an operator $u \in L(E, F)$ the following are equivalent:*

1. *u is of type (p, ∞);*
2. *for any E-r.v. X, with*

$$C_1 \equiv \sup_{n\geqslant 1}\big[\sup_{s\leqslant a_X(n)} s^p \mathrm{P}(\|X\| > s)/a_X^p(n)\, \mathrm{P}(\|X\| > a_X(n))\big] < \infty$$

and $\mathscr{L}(X) \in \mathrm{RV}(\boldsymbol{B}, p, F)$, the relation $\mathscr{L}(u(X)) \in \mathrm{DA}_p(F, u(F)/u(F)(B^c))$ holds;

3. *$u(\mathrm{RV}(E, p, F) \cap L^c_{p,\infty}(E)) \subset \mathrm{DNA}_p(F, cu(F))$;*
4. *$u(L^0_{p,\infty}(E)) \subset \mathrm{WLLN}_p(F)$.*

Proof. $1 \Rightarrow 2$. It will be shown that

$$\sup_{n\geqslant 1}\big[\sup_{1\leqslant s\leqslant n} a_X^p(s) s^{-1}/a_X^p(n) n^{-1}\big] < \infty. \tag{4.8}$$

We define the function b by formula (4.6), which, by the same reasons, belongs to $\mathrm{RV}(p)$ and $b(a_X(t)) \sim t$ as $t \to \infty$. Put $R(t) = \mathrm{P}(\|X\| > t)$. Then, by lemma 3.3,

$$C_2 \equiv \sup_{s\geqslant 1}[sR(a(s))]^{-1} < \infty.$$

Therefore we have

$$\sup_{1\leqslant s\leqslant n} a_X^p(s)/s \leqslant C_2^p \sup_{1\leqslant u\leqslant a_X(n)} u^p R(u) \leqslant C_2^p C_1 a_X^p(n) R(a_X(n))$$

for all $n \geqslant 1$, which implies (4.8). Put

$$C_3 \equiv \sup_{t>0} t/a_X(b(t)).$$

Then, by (4.8) and because $b(a_X(n)) \leqslant n$ for all $n \geqslant 1$, the estimate

$$(4.9) \qquad \sup_{0<t\leqslant a_X(n)} t^p \mathrm{P}(q_M(X) > t) \leqslant \sup_{t>0} b(t)\, \mathrm{P}(q_M(X) > t) \sup_{t\leqslant a_X(n)} t^p/b(t)$$
$$\leqslant \sup_{t>0} b(t)\, \mathrm{P}(q_M(X) > t)\, C_3^p \sup_{1\leqslant u\leqslant n} a_X^p(u)/u$$

holds for all $n \geqslant 1$ and any closed subspace $M \subset E$. As in the proof of theorem 4.6, we conclude that for fixed $\varepsilon > 0$ there exists an $M \in \mathscr{F}(E)$ such that

$$(4.10) \qquad \begin{gathered} \sup_{t>0} b(t)\, \mathrm{P}(q_M(X) > t) \leqslant 2^{-1} c\varepsilon^{1+p} b_{p,\infty}^{-1}(u), \\ \sup_{n\geqslant 1} n\mathrm{P}(q_M(X) > a_X(n)) \leqslant \varepsilon/2 \end{gathered}$$

for some constant c. Put $G = u(M)$; then, by (4.9), (4.8) and (4.10),

$$\mathrm{P}\big(q_G(S_n(u(X))) > \varepsilon a_X(n)\big) \leqslant n\mathrm{P}(q_M(X) > a_X(n)) +$$
$$+\varepsilon^{-p} b_{p,\infty}(u)\, n a_X^{-p}(n) \sup_{t\leqslant a_X(n)} t^p \mathrm{P}(q_M(X) > t) \leqslant \varepsilon$$

for all $n \geqslant 1$ and any symmetric E-r.v. Now we conclude, as in theorem 4.6, that, as $n \to \infty$,

$$\mathscr{L}\big(a_X^{-1}(n)[S_n(u(X)) - n\mathrm{E}u(X)\, I_{(\|u(X)\|\leqslant a_X(n))}]\big) \xrightarrow{w} e_s(u(F)).$$

It easy to see that $n\mathscr{L}(u(X)/a_X(n)) \xrightarrow{v} u(F)$ as $n \to \infty$. Therefore, by corollary 3.7, $a_{u(X)}(n) \sim a_X(n)\, [u(F)(B^c)]^{1/p}$ as $n \to \infty$. Now, a straightforward computation with the help of proposition 6.1.5 in [22] assures us that 2 holds.

