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ON IMPROVING UNBIASED ESTIMATORS BY MULTIPLICATION WITH A MATRIX

BY

STEFA ZONTEK (WROCŁAW)

Abstract. Consider a random n-vector Y with a mean vector μ and finite second moments. Under some assumptions on the model there is constructed a class of linear estimators as good as a given unbiased linear estimator of parametric function $C'\mu$. For some parametric functions there are identified those estimators in the constructed class which are admissible for $C'\mu$.

1. Introduction and notation. Hodges and Lehmann [2] gave an example which illustrates that the best unbiased estimator can be inadmissible. This inspired a number of authors to examine when linear unbiased estimators of a parametric function are admissible in the class of linear estimators as well as to find an admissible linear estimator dominating a given unbiased linear estimator in the opposite case. So far an explicite characterization of all admissible linear estimators dominating an unbiased linear estimator is given only for models with at most two parameters. For more complicated models such a characterization is difficult.

Consider a random vector Y with a mean vector μ and finite second moments. In this paper we find a class of admissible linear estimators based on Y dominating an unbiased linear estimator for some linear functions of μ.

Firstly we describe a class of linear estimators better than Y with respect to the mean squared error matrix and use them to construct linear estimators as good as an unbiased estimator of $C'\mu$ with respect to the mean squared error. Next, using a characterization of admissible linear estimators, we indicate those estimators in the resulting class which are admissible for $C'\mu$.

In particular, we consider the balanced random multi-way ANOVA model and characterize all invariant quadratic estimators that are better than the best invariant unbiased quadratic estimator for the vector of variance components with respect to the mean squared error matrix (as it is known, an invariant quadratic estimation of variance components can be reduced to a linear estimation based on a vector of given quadratic forms). Next we establish a class of parametric functions of variance components such that for every function in this class one can construct an admissible estimator that dominates its best unbiased estimator.

Throughout the paper we will use the following notation:

$\mathcal{M}_{n\times m}$ denotes the class of $(n\times m)$-matrices (we also write $\mathcal{M}_n$ instead of $\mathcal{M}_{n\times n}$).

$\mathcal{R}(A)$, where A is any matrix in $\mathcal{M}_{n\times m}$, denotes the linear subspace generated by columns of A; A', A^- and A^+ denote the transpose, the g-inverse and the Moore-Penrose g-inverse of A, respectively.

tr A stands for the trace of a matrix A in $\mathcal{M}_n$.

I_n (or simply I) denotes the unit matrix.

$A_d = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ for any matrix $A = (a_{ij})$ in $\mathcal{M}_n$.

$A \leqslant B$ or $A < B$ $(A, B \in \mathcal{M}_n)$ denote that $B - A$ is non-negative definite (n.n.d.) or positive definite (p.d.), respectively.

$\theta = (\theta_1, \ldots, \theta_n)' > 0$ means that all the coordinates of θ are positive.

2. Preliminaries. We recall now some definitions which will be used throughout the paper and establish a lemma which extends a result of Perlman [8] on improving unbiased estimators by multiplication with an appropriate constant.

We consider a random n-vector Y with expectation $\mu_\theta = \mathrm{E}Y$ and covariance matrix $V_\theta = \operatorname{cov} Y$, $\theta\in\Theta$. It is desired to estimate $C'\mu_\theta$, where $C\in\mathcal{M}_{n\times m}$, by estimators of the form $L'Y$ with L in $\mathcal{M}_{n\times m}$. We use two kinds of risks: the mean squared error matrix

$$M(L'Y;\, C'\mu_\theta) = \mathrm{E}(L'Y - C'\mu_\theta)(L'Y - C'\mu_\theta)' = L'V_\theta L + (L-C)'\mu_\theta\mu_\theta'(L-C) \tag{2.1}$$

and the mean squared error

$$R(L'Y;\, C'\mu_\theta) = \operatorname{tr} M(L'Y;\, C'\mu_\theta). \tag{2.2}$$

An estimator $L_1'Y$ is said to be *m-as good as* $L_2'Y$ if $M(L_2'Y;\, C'\mu_\theta) - M(L_1'Y;\, C'\mu_\theta)$ is n.n.d. for every $\theta\in\Theta$, and it is said to be *m-better than* $L_2'Y$ if, in addition, $M(L_2'Y;\, C'\mu_\theta) - M(L_1'Y;\, C'\mu_\theta) \neq 0$ for at least one $\theta\in\Theta$.

We say that $L_1'Y$ is *as good as* $L_2'Y$ if $R(L_2'Y;\, C'\mu_\theta) - R(L_1'Y;\, C'\mu_\theta) \geqslant 0$ for every $\theta\in\Theta$, and that it *dominates* $L_2'Y$ if, in addition, a strict inequality holds for at least one $\theta\in\Theta$. An estimator $L'Y$ is *admissible* for $C'\mu_\theta$ if there is no other linear estimator which is better than $L'Y$.

The method of constructing the estimators that dominate an unbiased estimator is based on the following properties of the risk functions (2.1) and (2.2).

If $B'Y$ is m-better than Y for μ_θ, then, for every matrix $C \in \mathcal{M}_{n\times m}$, the estimator $C'B'Y$ is as good as the unbiased estimator $C'Y$ for $C'\mu_\theta$, and there exists a $C \in \mathcal{M}_{n\times m}$ such that the estimator $C'B'Y$ dominates $C'Y$. Moreover, if $B'Y$ is admissible for μ_θ, then $C'B'Y$ is admissible for $C'\mu_\theta$.

