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# ON THE LAW OF LARGE NUMBERS OF THE HSU-ROBBINS TYPE

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*Abstract.* There are given the laws of large numbers of the Hsu-Robbins type which generalize some results of [1] and [2].

**1. Introduction.** Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables, and let $S_n = X_1 + \ldots + X_n$ $(n \geqslant 1)$. In studying the rate of convergence in the weak law of large numbers, the convergence of the series

(1) $$\sum_{n=1}^{\infty} \mathrm{P}[|S_n| \geqslant n\varepsilon],$$

for some $\varepsilon > 0$, was found to be connected with the existence of the second moment of $X$ (cf. [6], [3] or [7]). Some conditions, which guarantee the convergence of series (1) in the case of nonidentically distributed random variables, have been given in [2]. The paper [1] considers the convergence of series of type (1) with the index of summation restricted to a subsequence. That problem was reduced to investigating the convergence of series

(2) $$\sum_{n=1}^{\infty} c_n \mathrm{P}[|S_n| \geqslant \varepsilon b_n],$$

where $\{c_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ are sequences of positive integers such that $1 \leqslant c_n < \infty$, $n \geqslant 1$, $1 \leqslant b_1 < b_2 < \ldots$ From Theorem 1 of [1] it follows that if $X_1, X_2, \ldots$ are independent random variables with $\mathrm{E}X_i = 0$, $i \geqslant 1$, and for some sequence $\{\lambda_n, n \geqslant 1\}$ with $0 < \lambda_n \leqslant 1$, we have $\mathrm{E}|X_i|^{1+\lambda} < \infty$, $i \geqslant 1$, where $\lambda = \sup_n \lambda_n$, and, moreover, the sequences $\{c_n\}$ and $\{b_n\}$ satisfy the condition

(3) $$\sum_{n=1}^{\infty} c_n b_n^{-1-\lambda_n} \sum_{i=1}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n} < \infty,$$

then, for every $\varepsilon > 0$,

(4) $$\sum_{n=1}^{\infty} c_n \mathrm{P}[|S_{b_n}| \geqslant b_n \varepsilon] < \infty .$$

Note that if $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathrm{E}X_1 = 0$ and $\mathrm{E}X_1^2 = \sigma^2 < \infty$, then series (3) diverges for $c_n = 1$ $(n \geqslant 1)$, $b_n = n$ $(n \geqslant 1)$ and $\lambda_n = 1$ $(n \geqslant 1)$, though the series (1) (or (4)) converges by the Hsu-Robbins theorem [6]. Thus we conclude that condition (3) is too strong for (4).

This paper investigates the convergence of the series (4) under weaker conditions than condition (3). Moreover, we extend our result to 2-dimensional arrays of independent random variables.

**2. Sufficient conditions for complete convergence with the index of summation restricted to subsequences.** Let $\{X_i, i \geqslant 1\}$ be a sequence of independent random variables, $\mathrm{E}X_i = 0$, $i \geqslant 1$. Let $\delta > 0$ and put $X_i' = X_i I_{[|X_i| < b_n \delta]}$ and $X_i'' = X_i I_{[|X_i| \geqslant b_n \delta]}$, $1 \leqslant i \leqslant b_n$, where $I[\cdot]$ denotes the indicator function and $\{b_n, n \geqslant 1\}$ is a strictly increasing sequence of positive integers. Write

$$Y_i = X_i' - \mathrm{E}X_i', \quad S_k' = \sum_{i=1}^{k} X_i, \quad S_k^* = \sum_{i=1}^{k} Y_i .$$

Note that

(5) $$\Big(\sum_{i=1}^{b_n} x_i\Big)^4 = \sum_{i=1}^{b_n} x_i^4 + 6 \sum_{i=2}^{b_n} x_i^2 \sum_{j=1}^{i-1} x_j^2 + 12 \sum_{i=1}^{b_n} x_i^2 \sum_{\substack{j=2\\ j\neq i}}^{b_n} x_j \sum_{\substack{k=1\\ k\neq i}}^{j-1} x_k + 4 \sum_{i=2}^{b_n} x_i \sum_{j=1}^{i-1} x_j^3 + 4 \sum_{i=2}^{b_n} x_i^3 \sum_{j=1}^{i-1} x_j + 24 \sum_{i=4}^{b_n} x_i \sum_{j=3}^{i-1} x_j \sum_{k=2}^{j-1} x_k \sum_{l=1}^{k-1} x_l .$$

