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LOWER BOUNDS FOR DISCRETE APPROXIMATIONS TO SUMS OF m-DEPENDENT RANDOM VARIABLES

BY

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Abstract. Lower bounds for the second order Poisson, compound Poisson, negative binomial and binomial approximations to the sum of 1-dependent random variables are obtained for the Kolmogorov and local metrics. The results are then applied to sums of independent indicators and runs statistics.

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1. INTRODUCTION

For the most part, work on discrete approximations deals only with upper bounds. However, lower bounds are also of interest as they often provide some insight as to the correct order of the upper bounds. Unfortunately, lower bounds, especially for dependent random variables, are scarce in the literature. This problem for the case of Poisson approximation to Bernoulli summands is investigated in detail in [2, Chapter 3]. However, the Poisson distribution has just one parameter which is usually chosen to match the mean of the approximated distribution. Our aim in this paper is to obtain lower bounds for two-parametric discrete approximations to sums of weakly dependent discrete random variables. These results complement the upper bounds derived in [5].

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. A sequence $\{X_k\}_{k\geqslant 1}$ of random variables is called *m-dependent* if, for $1 < s < t < \infty$, $t-s > m$, the sigma-algebras generated by $X_1, \dots, X_s$ and $X_t, X_{t+1}, \dots$ are independent. We can reduce each sum of m-dependent variables to a sum of 1-dependent ones, by grouping consecutive m summands. Therefore, we consider hereafter, without loss of generality, sums $S_n = X_1 + \dots + X_n$ of non-identically-distributed 1-dependent random variables concentrated on $\mathbb{Z}_+$. We denote the distribution function and the characteristic function of S_n by $F_n(x)$ and $\widehat{F}_n(t)$, respectively. Similarly, for a signed measure M concentrated on $\mathbb{Z}_+$, let $\widehat{M}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}tk} M\{k\}$ denote its Fourier–Stieltjes transform. The local, Kolmogorov and total variation norms are defined as

$$\|M\|_\infty = \sup_k |M\{k\}|, \qquad |M|_K = \sup_k |M\{[0,k]\}|, \qquad \|M\| = \sum_{k=0}^{\infty} |M\{k\}|,$$

respectively. Observe that lower bounds for $|M|_K$ are also lower bounds for $\|M\|$, since $|M|_K \leqslant \|M\|$.

Let I_a denote the distribution concentrated at a real a and set $I = I_0$. Henceforth, the products and powers of measures are understood in the convolution sense. Further, for a measure M, we set $M^0 = I$ and $\exp\{M\} = \sum_{k=0}^{\infty} M^k/k!$.

Next, we introduce the two-parametric approximations considered in this paper. Let

$$\Gamma_{1n} = \mathrm{E}\, S_n, \qquad \Gamma_{2n} = \frac{1}{2}(\mathrm{Var}\, S_n - \mathrm{E}\, S_n).$$

We denote by $\Pi + \Pi_1 = \exp\{\Gamma_{1n}(I_1 - I)\} + \exp\{\Gamma_{1n}(I_1 - I)\}\Gamma_{2n}(I_1 - I)^2)$ the standard second order Poisson approximation. Its Fourier–Stieltjes transform is $\widehat{\Pi}(t) + \widehat{\Pi}_1(t) = \exp\{\Gamma_{1n} z(t)\}(1 + \Gamma_{2n} z^2(t))$; here and henceforth, $z(t) = \mathrm{e}^{\mathrm{i}t} - 1$. Let $\mathrm{G} = \exp\{\Gamma_{1n}(I_1 - I) + \Gamma_{2n}(I_1 - I)^2\}$ be the two-parametric compound Poisson measure with Fourier–Stieltjes transform $\widehat{\mathrm{G}}(t) = \exp\{\Gamma_{1n} z(t) + \Gamma_{2n} z^2(t)\}$. Approximations to G have been used in many papers: see, for example, [1], [8], and the references therein. Note that when $\Gamma_{2n} < 0$, the measure G becomes a signed measure. If $\mathrm{E}\, S_n < \mathrm{Var}\, S_n$, it is natural to consider the approximation to the negative binomial distribution, defined as

$$\mathrm{NB}\{j\} = \frac{\Gamma(r+j)}{j!\Gamma(r)}\, \overline{q}^r (1-\overline{q})^j, \qquad \frac{r(1-\overline{q})}{\overline{q}} = \Gamma_{1n}, \qquad r\left(\frac{1-\overline{q}}{\overline{q}}\right)^2 = 2\Gamma_{2n},$$

where $j \in \mathbb{Z}_+$, $r > 0$, $0 < \overline{q} < 1$ and $\Gamma(\cdot)$ is the gamma function.