LEMMA 2.1. *The estimator $B'Y$ is m-better than Y for μ_θ iff, for every* $\theta \in \Theta$,

(i) $V_\theta - B'V_\theta B \geqslant 0$;

(ii) $(I-B)'\mu_\theta \in \mathcal{R}(V_\theta - B'V_\theta B)$;

(2.3) (iii) $\mu_\theta'(I-B)(V_\theta - B'V_\theta B)^-(I-B)'\mu_\theta \leqslant 1$;

(iv) *there exists a point* $\theta_0 \in \Theta$ *such that* $\operatorname{rank}(V_{\theta_0} - B'V_{\theta_0}B) \geqslant 1$, *and* (iii) *holds with the strict inequality when* $\operatorname{rank}(V_{\theta_0} - B'V_{\theta_0}B) = 1$.

Proof. By definition, $B'Y$ is m-as good as Y iff, for all $\theta \in \Theta$,

$$(I-B)'\mu_\theta\mu_\theta'(I-B) \leqslant V_\theta - B'V_\theta B, \tag{2.4}$$

and $B'Y$ is m-better than Y if, in addition, (2.4) holds with the strict inequality for at least one point in Θ.

By Theorem 1 due to Stępniak [9], we infer that (2.4) is equivalent to (2.3), (i)–(iii). Thus there remains to show that if B fulfills (2.4), then (2.3), (iv), holds iff there is the strict inequality in (2.4) for a point $\theta_0 \in \Theta$. Clearly, we have to show this only for the case where $\operatorname{rank}(V_{\theta_0} - B'V_{\theta_0}B) = 1$. From (2.4) it then follows that there exists a number $c > 1$ such that

$$c(I-B)'\mu_{\theta_0}\mu_{\theta_0}'(I-B) = V_{\theta_0} - B'V_{\theta_0}B.$$

Evidently, (2.4) is the strict inequality at θ_0 iff $c > 1$. Since also $c > 1$ iff (2.3), (iii), is the strict inequality at θ_0, Lemma 2.1 is established.

If $B = bI$, $b \in \mathcal{R}$, and $V_\theta > 0$ for all $\theta \in \Theta$, then conditions (2.3), (i)–(iii), reduce to the well-known Perlman's [8] conditions: (i) $b \in (-1, 1)$ and (ii) $\mu_\theta' V_\theta^{-1}\mu_\theta \leqslant (1+b)/(1-b)$ for all $\theta \in \Theta$.

3. Characterization of estimators that are *m*-better than *Y* under some additional assumptions imposed on the model. We assume now that the mean vector μ_θ and the covariance matrix V_θ have for all $\theta \in \Theta$ the structure

$$\mu_\theta = AH'\theta, \qquad V_\theta = A(H'\theta\theta'H)_d, \tag{3.1}$$

and that $A = \operatorname{diag}(a_1, \ldots, a_n)$, $a_i > 0$, $H = (h_{ij})$ is a $(k\times n)$-matrix $(k \leqslant n)$ with non-negative entries but without zero columns and that $\Theta = \{\theta\colon \theta \in \mathcal{R}^k, \theta > 0\}$.

It is well known (see Section 4) that under some assumptions the invariant quadratic estimation of variance components in a normal mixed linear model leads to a linear estimation within models having structure (3.1). Also, if Y is a vector of n independent random variables and if the i-th variable ($i = 1, \ldots, n$) has a Gamma distribution $\Gamma(\alpha_i, 1/\theta_i)$, then (3.1) holds with $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$, $H = I$ and $\Theta = \{\theta: \theta \in R^n, \theta > 0\}$.

Now we deduce from Lemma 2.1 the necessary and sufficient conditions for an estimator $B'Y$ to be m-better than Y under several assumptions imposed on the matrix H.

First let us show that if B is a diagonal matrix, say $B = \operatorname{diag}(b_1, \ldots, b_n)$, then the necessary and sufficient conditions resulting from Lemma 2.1 do not depend on H.

THEOREM 3.1. *If $B = \operatorname{diag}(b_1, \ldots, b_n)$, then $B'Y$ is m-better than Y iff*

(i) $b_i = [(a_i-1)/(a_i+1), 1]$, $\quad i = 1, \ldots, n$,

(3.2) (ii) $\sum_{i=1}^{n} a_i(1-b_i)/(1+b_i) \leqslant 1$,

(iii) $B \neq I$ *and* $B \neq B_i = \operatorname{diag}\left(1, \ldots, 1, \frac{a_i-1}{a_i+1}, 1, \ldots, 1\right)$.

Proof. Since V_θ is a diagonal matrix for all $\theta \in \Theta$, it is easy to see that if B is a diagonal matrix, then (2.3), (i) and (ii), are equivalent to

$$b_i \in (-1, 1] \quad \text{for } i = 1, \ldots, n, \tag{3.3}$$

and (2.3), (iii), is equivalent to (3.2), (ii). Moreover, since the function $f(x) = a(1-x)/(1+x)$, where $a > 0$, is decreasing, positive on $(-1, 1]$ and taking the value 1 for $(a-1)/(a+1)$, conditions (3.2), (ii), and (3.3) imply (3.2), (i).

If BY is m-better than Y, then $B \neq I$ and $B \neq B_i$ because in this case $M(Y; AH'\theta) - M(BY; AH'\theta) \equiv 0$. It remains to show that if two or more diagonal elements of a matrix B, satisfying (3.2), (i), are not equal to 1, then $M(Y; AH'\theta) - M(BY; AH'\theta) \neq 0$ for some $\theta \in \Theta$. This follows from the observation that $\operatorname{rank}(V_\theta - BV_\theta B) \geqslant 2$ for all $\theta \in \Theta$.