Using (3) we get

$$\mathrm{E}(S_{b_n}^*)^4 = \sum_{i=1}^{b_n} \mathrm{E}Y_i^4 + 6 \sum_{i=2}^{b_n} \sigma^2 X_i' \sum_{j=1}^{i-1} \sigma^2 X_j'$$

and, moreover, we obtain the inequality

(6) $$\begin{aligned} &\mathrm{E}[S_{b_n}^4/(S_{b_n}^4 + b_n^4 \varepsilon^4)] \\ &\leqslant \mathrm{E}[S_{b_n}^4/(S_{b_n}^4 + b_n^4 \varepsilon^4)] I_{[S_{b_n} = S_{b_n}']} + \mathrm{P}[S_{b_n} \neq S_{b_n}'] \\ &\leqslant \mathrm{E}(S_{b_n}')^4/b_n^4 \varepsilon^4 + \sum_{i=1}^{b_n} \mathrm{P}[|X_i| \geqslant b_n \delta] \\ &\leqslant 8\big(\mathrm{E}(S_{b_n}^*)^4/b_n^4 \varepsilon^4 + (\mathrm{E}S_{b_n}')^4/b_n^4 \varepsilon^4\big) + \sum_{i=1}^{b_n} \mathrm{P}[|X_i| \geqslant b_n \delta] \\ &= 8 b_n^{-4} \varepsilon^{-4} \Big[\sum_{i=1}^{b_n} \mathrm{E}Y_i^4 + 6 \sum_{i=2}^{b_n} \sigma^2 X_i' \sum_{j=1}^{i-1} \sigma^2 X_j' + (\mathrm{E}S_{b_n}')^4\Big] + \sum_{i=1}^{b_n} \mathrm{P}[|X_i| \geqslant b_n \delta]. \end{aligned}$$

Hence, one can get the following estimate:

(7) $$P[|S_{b_n}| \geq b_n\varepsilon] \leq 2E[S_{b_n}^4/(S_{b_n}^4+b_n^4\varepsilon^4)]$$
$$\leq 16b_n^{-4}\varepsilon^{-4}\Big[\sum_{i=1}^{b_n} EY_i^4+6\sum_{i=2}^{b_n}\sigma^2 X_i'\sum_{j=1}^{i-1}\sigma^2 X_j'+(ES_{b_n}')^4\Big]+2\sum_{i=1}^{b_n} P[|X_i|\geq b_n\delta].$$

Taking into account (7), we can state the following

THEOREM 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables with* $EX_n = 0$, $n \geq 1$. *Suppose that* $\{c_n, n \geq 1\}$ *is a sequence of positive real numbers and* $\{b_n, n \geq 1\}$ *is a strictly increasing sequence of positive integers. If*

(i) $$\sum_{n=1}^{\infty} c_n b_n^{-4}\sum_{i=1}^{b_n} EY_i^4 < \infty,$$

(ii) $$\sum_{n=1}^{\infty} c_n b_n^{-4}\sum_{i=2}^{b_n}\sigma^2 X_i'\sum_{j=1}^{i-1}\sigma^2 X_j' < \infty,$$

(iii) $$\sum_{n=1}^{\infty} c_n b_n^{-4}(ES_{b_n}')^4 < \infty,$$

(iv) $$\sum_{n=1}^{\infty} c_n\sum_{i=1}^{b_n} P[|X_i|\geq b_n\delta] < \infty,$$

*then the sequence* $\{X_n, n \geq 1\}$ *satisfies* (4).