Note that

$$\widehat{\mathrm{NB}}(t) = \left(\frac{\overline{q}}{1 - (1-\overline{q})\mathrm{e}^{\mathrm{i}t}}\right)^r = \left(1 - \frac{(1-\overline{q}) z(t)}{\overline{q}}\right)^{-r}.$$

The negative binomial approximation has been successfully applied to a special case of k-dependent indicators in [11]. If $\mathrm{Var}\, S_n < \mathrm{E}\, S_n$, we consider the approximation to the binomial distribution defined by

$$\begin{aligned} &\mathrm{Bi} = ((1-\overline{p})I + \overline{p} I_1)^n, \qquad N = \lfloor \tilde{N} \rfloor, \qquad \tilde{N} = \frac{\Gamma_{1n}^2}{2|\Gamma_{2n}|}, \qquad \bar{p} = \frac{\Gamma_{1n}}{N}, \\ &\widehat{\mathrm{Bi}}(t) = (1 + \overline{p} z(t))^N. \end{aligned} \tag{1.1}$$

Here, we use $\lfloor \tilde{N} \rfloor$ to denote the integer part of $\tilde{N}$, that is, $\tilde{N} = \lfloor \tilde{N} \rfloor + \epsilon$ for some $0 \leqslant \epsilon < 1$. Note that the symbols $\overline{q}$ and $\overline{p}$ are unrelated and $\overline{q} + \overline{p} \neq 1$ in general. The binomial approximation to dependent indicators has been discussed in [9].

We define the jth factorial moment of X_k by

$$\nu_j(k) = \mathrm{E}\, X_k(X_k - 1) \cdots (X_k - j + 1) \quad (k = 1, \dots, n;\ j = 1, 2, \dots).$$

For simplicity, we assume that $X_k \equiv 0$ and $\nu_j(k) = 0$ if $k \leqslant 0$ and $\sum_k^n = 0$ if $k > n$. Next, we introduce some technical notations which are used in the proofs. Let $\{Y_k\}_{k\geqslant 1}$ be a sequence of arbitrary real- or complex-valued random variables. We assume that $\widehat{\mathrm{E}}(Y_1) = \mathrm{E}\, Y_1$ and, for $k \geqslant 2$, define $\widehat{\mathrm{E}}(Y_1, \dots, Y_k)$ by

$$\widehat{\mathrm{E}}(Y_1, Y_2, \dots, Y_k) = \mathrm{E}\, Y_1 Y_2 \cdots Y_k - \sum_{j=1}^{k-1} \widehat{\mathrm{E}}(Y_1, \dots, Y_j)\, \mathrm{E}\, Y_{j+1} \cdots Y_k.$$

For $k \geqslant 2$, let

$$\begin{aligned}
\widehat{\mathrm{E}}^+(X_1) = \mathrm{E}\, X_1, \quad \widehat{\mathrm{E}}^+(X_1, X_2) &= \mathrm{E}\, X_1 X_2 + \mathrm{E}\, X_1\, \mathrm{E}\, X_2, \\
\widehat{\mathrm{E}}^+(X_1, \dots, X_k) &= \mathrm{E}\, X_1 \dots X_k \\
&\quad + \sum_{j=1}^{k-1} \widehat{\mathrm{E}}^+(X_1, \dots, X_j)\, \mathrm{E}\, X_{j+1} X_{j+2} \cdots X_k, \\
\widehat{\mathrm{E}}_2^+(X_{k-1}, X_k) &= \widehat{\mathrm{E}}^+(X_{k-1}(X_{k-1} - 1), X_k) \\
&\quad + \widehat{\mathrm{E}}^+(X_{k-1}, X_k(X_k - 1)), \\
\widehat{\mathrm{E}}_2^+(X_{k-2}, X_{k-1}, X_k) &= \widehat{\mathrm{E}}^+\big(X_{k-2}(X_{k-2} - 1), X_{k-1}, X_k\big) \\
&\quad + \widehat{\mathrm{E}}^+\big(X_{k-2}, X_{k-1}(X_{k-1} - 1), X_k\big), \\
\widehat{\mathrm{E}}_3^+(X_{k-1}, X_k) &= \widehat{\mathrm{E}}^+\big(X_{k-1}(X_{k-1} - 1)(X_{k-1} - 2), X_k\big) \\
&\quad + \widehat{\mathrm{E}}^+\big(X_{k-1}(X_{k-1} - 1), X_k(X_k - 1)\big) \\
&\quad + \widehat{\mathrm{E}}^+\big(X_{k-1}, X_k(X_k - 1)(X_k - 2)\big)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{3n} = \frac{1}{6} &\sum_{k=1}^{n} \big(\nu_3(k) - 3\nu_1(k)\nu_2(k) + 2\nu_1^3(k)\big) \\
&- \sum_{k=2}^{n} (\nu_1(k-1) + \nu_1(k))\widehat{\mathrm{E}}(X_{k-1}, X_k) \\
&+ \frac{1}{2} \sum_{k=2}^{n} \big(\widehat{\mathrm{E}}(X_{k-1}(X_{k-1} - 1), X_k) + \widehat{\mathrm{E}}(X_{k-1}, X_k(X_k - 1))\big) \\
&+ \sum_{k=3}^{n} \widehat{\mathrm{E}}(X_{k-2}, X_{k-1}, X_k),
\end{aligned}$$