If $B = bI$, $b \in \mathscr{R}$, then BY is m-better than Y iff $b \in [(a-1)/(a+1), 1)$, where $a = \sum a_i$ ($i = 1, \ldots, n$). This result was obtained by LaMotte [6] for the variance component model.

To formulate the next theorem we need the following notation. For the matrix H appearing in (3.1) define, for $i = 1, \ldots, k$ and $j = 1, \ldots, n$:

$$S_i = \{s: h_{is} \neq 0, 1 \leqslant s \leqslant n\}, \quad T_j = \{t: h_{tj} = 0, 1 \leqslant t \leqslant k\},$$

$$\bar{S}_j = \bigcup_{q \in T_j} S_q.$$

THEOREM 3.2. (i) *The class of estimators which are m-better than Y coincides with the class of estimators given by* (3.2) *iff, for all* $1 \leqslant j \leqslant n$,

$$\bar{S}_j = \{1, \ldots, j-1, j+1, \ldots, n\}. \tag{3.4}$$

(ii) *If H is a diagonal matrix, then condition* (3.4) *is fulfilled.*

LEMMA 3.3. *If* $B'Y$ *is m-better than* Y, *then* $b_{ij} = 0$ *for* $i \in \bar{S}_j$.

If, in addition, $b_{i_0 i_0} = 1$, *where* $1 \leqslant i_0 \leqslant n$, *then* $b_{i_0 j} = b_{j i_0} = 0$ *for all* $j \neq i_0$.

Indeed, by Theorem 2.3 the diagonal elements of $V_\theta - B' V_\theta B$ are non-negative for all $\theta \in \Theta$, i.e.

$$a_j v_j^2 - \sum_{i=1}^{n} a_i b_{ij}^2 v_i^2 \geqslant 0 \quad \text{for } \theta \in \Theta, \tag{3.5}$$

where $\operatorname{diag}(v_1^2, \ldots, v_n^2) = (H'\theta\theta' H)_d$. Since $v_j = \sum h_{tj}\theta_t$ $(t = 1, \ldots, k)$ does not depend on θ_t for $t \in T_j$, we have $b_{ij} = 0$ for $i \in \bigcup S_t$ $(t \in T_j)$. If a diagonal element of B is equal to 1, say $b_{11} = 1$, then $b_{j1} = 0$ for $j = 2, \ldots, n$ by (3.5).

Condition (2.1), (ii), implies now that all elements of $V_\theta - B' V_\theta B$ in the first row are equal to 0, so that

$$\sum_{s=1}^{n} a_s b_{sj} b_{s1} v_s^2 = 0 \quad \text{for } \theta \in \Theta \text{ and } 2 \leqslant j \leqslant n.$$

Since $b_{11} = 1$, we conclude that $b_{1j} = 0$ for $j = 2, \ldots, n$, and this completes the proof of Lemma 3.3

Note that, by the definition of the set $\bar{S}_j$, $i \in \bar{S}_j$ iff $h_{sj} = 0$ and $h_{si} \neq 0$, where $1 \leqslant s \leqslant n$. This implies that $i \in \bar{S}_j$ iff

$$\sup\{v_i/v_j \colon (v_1, \ldots, v_n)' = H'\theta, \theta \in \Theta\} = \infty.$$

Proof of Theorem 3.2. If (3.4) holds and if $B'Y$ is m-better than Y, then, by Lemma 3.3, B is a diagonal matrix.

If (3.4) does not hold, then for $i = 1$ and $j = 2$, say, we have $1 \notin \bar{S}_2$. Define $B = (b_{ij})$ by

$$b_{ij} = \begin{cases} p_i & \text{for } i = 1, 2, \\ 2p_2/(1+a_1)\,x & \text{for } i = 1, j = 2, \\ 1 & \text{for } i = j = 3, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $p_i = a_i/(1+a_i)$ for $i = 1, 2$, and $x = \sup\{v_1/v_2 \colon (v_1, \ldots, v_n)' = H'\theta, \theta \in \Theta\} < \infty$. A simple algebra shows that, for $\theta \in \Theta$,

$$(H'\theta\theta' H)_d^{-1/2} [M(Y; AH'\theta) - M(B'Y; AH'\theta)] (H'\theta\theta' H)_d^{-1/2} = (m_{ij}),$$

where

$$m_{ij} = \begin{cases} p_1 & \text{for } i = j = 1, \\ -p_1 p_2 & \text{for } i \neq j = 1, 2, \\ p_2 + 4p_1 p_2^2 (v_1/v_2 x - (v_1/v_2 x)^2) & \text{for } i = j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix (m_{ij}) is n.n.d. for all $v_1/v_2 \in (0, x)$. Thus there exists an estimator $B'Y$ with a non-diagonal matrix B dominating Y. This completes the proof of part (i).

If H is a diagonal matrix, then $S_i = \{i\}$ and $T_j = \{1, \ldots, j-1, j+1, \ldots, n\}$, hence $\bar{S}_j = T_j$, which proves part (ii).

Notice that if a model satisfies condition (3.4), then the corresponding matrix H has not to be diagonal.