$2 \Rightarrow 3$. By corollaries 3.5 and 3.7, $a_{u(X)}(n) \sim [ncu(F)(B^c)]^{1/p}$ as $n \to \infty$, where

$$c = \lim_{t\to\infty} t^p \mathrm{P}(\|X\| > t).$$

An application of proposition 6.1.5 of [22] shows that 3 holds.

$3 \Rightarrow 4$. Following the proof of the analogous implication in [25] and [12], we define the E-r.v. $Y = rX + \eta x$, where $x \in S$, r is a Rademacher $\boldsymbol{R}^1$-r.v., η is an $\boldsymbol{R}^1$-r.v. with ch.f. $\mathrm{E}e^{it\eta} = \exp\{-c(p)|t|^p\}$,

$$c(p) = \int_0^\infty (\cos u - 1)\, \gamma_p(du),$$

the E-r.v. X belongs to $L^0_{p,\infty}(E)$ and all r.v.'s are independent. Let $\Gamma = 2^{-1}(\delta_x+\delta_{-x})$ and $F = i(\gamma_p \otimes \Gamma)$. Then in the papers mentioned above it is proved that $Y \in \mathrm{RV}(E, p, pF)$ and $t^p \mathrm{P}(\|Y\| > t) \to F(B^c) = p^{-1}$ as $t \to \infty$. Therefore, by the assumptions,

$$\mathscr{L}\big(n^{-1/p} S_n\big(ru(X)\big)+u(x)\eta\big) \xrightarrow{w} e\big(u(F)\big) \quad \text{as } n \to \infty.$$

Since

$$e\big(u(F)\big)\hat{}\,(y^*) = \exp\{-c(p)|\langle u(x), y^*\rangle|^p\} = \mathrm{E}\exp\{i\langle u(x)\eta, y^*\rangle\},$$

we have $ru(X) \in \mathrm{WLLN}_p(F)$, which, by corollary 3.2 in [33], implies $u(X) \in \mathrm{WLLN}_p(F)$.

$4 \Rightarrow 1$. This implication is proved in [33], which completes the proof of theorem 4.7.

5. Applications. We will apply our results in order to get sufficient conditions for stochastic processes to have a version with continuous sample paths satisfying some maximal inequality and to characterize domains of attraction of p-stable processes. For a stable process with index $1 \leqslant p < 2$ sufficient (and necessary, for the strongly stationary case) continuity conditions are due to Marcus and Pisier [26, 27], which extends the Dudley-Fernique theorem for Gaussian processes, i.e. the case $p = 2$. The paper [27] contains among others the correct generalization to domains of normal attraction of the $C(S)$ central limit theorem.

Supplementary results are given below. Our approach goes back to Zinn [42] and was applied in this context by Araujo and Marcus [7], Giné and Marcus [13, 14]. The continuity conditions given below can be applied to processes being not necessary p-stable. They essentially concide with those for the stable case. The second statement is an extension of the Marcus and Pisier result in [27], related to the limit theorem for the case of not necessarily normal attraction.

Let $(T, \mathscr{T})$ be a compact metric space and $C(T)$ denote the Banach space of continuous functions on T with sup norm. For a τ-continuous pseudo-metric ϱ on T let $N(T, \varrho; \varepsilon)$ denote the minimal number of ϱ-balls of radius $\varepsilon > 0$ with centers in T, which are necessary in order to cover T. We define

$$\mathscr{I}_q(\varrho) = \int_0^1 \log^{1/q} N(T, \varrho; \varepsilon)\, d\varepsilon \quad \text{for } 2 \leqslant q < \infty,$$

$$\mathscr{I}_\infty(\varrho) = \int_0^1 \log\log N(T, \varrho; \varepsilon)\, d\varepsilon.$$

For fixed $t_0 = T$ put

$$\mathrm{Lip}_\varrho(T) = \{x \in C(T)\colon \|x\|_\varrho = |x(t_0)| + \sup_{s,t \in T} |x(s)-x(t)|/\varrho(s, t) < \infty\}$$

and $S = \{x \in C(T): \|x\|_\varrho = 1\}$. Let m be a finite Borel measure on the unit sphere U of $C(T)$. Actually in [27] (cf. corollary 3.2 in [13]) it is proved that if $\mathscr{I}_q(\varrho) < \infty$ and

$$\int_U \|x\|_\varrho^p \, m(dx) < \infty \tag{5.1}$$

for $p \in [1, 2)$ and $1/p + 1/q = 1$, then there exists a p-stable p.m. on $C(T)$ with spectral measure m. Put

$$\Gamma(W) = \int_U \|y\|_\varrho^p I_W(y/\|y\|_\varrho) \, m(dy) \tag{5.2}$$

for all Borel sets W on S. Then, by (5.1),

$$\Gamma(S) = \int_{\{y \in U: \|y\|_\varrho < \infty\}} \|y\|_\varrho^p \, m(dy) = \int_U \|y\|_\varrho^p \, m(dy) < \infty . \tag{5.3}$$