For independent identically distributed random variables we can state

COROLLARY 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with* $EX_1 = 0$. *Suppose that* $\{c_n, n \geq 1\}$ *is a sequence of positive real numbers and* $\{b_n, n \geq 1\}$ *is a strictly increasing sequence of positive integers. If, for any given* $\delta > 0$,

$(i)_1$ $$\sum_{n=1}^{\infty} c_n b_n^{-3}E|X_1|^4 I_{[|X_1|<b_n\delta]} < \infty,$$

$(ii)_1$ $$\sum_{n=1}^{\infty} c_n b_n^{-2}\sigma^4(X_1 I_{[|X_1|<b_n\delta]}) < \infty,$$

$(iii)_1$ $$\sum_{n=1}^{\infty} c_n E^4(X_1 I_{[|X_1|<b_n\delta]}) < \infty,$$

$(iv)_1$ $$\sum_{n=1}^{\infty} c_n b_n P[|X_1|\geq b_n\delta] < \infty,$$

*then the sequence* $\{X_n, n \geq 1\}$ *satisfies* (4).

Proof. It is not difficult to verify that under the conditions of Corollary 1 conditions (i)-(iv) reduce to $(i)_1$-$(iv)_1$.

We now note that condition (3), p. 85, implies the statement of Theorem 1.

COROLLARY 2. *Let* $X_1, X_2, \ldots$ *be a sequence of independent random variables with* $\mathrm{E}X_i = 0$.

*Assume that, for some sequence* $\{\lambda_n\}$ *with* $0 < \lambda_n \leqslant 1$, *we have* $E|X_i|^{1+\lambda} < \infty$, $i \geqslant 1$, *where* $\lambda = \sup_n \lambda_n$, *and the sequences* $\{c_n, n \geqslant 1\}$ *and* $\{b_n, n \geqslant 1\}$ *satisfy condition* (3). *Then* (4) *holds.*

Proof. It is enough to prove that (3) implies conditions (i)-(iv). Indeed, by (3), we have

$$\text{(i)}\qquad \sum_{n=1}^{\infty} c_n b_n^{-4} \sum_{i=1}^{b_n} \mathrm{E}Y_i^4 \leqslant 8 \sum_{n=1}^{\infty} c_n b_n^{-4} \sum_{i=1}^{b_n} \mathrm{E}(X_i')^4$$

$$= 16 \sum_{n=1}^{\infty} c_n b_n^{-4} \sum_{i=1}^{b_n} \mathrm{E}X_i^4 I_{[|X_i|<b_n\delta]}$$

$$\leqslant 16\delta^{3-\lambda} \sum_{n=1}^{\infty} c_n b_n^{-1-\lambda_n} \sum_{i=1}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n} < \infty;$$

$$\text{(ii)}\qquad \sum_{n=1}^{\infty} c_n b_n^{-4} \sum_{i=2}^{b_n} \sigma^2 X_i' \sum_{j=1}^{i-1} \sigma^2 X_j'$$

$$\leqslant \sum_{n=1}^{\infty} c_n b_n^{-4} \sum_{i=2}^{b_n} \mathrm{E}|X_i|^2 I_{[|X_i|<b_n\delta]} \sum_{j=1}^{i-1} \mathrm{E}|X_j|^2 I_{[|X_j|<b_n\delta]}$$

$$\leqslant \delta^{2(1-\lambda)} \sum_{n=1}^{\infty} c_n b_n^{-2(1+\lambda_n)} \sum_{i=2}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n} \sum_{j=1}^{i-1} \mathrm{E}|X_j|^{1+\lambda_n}$$

$$\leqslant \delta^{2(1-\lambda)} \sum_{n=1}^{\infty} c_n b_n^{-2(1+\lambda_n)} \Big(\sum_{i=1}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n}\Big)^2 < \infty;$$

$$\text{(iii)}\qquad \sum_{n=1}^{\infty} c_n b_n^{-4} (\mathrm{E}S_{b_n}')^4 = \sum_{n=1}^{\infty} c_n b_n^{-4} \Big(-\sum_{i=1}^{b_n} \mathrm{E}X_i I_{[|X_i|\geqslant b_n\delta]}\Big)^4$$

$$\leqslant \sum_{n=1}^{\infty} c_n b_n^{-4(1+\lambda_n)} \Big(\sum_{i=1}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n}\Big)^4 < \infty;$$

$$\text{(iv)}\qquad \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} \mathrm{P}[|X_i| \geqslant b_n\delta] \leqslant \delta^{1-\lambda} \sum_{n=1}^{\infty} c_n b_n^{-1-\lambda_n} \sum_{i=1}^{b_n} \mathrm{E}|X_i|^{1+\lambda_n} < \infty.$$