4. Balanced random multi-way ANOVA model. Let

$$X = (X_{1\ldots 11}, X_{1\ldots 12}, \ldots, X_{p_1 \ldots p_{n-1} p_n})'$$

be a $(p = p_1 \cdot \ldots \cdot p_n)$-vector of random variables such that

$$X_{s_1 \ldots s_n} = \beta + \alpha_{s_1}^{(1)} + \ldots + \alpha_{s_{n-1}}^{(n-1)} + e_{s_1 \ldots s_n}, \qquad s_j = 1, \ldots, p_j, \; j = 1, \ldots, n,$$

where $p_1, \ldots, p_{n-1} \geq 2$ and $p_n \geq 1$. Here β is an unknown constant, and the α's and e's are unobservable random variables with the zero mean. Assume also that

$$\operatorname{cov}(\alpha_l^{(i)}, \alpha_s^{(i)}) = \begin{cases} \theta_i & \text{for } i = j = 1, \ldots, n-1, \; l = s = 1, \ldots, p_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\operatorname{cov}(e_{s_1 \ldots s_n}, \alpha_s^{(i)}) = 0 \quad \text{for } i = 1, \ldots, n-1, \; s_j = 1, \ldots, p_j, \; j = 1, \ldots, n$$

and

$$\operatorname{cov}(e_{s_1 \ldots s_n}, e_{l_1 \ldots l_n}) = \begin{cases} \theta_n & \text{for } s_i = l_i = 1, \ldots, p_i, \; i = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta = (\theta_1, \ldots, \theta_n)' > 0$. Then

$$\mathrm{E}X = \mathbf{1}_p \quad \text{and} \quad \operatorname{cov} X = \sum_{i=1}^{n} \theta_i V_i,$$

where

$$V_1 = I_{p_1} \otimes J_{p_2 \cdot \ldots \cdot p_n},$$
$$V_i = J_{p_1 \cdot \ldots \cdot p_{i-1}} \otimes I_{p_i} \otimes J_{p_{i+1} \cdot \ldots \cdot p_n}, \qquad i = 2, \ldots, n-1,$$
$$V_n = I_p.$$

$\mathbf{1}_s$ denotes, as usually, the s-vector of 1's, while $J_s = \mathbf{1}_s \mathbf{1}_s'$.

Now define n matrices in $\mathscr{M}_n$ by

$$D_i = \frac{p_i}{p} V_i - \frac{1}{p} J_p, \quad i = 1, \ldots, n-1,$$

$$D_n = I_p - \sum_{i=1}^{n-1} \frac{p_i}{p} V_i + \frac{1}{p} J_p.$$

Notice that $D_1, \ldots, D_n$ are idempotent matrices and that they form an orthogonal basis for the quadratic subspace spanned by $MV_1 M, \ldots, MV_n M$, where $M = I_p - p^{-1} J_p$.

Next define a random n-vector by $Y = (X' D_1 X, \ldots, X' D_n X)'$ and note that the mean vector and the covariance matrix of Y have the structure (3.1) and that $A = \operatorname{diag}(r_1/2, \ldots, r_n/2)$, where $r_i = 2a_i = \operatorname{tr} D_i$, $i = 1, \ldots, n$, and

$$H = \begin{bmatrix} p/p_1 & 0 & \ldots & 0 & 0 \\ 0 & p/p_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p/p_{n-1} & 0 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix},$$

while $\Theta = \{\theta \in \mathscr{R}^n \colon \theta > 0\}$.

We now establish a theorem which gives the necessary and sufficient conditions for a linear estimator based on Y to be m-better than Y for $\mu_\theta = \mathrm{E} Y$. To formulate this theorem we need the following notation.

Let $\mathscr{B}$ denote the set of matrices fulfilling the necessary conditions for $B'Y$ to be m-better than Y given by Lemma 3.3, i.e. let

$$\mathscr{B} = \{B = (b_{ij}) \in \mathscr{M}_n \colon b_{ij} = 0 \text{ for } i \in \bar{S}_j\} \cap$$
$$\cap \{B \in \mathscr{M}_n \colon B'(I_n - D_B) = (I_n - D_B) B = I_n - D_B\},$$

where $D_B = (B - I_n)_d (B - I_n)_d^+$. It is easy to see that $i \in \bar{S}_j$ iff $h_{ij} = 0$.

Let $B_0 = (b_{ij}^0) = D_B^0 B (D_B^0)'$, where D_B^0 is a $(k \times n)$-matrix obtained by cancelling all the zero rows in D_B, while $k = \operatorname{tr} D_B$. Finally, define $Q_B = (q_{ij})$ by

$$q_{ij} = \begin{cases} 1/\prod\limits_{\substack{l=1 \\ l \neq i,j}}^{k-1} w_l & \text{for } i \neq j = 1, \ldots, k-1, \\ (1 - \sum\limits_{\substack{l=1 \\ l \neq i}}^{k-1} w_l)/\prod\limits_{\substack{l=1 \\ l \neq i}}^{k-1} w_l & \text{for } i = j = 1, \ldots, k-1, \end{cases}$$

where $w_i = a_i(1-b_{ii}^0)/(1+b_{ii}^0)$, $i = 1, \ldots, k$. Define the sum and the product over the empty set as 0 and 1, respectively.

THEOREM 4.1. *An estimator $B'Y$ is m-better than Y iff*

(i) $B \in \mathscr{B}$;

(ii) $b_{ii} \in [(a_i-1)/(a_i+1), 1]$, $i = 1, \ldots, n$;

(iii) $\mathscr{U} = \{u = (u_1, \ldots, u_{k-1})': u_i = -1$ *or* $u_i = u_i^0 - 1$,

(4.1) $i = 1, \ldots, k-1\} \subset \{z \in \mathscr{R}^{k-1}: z'Qz \leqslant q_0\}$, *where*

$$u_i^0 = b_{ki}^0/(1-b_{ii}^0)(1-b_{kk}^0), \quad i = 1, \ldots, k-1,$$

$$q_0 = (1-w_k)\Big(1-\sum_{l=1}^{k-1} w_l\Big)\Big/\prod_{l=1}^{k} w_l;$$

(iv) $B \neq I_n$, $B \neq B_i$, $i = 1, \ldots, n$.