For the measure Γ we can define a σ-finite measure $F = i(\gamma_p \otimes \Gamma)$ as in section 3. Let $\tilde{F}$ denote an image measure of F under the inclusion of $\mathrm{Lip}_\varrho(T)$ in $C(T)$. By (5.2) we have

$$\Gamma_{\tilde{F}}(A) = p\tilde{F}\{(1, \infty) \times A\} = \int_S \|x\|^p I_A(x/\|x\|) \, \Gamma(dx) = m(A)$$

for all Borel sets A on U. We conclude that the above continuity result is actually a generalization of a result due to Mouchtari and de Acosta mentioned in the intriduction, because the inclusion map of $\mathrm{Lip}_\varrho(T)$ in $C(T)$, under the condition $\mathscr{I}_p(\varrho) < \infty$, is of type (p, ∞) (see theorem 1.2 in [27]). By virtue of an argument in theorem 2 [42], the next corollary is a simple translation of theorem 2.12 (see also remark 2.13.2)) into the language of stochastic processes.

Corollary 5.1. *With the above notations, for a σ-finite Borel measure F on $\mathrm{Lip}_\varrho(T)$ such that*

$$\sup_{t \geq 1} t^p F\{x: \|x\|_\varrho > t\} < \infty \quad \textit{and} \quad \lim_{t \to \infty} t^p F\{x: \|x\|_\varrho > t\} < \infty,$$

the finiteness of $\mathscr{I}_q(\varrho)$ implies that there exists a stochastic process $\{X(t); t \in T\}$ on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with continuous sample paths and with the L.m. F of the induced p.m. on $C(T)$, satisfying the inequality

$$\sup_{s>0} s^p \, \mathrm{P}\{\sup_{t \in T} |X(t)| > s\} \leq c_p \sup_{s>0} s^p F\{x: \|x\|_\varrho > s\}$$

for some constant c_p depending only on p.

The next statement gives conditions for a process to belong to the domain of attraction of a p-stable process with general norming constants. This is a consequence of theorem 4.7.

COROLLARY 5.2. *With the above notations, assume that* $\mathscr{J}_q(\varrho) < \infty$ *and* $\{X(t);\, t \in T\}$ *is a sample continuous process on* T. *If*

(i) *there exists a Borel* σ-*finite measure* F *on* $\mathrm{Lip}_\varrho(T)$ *with* $F\{x\colon \|x\|_\varrho > t\} = t^{-p} F\{x\colon \|x\|_\varrho > 1\}$ *for all* $t > 0$, *and*

$$\mathscr{L}(X)(t\,\cdot)/\mathscr{L}(X)\{x\colon \|x\|_\varrho > t\} \xrightarrow{v} F \quad \text{as } t \to \infty \text{ in } \mathrm{Lip}_\varrho(T);$$

(ii) $\sup_{n \geqslant 1} \sup_{s \leqslant a_X(n)} s^p\, \mathrm{P}(\|X\|_\varrho > s)/a_X^p(n)\, \mathrm{P}(\|X\|_\varrho > a_X(n)) < \infty$, *where*

$$a_X(n) = \sup\{s > 0\colon n\mathrm{P}(\|X\|_\varrho > s) \geqslant 1\},$$

then there exists a p-stable stochastic process $\{Y(t);\, t \in T\}$ *with continuous sample paths and with the L.m.* F *of an induced p.m. on* $C(T)$ *satisfying*

$$\mathscr{L}([S_n(X) - n\mathrm{E}XI_{(\|X\| \leqslant \tilde{a}_X(n))}]/\tilde{a}_X(n)) \xrightarrow{w} \mathscr{L}() \quad \text{as } n \to \infty,$$

in $C(T)$, *where* $\tilde{a}_X(n) = \sup\{s > 0\colon n\mathrm{P}(\|X\| > s) \geqslant 1\}$.

Remark 5.3. A complete description of the domain of attraction, i.e. without condition (ii) in corollary 5.2 but under the more restrictive entropy condition $\mathscr{J}_2(\varrho) < \infty$, is given in [13]. It is easy to remove condition (ii), when $\mathscr{J}_{q'}(\varrho) < \infty$ for some $2 \leqslant q' < q$.

Remark 5.4 (Added in proof). Results on the DNA in $C(S)$, obtained by the method of majorizing measures, are presented in recent D. Juknevičienė paper (Liet. mat. rink., 1987, XXVI, p. 362–373).

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