It easy to get the following

COROLLARY 3. *Assume that* $\{X_n, n \geqslant 1\}$ *is a sequence of i.i.d. random variables with* $\mathrm{E}X_1 = 0$ *and such that* $\mathrm{E}|X_1|^{1+\lambda} < \infty$ *for some* $\lambda$, $0 < \lambda \leqslant 1$. *If* *the sequences* $\{c_n, n \geqslant 1\}$ *and* $\{b_n, n \geqslant 1\}$ *satisfy the condition*

$$\sum_{n=1}^{\infty} c_n b_n^{-\lambda} < \infty, \tag{8}$$

*then* (4) *holds.*

**3. Complete convergence for 2-dimensional arrays of independent random variables.** Let now $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ be a double sequence of independent random variables. The aim of this section is to extend Theorem 1 to double sequences of independent random variables.

Assume that $\{c_{mn}, m \geqslant 1, n \geqslant 1\}$ is a sequence of positive real numbers, and $\{a_n, n \geqslant 1\}$, $\{b_n, n \geqslant 1\}$ be strictly increasing sequences of positive integers.

Let $\delta > 0$, and put

$$X'_{ij} = X_{ij} I_{[|X_{ij}| < a_m b_n \delta]}, \; X''_{ij} = X_{ij} I_{[|X_{ij}| \geqslant a_m b_n \delta]} \qquad (1 \leqslant i \leqslant a_m, \, 1 \leqslant j \leqslant b_n),$$

$$Y_{ij} = X'_{ij} - \mathrm{E}X'_{ij}, \qquad S'_{kl} = \sum_{i=1}^{k} \sum_{j=1}^{l} X_{ij}, \qquad S^*_{kl} = \sum_{i=1}^{k} \sum_{j=1}^{l} Y_{ij}.$$

Note that

$$\begin{aligned}
(\sum_{i=1}^{a_m} \sum_{j=1}^{b_n} x_{ij})^4 = & \sum_{i=1}^{a_m} (\sum_{j=1}^{b_n} x_{ij}^4) + \\
& + 6 \sum_{i=1}^{a_m} (\sum_{j=2}^{b_n} x_{ij}^2 \sum_{k=1}^{j-1} x_{ik}^2) + 12 \sum_{i=1}^{a_m} (\sum_{j=1}^{b_n} x_{ij}^2 \sum_{\substack{k=2 \\ k \neq j}}^{b_n} x_{ik} \sum_{\substack{l=1 \\ l \neq j}}^{k-1} x_{il} + \\
& + 4 \sum_{i=1}^{a_m} (\sum_{j=2}^{b_n} x_{ij} \sum_{k=1}^{j-1} x_{ik}^3) + 4 \sum_{i=1}^{a_m} (\sum_{j=2}^{b_n} x_{ij}^3 \sum_{k=1}^{j-1} x_{ik}) + \\
& + 24 \sum_{i=1}^{a_m} (\sum_{j=4}^{b_n} x_{ij} \sum_{k=3}^{j-1} x_{ik} \sum_{l=2}^{k-1} x_{il} \sum_{s=1}^{l-1} x_{is}) + \\
& + 6 \sum_{i=2}^{a_m} (\sum_{j=1}^{b_n} x_{ij}^2) \sum_{k=1}^{i-1} (\sum_{l=1}^{b_n} x_{kl}^2) + \\
& + 12 \sum_{i=2}^{a_m} (\sum_{j=1}^{b_n} x_{ij}^2) \sum_{k=1}^{i-1} (\sum_{l=2}^{b_n} x_{kl} \sum_{t=1}^{l-1} x_{kt}) + \\
& + 12 \sum_{i=2}^{a_m} (\sum_{j=2}^{b_n} x_{ij} \sum_{s=1}^{j-1} x_{is}) \sum_{k=1}^{i-1} (\sum_{l=1}^{b_n} x_{kl}^2) + \\
& + 24 \sum_{i=2}^{a_m} (\sum_{j=2}^{b_n} x_{ij} \sum_{s=1}^{j-1} x_{is}) \sum_{k=1}^{i-1} (\sum_{l=2}^{b_n} x_{kl} \sum_{t=1}^{l-1} x_{kt}) + \\
& + 12 \sum_{i=1}^{a_m} (\sum_{j=1}^{b_n} x_{ij})^2 \sum_{\substack{k=2 \\ k \neq i}}^{a_m} (\sum_{j=1}^{b_n} x_{kj}) \sum_{\substack{l=1 \\ l \neq i}}^{k-1} (\sum_{j=1}^{b_n} x_{lj}) +
\end{aligned} \tag{9}$$