Proof. Assume $B \in \mathscr{B}$ and $b_{nn} = 1$. Then B is a diagonal matrix. Since $\mathscr{U} = \{-\mathbf{1}_k\}$, condition (4.1), (iii), is equivalent to (3.2), (ii). Theorem 3.1 implies the validity of our assertion.

Now assume $B \in \mathscr{B}$ and $b_{nn} \neq 1$. Multiplying $M(Y; \mu_\theta) - M(B'Y; \mu_\theta)$ from the right-hand side and from the left-hand side by $D_B^0(B - I_n)^+(H'\theta\theta'H)_d^{-1/2}A^{-1}$ and by the transposed matrix, respectively, and taking into account the identity $B(B-I_n)^+ = D_B - (I_n - B)^+$, we infer that $B'Y$ is m-as good as Y iff, for all $\theta \in \Theta$,

$$\Delta(\theta^*) = (\delta_{ij}) = D_B^0[F(\theta^*) + F'(\theta^*) - A^{-1}](D_B^0)' - \mathbf{1}_k\mathbf{1}_k' \geqslant 0,$$

where

$$\theta^* = (\theta_1^*, \ldots, \theta_{k-1}^*, 1)' = \theta_k D_B^0(H'\theta\theta'H)_d^{-1/2}\mathbf{1}_k \in (0, 1) \times \ldots \times (0, 1) \times \{1\}$$

and

$$F(\theta^*) = (H'\theta\theta'H)_d^{1/2}(I_n - B)^+(H'\theta\theta'H)_d^{-1/2}A^{-1}.$$

The equivalence follows from the fact that, for $B \in \mathscr{B}$,

$$\mathscr{R}\big(M(Y; \mu_\theta) - M(B'Y; \mu_\theta)\big) \subset \mathscr{R}(I_n - B) = \mathscr{R}\big((I_n - B)^+\big)$$

and that $\operatorname{rank}(D_B^0) = \operatorname{rank}(I_n - B)^+$.

Since $(I_n - B)^+ = (D_B^0)'(I_k - B_0)^{-1}D_B^0$, a simple algebra shows that

$$\delta_{ij} = \begin{cases} 1/w_i - 1 & \text{for } i = j = 1, \ldots, k, \\ u_i^0\theta_i^* - 1 & \text{for } i = k \text{ and } j = 1, \ldots, k-1 \\ & \text{or } i = 1, \ldots, k-1 \text{ and } j = k, \\ -1 & \text{otherwise.} \end{cases}$$

The i-th diagonal element of $\Delta(\theta^*)$ is non-negative iff $b_{ii}^0 \in [(a_i^0-1)/(a_i^0+1), 1)$ for $i = 1, \ldots, k$, where $\operatorname{diag}(a_1^0, \ldots, a_k^0) = D_B^0 A(D_B^0)'$. Hence we get

(4.1), (ii), by noting that each diagonal element of B which differs from 1 is a diagonal element of B_0.

Since the matrix Δ, obtained by cancelling the last row and the last column of $\Delta(\theta^*)$, does not depend on θ, we see that $\Delta(\theta^*) \geqslant 0$ for all $\theta \in \Theta$ iff

$$\det(\Delta(\theta^*)) = -\sum_{i,j=1}^{k-1} q_{ij}(u_i^0\theta_i - 1)(u_j^0\theta_j - 1) + q_0 \geqslant 0$$

for all $\theta \in \Theta$ and $\Delta \geqslant 0$.

By (4.1), (ii), and Theorem 1 due to Stępniak [9], the last inequality holds iff

$$\sum_{i=1}^{k-1} w_i \leqslant 1. \tag{4.2}$$

Since the determinant of $\Delta(\theta^*)$ is a polynomial of degree 2 with respect to $u_i^0\theta_i^* - 1$ and since the coefficient $-q_{ii}$ of $(u_i^0\theta_i^* - 1)^2$ is not positive by (4.1), (ii), and (4.2), we infer that $\det(\Delta(\theta^*)) \geqslant 0$ for every $\theta \in \Theta$ iff $\det(\Delta(\theta^*)) \geqslant 0$ for $\theta^* \in \{(u_1, \ldots, u_{k-1}, 1)' \colon u_i = 0$ or $1, i = 1, \ldots, k-1\}$ or, equivalently, iff (4.1), (iii), holds. Noting that

$$\Delta((0, \ldots, 0, 1)') = (1 - \sum_{i=1}^{k} w_i)/\prod_{i=1}^{k} w_i \geqslant 0$$

implies (4.2) by (4.1), (ii), we see that $B'Y$ is m-as good as Y iff (4.1), (i)–(iii), hold.

Repeating word by word the arguments used in the proof of Theorem 3.1, we arrive at condition (4.1), (iv).

5. Construction of admissible estimators which dominate unbiased estimators for some parametric functions. In cases where we can find an m-better estimation than Y, we can construct (as already indicated in Section 2) for any parametric function $C'EY$ an estimator which dominates an unbiased estimator $L'Y$ for $C'EY$. In fact, if $B'Y$ is m-better than Y, then, for any parametric function $C'EY$, the estimator $L'B'Y$ is as good as $L'Y$. Because it is desirable to have admissible estimators dominating unbiased estimators, we shall in this section indicate, for some models, parametric functions for which the resulting estimator $L'B'Y$ is also admissible.

To the end of this section we assume that the matrix H appearing in (3.1) is of full rank.