$$+4\sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n} x_{ij}\Big)\sum_{k=1}^{i-1}\Big(\sum_{j=1}^{b_n} x_{kj}\Big)^3+$$

$$+4\sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n} x_{ij}\Big)^3\sum_{k=1}^{i-1}\Big(\sum_{j=1}^{b_n} x_{kj}\Big)+$$

$$+24\sum_{i=4}^{a_m}\Big(\sum_{j=1}^{b_n} x_{ij}\Big)\sum_{k=3}^{i-1}\Big(\sum_{j=1}^{b_n} x_{kj}\Big)\sum_{l=2}^{k-1}\Big(\sum_{j=1}^{b_n} x_{lj}\Big)\sum_{s=1}^{l-1}\Big(\sum_{j=1}^{b_n} x_{sj}\Big).$$

Using (9) we get

$$\mathrm{E}(S^*_{a_m b_n})^4 = \sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{E}Y_{ij}^4+6\sum_{i=1}^{a_m}\Big(\sum_{j=2}^{b_n}\sigma^2 X'_{ij}\sum_{l=1}^{j-1}\sigma^2 X'_{il}\Big)+$$

$$+6\sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n}\sigma^2 X'_{ij}\Big)\sum_{k=1}^{i-1}\Big(\sum_{l=1}^{b_n}\sigma^2 X'_{kl}\Big).$$

Moreover, it is not difficult to see that

$$\mathrm{E}\big[S^4_{a_m b_n}/(S^4_{a_m b_n}+(a_m b_n)^4\varepsilon^4)\big]I_{[S_{a_m b_n}=S'_{a_m b_n}]}+\mathrm{P}[S_{a_m b_n}\neq S'_{a_m b_n}]$$

$$\leqslant 8(a_m b_n)^{-4}\varepsilon^{-4}\Big[\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{E}Y_{ij}^4+6\sum_{i=1}^{a_m}\Big(\sum_{j=1}^{b_n}\sigma^2 X'_{ij}\sum_{l=1}^{j-1}\sigma^2 X'_{il}\Big)+$$

$$+6\sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n}\sigma^2 X'_{ij}\Big)\sum_{k=1}^{i-1}\Big(\sum_{l=1}^{b_n}\sigma^2 X'_{kl}\Big)+\mathrm{E}^4 S'_{a_m b_n}\Big]+$$

$$+\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{P}[|X_{ij}|\geqslant a_m b_n\delta].$$

Hence, we have the estimate

$$\mathrm{P}[|S_{a_m b_n}|\geqslant a_m b_n\varepsilon]\leqslant 2\mathrm{E}[S^4_{a_m b_n}/(S^4_{a_m b_n}+a_m^4 b_n^4\varepsilon^4)] \tag{10}$$

$$\leqslant 16a_m^{-4}b_n^{-4}\varepsilon^{-4}\Big[\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{E}Y_{ij}^4+6\sum_{i=1}^{a_m}\Big(\sum_{j=2}^{b_n}\sigma^2 X'_{ij}\sum_{l=1}^{j-1}\sigma^2 X'_{il}\Big)+$$

$$+6\sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n}\sigma^2 X'_{ij}\Big)\sum_{k=1}^{i-1}\Big(\sum_{l=1}^{b_n}\sigma^2 X'_{kl}\Big)+\mathrm{E}^4 S'_{a_m b_n}\Big]+\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{P}[|X_{ij}|\geqslant a_m b_n\delta].$$

Taking into account (10) we can state the following

THEOREM 2. *Let* $\{X_{mn}, m\geqslant 1, n\geqslant 1\}$ *be a double sequence of independent random variables with* $\mathrm{E}X_{mn}=0$, $m\geqslant 1$, $n\geqslant 1$. *Suppose that* $\{c_{mn}, m\geqslant 1, n\geqslant 1\}$ *is a double sequence of positive real numbers and let* $\{a_m, m\geqslant 1\}$ *and* $\{b_n, n\geqslant 1\}$ *be strictly increasing sequences of positive integers. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{mn}(a_m b_n)^{-4}\Big(\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{E}Y_{ij}^4\Big)<\infty, \tag{I}$$

$$\text{(II)} \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}(a_m b_n)^{-4}\Big[\sum_{i=1}^{a_m}\Big(\sum_{j=2}^{b_n}\sigma^2 X'_{ij}\sum_{l=1}^{j-1}\sigma^2 X'_{il}\Big)+ + \sum_{i=2}^{a_m}\Big(\sum_{j=1}^{b_n}\sigma^2 X'_{ij}\Big)\sum_{k=1}^{i-1}\Big(\sum_{l=1}^{b_n}\sigma^2 X'_{kl}\Big)\Big] < \infty,$$

$$\text{(III)} \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}(a_m b_n)^{-4}(\mathrm{E}S'_{a_m b_n})^4 < \infty,$$

$$\text{(IV)} \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\sum_{i=1}^{a_m}\sum_{j=1}^{b_n}\mathrm{P}[|X_{ij}| \geqslant a_m b_n \delta] < \infty,$$

*then the double sequence* $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ *satisfies*

$$(11) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\mathrm{P}[|S_{a_m b_n}| \geqslant a_m b_n \varepsilon] < \infty.$$

For independent identically distributed random variables we can state the following

Corollary 1. *Let* $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ *be a double sequence of i.i.d. random variables with* $\mathrm{E}X_{11} = 0$. *Suppose that* $\{c_{mn}, m \geqslant 1, n \geqslant 1\}$ *is a double sequence of positive real numbers and let* $\{a_m, m \geqslant 1\}$ *and* $\{b_n, n \geqslant 1\}$ *be strictly increasing sequences of positive integers. If*

$$\text{(I)}_1 \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}(a_m b_n)^{-3}\mathrm{E}|X_{11}|^4 I_{[|X_{11}| < a_m b_n \delta]} < \infty,$$

$$\text{(II)}_1 \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}(a_m b_n)^{-2}\sigma^4(X_{11} I_{[|X_{11}| < a_m b_n \delta]}) < \infty,$$

$$\text{(III)}_1 \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}[\mathrm{E}X_{11} I_{[|X_{11}| < a_m b_n \delta]}]^4 < \infty,$$

$$\text{(IV)}_1 \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}(a_m b_n)\mathrm{P}[|X_{11}| \geqslant a_m b_n \delta] < \infty,$$

*then* (10) *holds.*

Proof of Corollary 1 follows from the considerations of the proof of Corollary 1 after Theorem 1.

Corollary 2. *Let* $\{X_{mn}, m \geqslant 1, n \geqslant 1\}$ *be a double sequence of i.i.d. random variables with* $\mathrm{E}X_{11} = 0$ *and* $\mathrm{E}|X_{11}|^2 \log^+ |X_{11}| < \infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{P}[|S_{mn}| \geqslant mn\varepsilon] < \infty.$$