We begin by recalling a result of Klonecki and Zontek [5].

Let Ω be the convex cone generated by $\{\theta\theta' \colon \theta \in \Theta\}$ and let $\bar{\mathscr{G}}$ be the closure of

$$\mathscr{G} = \{(H'\omega H)_d^{-1} H'\omega H \colon \omega \in \Omega\}. \tag{5.1}$$

THEOREM 5.1. *Within model* (3.1), *if*

$$L \in \mathscr{L}_C = \theta\{(I+GA)^{-1}GAC \colon G \in \bar{\mathscr{G}}\}, \tag{5.2}$$

then $L'Y$ *is admissible for* $C'\mu_\theta$.

Klonecki and Zontek [5] and Zontek [10] have also shown that, under some additional assumptions imposed on the model, the class of estimators $\{L'Y \colon L \in \mathscr{L}_C\}$ forms the minimal complete class. In particular, this is the case for the model (3.1) with $n \leqslant 4$, for models with diagonal matrix H and for models with H given by (4.1).

Assume that $\{L'Y \colon L \in \mathscr{L}_C\}$ forms the minimal complete class. Let $\mathscr{A}_C$ be the set of estimators for $C'\mu_\theta$ as good as $C'Y$ generated by estimators m-better than Y, i.e. let

$$\mathscr{A}_C = \{C'B'Y \colon B'Y \text{ is } m\text{-better than } Y\}.$$

Note that each estimator in $\mathscr{A}_C$, admissible for $C'\mu_\theta$, dominates $C'Y$.

LEMMA 5.2. *An estimator* $C'B'Y$ *in* $\mathscr{A}_C$ *is admissible for* $C'\mu_\theta$ *iff there exists a matrix* $G \in \bar{\mathscr{G}}$ *such that*

$$BC = GA(I-B)C. \tag{5.3}$$

Proof. By Theorem 5.1 the estimator $(BC)'Y$ is admissible for $C'\mu_\theta$ iff there exists a matrix $G \in \bar{\mathscr{G}}$ such that $BC = (I+GA)^{-1}GAC$ or, equivalently, if (5.3) holds.

COROLLARY 5.3. *Assume that* $C \in \mathscr{M}_{n\times 1}$. *An estimator* $bC'Y$ *in* $\mathscr{A}_C$ *is admissible for* $C'\mu_\theta$ *iff*

(i) $b \in [(a-1)/(a+1),\, 1)$, *where* $a = \sum_{i=1}^{n} a_i$;

(ii) *there exists a* $G \in \bar{\mathscr{G}}$ *such that* C *is the eigenvector of* GA *with the eigenvalue in* $[(a-1)/2,\, +\infty)$.

Proof. As noted in Section 3, an estimator bY is m-better than Y iff condition (i) of Corollary 5.3 holds. An application of Lemma 5.2 gives the second condition.

This result was first obtained by Gnot and Kleffe [1] for the model (3.1) with $H = (h_{ij}) \in M_{2\times n}$ such that $h_{11} = \ldots = h_{n1} = 1$ and $h_{12} > \ldots > h_{n-1,2} > h_{n2} = 0$.

Now we present some other examples of functions $C'\mu_\theta$ for which we can find the admissible estimators dominating $C'Y$.

(α) Assume that in the model described by (3.1) the matrix H is diagonal. If B_1 is a diagonal matrix, then $B_1 Y$ is admissible for μ_θ iff B_1

$=(\mathrm{I}+A)^{-1}A$. Hence, by Theorem 3.2, there exists an admissible estimator of μ_θ in $\mathscr{A}_I$ iff

(5.4) $$\sum_{i=1}^{n} a_i/(2a_i+1) \leqslant 1.$$

The last condition was obtained by Kleffe [3] under additional assumption that Y is a vector of independent random variables Y_i having the $(\theta_i/2a_i)\chi^2_{2a_i}$ distribution $(i = 1, \ldots, n)$.

Clearly, condition (5.4) is always satisfied if $n = 2$. If (5.4) does not hold, we can still construct admissible estimators dominating a given unbiased estimator for some particular functions. In fact, let N be a subset of $\{1, \ldots, n\}$ such that

(5.5) $$\sum_{i\in N} a_i/(2a_i+1) \leqslant 1$$

and let $D = \operatorname{diag}(d_1, \ldots, d_n)$ be a matrix given by

$$d_i = \begin{cases} 1 & \text{if } i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Now put $B_2 = D(I+A)^{-1}A+I-D$. Using (5.5), we see that the matrix B_2 fulfills (3.2). Moreover, condition (5.2) is satisfied for $G = I \in \overline{\mathscr{G}}$ and $C = D$. This shows that the estimator $C'DB_2Y$ is admissible for $C'D\mu_\theta$ and dominates the unbiased estimator $C'DY$ for any matrix $C \in \mathscr{M}_{n\times m}$.

Next we find all estimators admissible for a linear combination of two parameters, $f_1\theta_1+f_2\theta_2$ say, contained in the class $\mathscr{A}_{C_{12}}$, where $C_{12} = (f_1/a_1h_{11}, f_2/a_2h_{22}, 0, \ldots, 0)' \in \mathscr{R}^n$.

For the model (3.1) with the diagonal matrix H, if an estimator $B'Y$ is m-better than Y, then B is a diagonal matrix. Since the estimator $C'_{12}BY$ depends on the two first diagonal elements, we see that

$$\mathscr{A}_{C_{12}} = \{C'_{12}BY\colon B = \operatorname{diag}(b_1, b_2, 1, \ldots, 1) \text{ and } (b_1, b_2) \in \mathscr{B}_{12}\},$$

where

$$\begin{aligned}\mathscr{B}_{12} = \{(b_1, b_2)\colon b_2 \geqslant (a_1+a_2-1+(a_2-a_1-1)b_1)/(a_2-a_1+1 \\ +(a_1+a_2+1)b_1),\ 1 \geqslant b_i \geqslant (a_i-1)/(a_i+1),\ i = 1, 2\}\setminus \\ \setminus\{(1, 1), (1, (a_2-1)/(a_2+1)), (a_1-1)/(a_1+1), 1)\}.\end{aligned}$$

THEOREM 5.4. *Assume that* $(b_1, b_2) \in \mathscr{B}_{12}$. *An estimator* $C'_{12}BY$ *is admissible for* $C'_{12}\mu_\theta$ iff

(i) $b_2 \leqslant a_2b_1/((a_1+a_2+1)b_1-a_1)$ *and* $b_i \geqslant a_i/(a_i+1)$, $i = 1, 2$, *when* $c_1c_2 > 0$;

(ii) $b_i \leqslant a_i/(a_i+1)$, $i = 1, 2$, *when* $c_1c_2 < 0$;

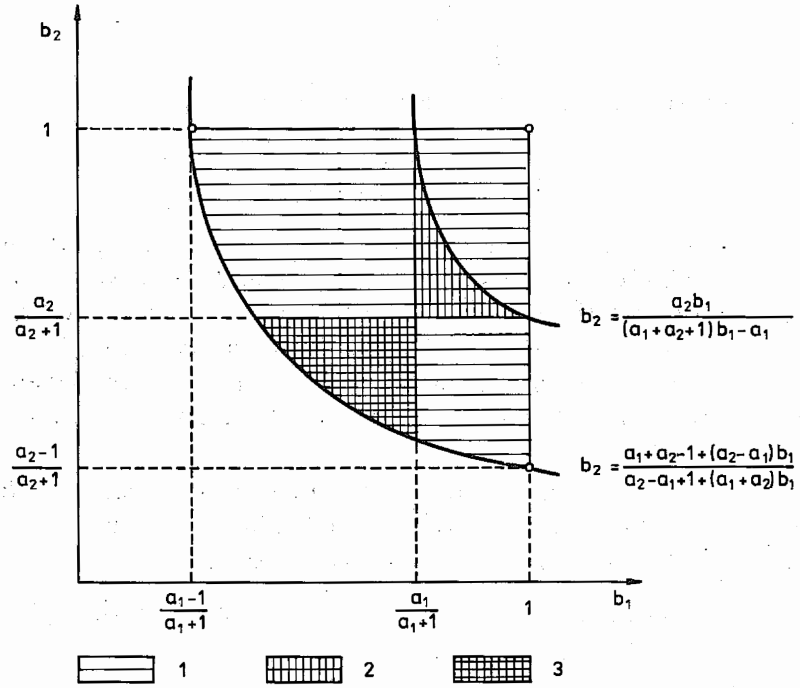
(iii) $b_1 = a_1/(a_1+1)$ *or* $b_2 = a_2/(a_2+1)$, *when* $c_1 \neq 0$ *and* $c_2 = 0$ *or* $c_1 = 0$ *and* $c_2 \neq 0$, *respectively.*

Proof. Assume that $G = (g_{ij}) \in \bar{\mathscr{G}}$. The matrix G satisfies (5.3) iff

$$g_{ij} = \big((1+a_i)b_i - a_i\big)c_i/(1-b_j)a_j c_j \geqslant 0, \quad i \neq j = 1, 2,$$
$$g_{ij} = 0, \quad i = 3, \ldots, n, j = 1, 2, \tag{5.6}$$
$$g_{12}g_{21} \leqslant 1.$$

We define $g_{ij} = 0$, when $b_j = 1$ *or* $c_j = 0$, $i \neq j = 1, 2$.

The first and the last inequality in (5.6) are equivalent to conditions (ii) and (i) of the theorem, respectively.



1 – the set B_{12} (BY is m-better than Y).
2 – $C'_{12}BY$ is admissible for $C'_{12}\mu_\theta$ and dominates $C'_{12}Y$, when $c_1c_2 > 0$.
3 – $C'_{12}BY$ is admissible for $C'_{12}\mu_\theta$ and dominates C'_{12}, when $c_1c_2 < 0$.

Similarly we can construct an admissible estimator for linear combination of four parameters $C'_{14}\mu_\theta$, say, where $C_{14} = (c_1, \ldots, c_4, 0, \ldots, 0)'$. We assume that there exists a permutation $c_{(1)}, \ldots, c_{(4)}$ of $c_1, \ldots, c_4$ such that $c_{(1)}c_{(2)} > 0$ and $c_{(3)}c_{(4)} \geqslant 0$. This assumption is always satisfied if one of the c's is equal to 0.

To simplify notation we assume that $c_1 c_2 > 0$ and $c_3 c_4 \geqslant 0$.

THEOREM 5.5. *If*

$$(5.7)\quad \begin{aligned} &\text{(i)}\quad b_{2+i} \leqslant a_{2+i} b_{1+i}/((a_{1+i}+a_{2+i}+1)\, b_{1+i}-a_{1+i}), \qquad i = 0, 2,\\ &\text{(ii)}\quad \sum_{i=1}^{4} (1-b_i)\, a_i/(1+b_i) \leqslant 1, \end{aligned}$$

then the estimator $C'_{14} BY$ is admissible for $C'_{14} \mu_\theta$ and dominates $C'_{14} Y$, where $B = \operatorname{diag}(b_1, \ldots, b_4, 1, \ldots, 1)$.