Proof. Letting $a_n = n$, $b_m = m$, $c_{mn} = 1$, $m \geqslant 1$ and $n \geqslant 1$ in Corollary 1, we see that $(\mathrm{I})_1$ - $(\mathrm{IV})_1$ are satisfied. Indeed, if we let $d_k$ to be the cardinality of $\{(m, n)\colon mn = k\}$, and $F(x)$ be the distribution of $X_{11}$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}{}'(mn)^{-3}\mathrm{E}|X_{11}|^4 I_{[|X_{11}|<mn]} = \sum_{k=1}^{\infty}(d_k/k^3)\mathrm{E}|X_{11}|^4 I_{[|X_{11}|<k]}$$
$$= \sum_{k=1}^{\infty}(d_k/k^3)\int_{-k}^{k} x^4\,dF(x)$$
$$\leqslant \sum_{i=1}^{\infty} 1/(i+1)\sum_{k=i+1}^{\infty}(d_k/k^2)\Big(\int_{-(i+1)}^{-i} x^4\,dF(x)+\int_{i}^{i+1} x^4\,dF(x)\Big)$$
$$\leqslant C\sum_{i=1}^{\infty}\log i/(i+1)^2\Big(\int_{-(i+1)}^{-i} x^4\,dF(x)+\int_{i}^{i+1} x^4\,dF(x)\Big)$$
$$\leqslant C\sum_{i=1}^{\infty}\log i\Big(\int_{-(i+1)}^{-i} x^2\,dF(x)+\int_{i}^{i+1} x^2\,dF(x)\Big)\leqslant C\mathrm{E}|X_{11}|^2\log^+|X_{11}|<\infty,$$

where

$$\sum_{k=i+1}^{\infty}(d_k/k^2) = O\big(\log i/(i+1)\big),$$

$\log^+ a = \log(\max(a, 1))$, and $C$ is a positive constant.

Thus $(\mathrm{I})_1$ is satisfied. Moreover, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-2}\sigma^4(X_{11}I_{[|X_{11}|<mn]})<\infty,$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{E}^4 X_{11} I_{[|X_{11}|<mn]} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{E}^4 X_{11} I_{[|X_{11}|\geqslant mn]}$$
$$\leqslant \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathrm{E}^4 X_{11}^2/(mn) I_{[|X_{11}|\geqslant mn]} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-4}\mathrm{E}^4 X_{11}^2 I_{[|X_{11}|\geqslant mn]}<\infty$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn\,\mathrm{P}[|X_{11}|\geqslant mn] = \sum_{k=1}^{\infty} kd_k\,\mathrm{P}[|X_{11}|\geqslant k]$$
$$= \sum_{i=1}^{\infty}\Big(\sum_{k=1}^{i} kd_k\Big)\Big(\int_{-(i+1)}^{-i} dF(x)+\int_{i}^{i+1} dF(x)\Big)$$
$$\leqslant \sum_{i=1}^{\infty} i\sum_{k=1}^{i} d_k\Big(\int_{-(i+1)}^{-i} dF(x)+\int_{i}^{i+1} dF(x)\Big)$$
$$\leqslant C\sum_{i=1}^{\infty} i^2\log i\Big(\int_{-(i+1)}^{-i} dF(x)+\int_{i}^{i+1} dF(x)\Big)\leqslant C\mathrm{E}X_{11}^2\log^+|X_{11}|<\infty,$$

where $\sum_{k=1}^{i} d_k \sim i\log i$ ([5], p. 263), $C$ is a positive constant.

The last estimates complete the proof of Corollary 2.

Our considerations lead us to the following extension of Theorem 1 [1] to the case of double sequences of independent random variables:

THEOREM 3. *Let* $\{X_{mn}, m \geq 1, n \geq 1\}$ *be a double sequence of independent random variables with* $\mathrm{E}X_{mn} = 0$, $m \geq 1$, $n \geq 1$, *and let, for some sequence* $\{\lambda_{mn}, m \geq 1, n \geq 1\}$ *with* $0 < \lambda_{mn} \leq 1$, *be* $\mathrm{E}|X_{mn}|^{1+\lambda} < \infty$, $m \geq 1$, $n \geq 1$, *where* $\lambda = \sup_{m,n} \lambda_{mn}$. *Suppose that* $\{c_{mn}, m \geq 1, n \geq 1\}$ *is a double sequence of positive real numbers and let* $\{a_m, m \geq 1\}$ *and* $\{b_n, n \geq 1\}$ *be strictly increasing sequences of positive integers. If*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} (a_m b_n)^{-1-\lambda_{mn}} \sum_{i=1}^{a_m} \sum_{j=1}^{b_n} \mathrm{E}|X_{ij}|^{1+\lambda_{mn}} < \infty,$$

*then* (10) *holds.*

The results of Section 3 generalize results of [4].

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