Proof. The matrix

$$G = \begin{bmatrix} 1 & g_{12} & & & & & \\ g_{21} & 1 & & & 0 & & \\ & & 1 & g_{34} & & & \\ & & g_{43} & 1 & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

belongs to $\mathscr{G}$ iff $g_{12}, g_{21}, g_{34}, g_{43} \geqslant 0$, $g_{12} g_{21} \leqslant 1$ and $g_{34} g_{43} \leqslant 1$.

Now, by similar arguments as in the proof of Theorem 5.4, we obtain that the estimator $C'_{14} BY$ is admissible for $C'_{14} \mu_\theta$ and dominates $C'_{14} Y$.

$$b_i = \begin{cases} (a_1+a_2)/(a_1+a_2+1) & \text{for } i = 1, 2,\\ (a_3+a_4)/(a_3+a_4+1) & \text{for } i = 3, 4 \text{ and } c_3 c_4 > 0,\\ a_i/(a_i+1) & \text{for } c_3 c_4 = 0 \text{ and } c_i \neq 0,\ i = 3, 4,\\ 1 & \text{otherwise.} \end{cases}$$

(β) The balanced random multi-way ANOVA model. For the balanced random one-way ANOVA model ($n = 2$) LaMotte [7] has characterized the class of all admissible estimators for $C' \mu_\theta$, $C \in \mathscr{R}^2$, dominating the unbiased estimator $C' Y$. Klonecki and Zontek [5] have indicated a simple subclass of these estimators. They have showed that an admissible estimator $L' Y$ for μ_θ is m-better than Y iff

$$L \in \mathscr{L} = \left\{(I_2+GA)^{-1}\colon G = \begin{bmatrix} 1 & 0 \\ g & 1 \end{bmatrix}, g \in \left[1,\ 1+\left(\frac{(1+a_1)(1+a_2)}{a_1 a_2}\right)^{-1/2}\right]\right\}$$

and they have noted that, for every $C \in \mathscr{R}^2$, the estimator $C' L' Y$ with $L \in \mathscr{L}$ is admissible for $C' \mu_\theta$ and dominates $C' Y$. This result can be easily extended to the considered model with $n > 2$. Let $C_{in} = (e_i, e_n)$, where $e_j = (0, \ldots, 0, 1, \ldots, 0)'$, denotes the j-th unit vector in $\mathscr{R}^n$.

COROLLARY 5.6. *If*

$$L \in \left\{ (I_2 + GA_{in})^{-1} GA_{in} \colon G = \begin{bmatrix} 1 & 0 \\ g_i & 1 \end{bmatrix},\ g_i \in \left[1,\ 1 + \left((a_i+1)(a_n+1)/a_i a_n\right)^{-1/2}\right] \right\},$$

where $A_{in} = \mathrm{diag}(a_i, a_n)$, $i = 1, \ldots, n-1$, *then the estimator* $L'C'_{in} Y$ *is admissible for* $C'_{in}\mu_\theta$ *and is m-better than* $C'_{in} Y$.

The proof is based on Theorem 4.1 and Lemma 5.2 and is similar to that for the case $n = 2$.

It is interesting to note that for the balanced random two-way ANOVA model ($n = 3$) there exists an admissible estimator of μ_θ which is m-better than Y for some p_1, p_2 and p_3. In fact, by virtue of Theorems 4.1 and 5.1, the admissible estimator $(I_3 + AG'_0)^{-1} AG'_0 Y$ for μ_θ, where

$$G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

is m-better than Y iff $(p_1, p_2, p_3) \in \{(2, 2, 1), (2, 3, 1), (3, 2, 1), (4, 2, 1), (2, 4, 1), (2, 2, 2)\}$.

Now we show an analogous result to Theorems 5.4 and 5.5 for the considered model.

THEOREM 5.7. *Assume that* $n > 2$ *and that* $(b_1, b_2) \in \mathscr{B}_{12}$. *If* $c_1 c_2 < 0$ *and if* $b_i \leqslant a_i/(a_i+1)$, $i = 1, 2$, *then the estimator* $C'_{12} BY$ *with* $B = (b_1, b_2, 1, \ldots, 1)$ *is admissible for* $C'_{12}\mu_\theta$ *and dominates* $C'_{12} Y$.

Proof. First note that the matrix

$$G = \begin{bmatrix} 1 & g_{12} & 0 & \ldots & 0 \\ g_{21} & 1 & 0 & \ldots & 0 \\ g_1 & g_2 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ g_1 & g_2 & 1 & \ldots & 1 \end{bmatrix} \tag{5.8}$$

belongs to $\mathscr{G}$ iff $g_{12}, g_{21} \geqslant 0$, $g_{12} g_{21} \leqslant 1$ and $g_1, g_2 \geqslant 1$.

If the assumptions of the theorem are fulfilled, then there exists a matrix G in $\mathscr{G}$ satisfying (5.3) and (5.8) (compare the proof of Theorem 5.4).

THEOREM 5.8. *Assume that* $n > 4$ *and that* $c_1 c_2 > 0$, $c_3 c_4 \geqslant 0$ *and* $c_1 c_3 < 0$. *If* $b_1, \ldots, b_4$ *satisfy* (5.7), *then the estimator* $C'_{14} BY$ *is admissible for* $C'_{14}\mu_\theta$ *and dominates* $C'_{14} Y$.

We omit the proof.

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Polish Academy of Sciences
Institute of Mathematics
ul. Kopernika 18
51–617 Wrocław, Poland

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