$$
\begin{aligned}
R_0 &= \sum_{k=1}^{n} \{\nu_2(k) + \nu_1^2(k) + \mathrm{E}\, X_{k-1}X_k\},\\
R_1 &= \sum_{k=1}^{n} \big\{\nu_1^3(k) + \nu_1(k)\nu_2(k) + \nu_3(k)\\
&\quad + [\nu_1(k-2) + \nu_1(k-1) + \nu_1(k)]\, \mathrm{E}\, X_{k-1}X_k\\
&\quad + \widehat{\mathrm{E}}_2^+(X_{k-1}, X_k) + \widehat{\mathrm{E}}^+(X_{k-2}, X_{k-1}, X_k)\big\},\\
R_2 &= \sum_{k=1}^{n} \Big\{\nu_1^4(k) + \nu_2^2(k) + \nu_4(k)\\
&\quad + [\nu_1(k-1) + \nu_1(k)][\nu_3(k) + \widehat{\mathrm{E}}_2^+(X_{k-1}, X_k)]\\
&\quad + (\mathrm{E}\, X_{k-1}X_k)^2 + \sum_{l=0}^{3} \nu_1(k-l)\widehat{\mathrm{E}}^+(X_{k-2}, X_{k-1}, X_k)\\
&\quad + \widehat{\mathrm{E}}_2^+(X_{k-2}, X_{k-1}, X_k) + \widehat{\mathrm{E}}_3^+(X_{k-1}, X_k)\\
&\quad + \widehat{\mathrm{E}}^+(X_{k-3}, X_{k-2}, X_{k-2}, X_k)\Big\}.
\end{aligned}
$$

It is shown in [5] that

$$
\begin{aligned}
\Gamma_{1n} &= \sum_{k=1}^{n} \nu_1(k),\\
\Gamma_{2n} &= \frac{1}{2}\sum_{k=1}^{n} (\nu_2(k) - \nu_1^2(k)) + \sum_{k=2}^{n} \widehat{\mathrm{E}}(X_{k-1}, X_k).
\end{aligned} \tag{1.2}
$$

As our method of proof does not yield small constants, we focus on the order of the accuracy of approximation. The symbol C denotes generic (but different) positive absolute constants. Sometimes, we supply C with indices, to avoid ambiguity. Let θ denote a real or complex number satisfying $|\theta| \leqslant 1$.

2. THE MAIN RESULTS

In this paper, we use the assumptions from [5]:

$$
\nu_1(k) \leqslant 1/100, \qquad \nu_2(k) \leqslant \nu_1(k), \qquad \nu_4(k) < \infty \qquad (k = 1, \dots, n), \tag{2.1}
$$

$$
\sum_{k=1}^{n} \nu_2(k) \leqslant \frac{\Gamma_{1n}}{20}, \qquad \sum_{k=2}^{n} |\mathrm{Cov}(X_{k-1}, X_k)| \leqslant \frac{\Gamma_{1n}}{20}. \tag{2.2}
$$

Though we assume small means and even smaller second factorial moments, they nevertheless include the case where all moments are of the constant order.

Our primary goal is to obtain estimates with the so-called ‘magic factor’, which corresponds to the case $\Gamma_{1n} \geqslant 1$ (see [2, Introduction]).

THEOREM 2.1. *Suppose that* $\Gamma_{1n} \geqslant 1$ *and conditions* (2.1) *and* (2.2) *hold. Then, for each* $n \geqslant 1$ *and any* $\alpha \geqslant 1$,

$$|F_n - \Pi - \Pi_1|_K \geqslant \frac{C_1\Gamma_{2n}^2}{\alpha^4\Gamma_{1n}^2} - \frac{R_1}{\alpha^3\Gamma_{1n}\sqrt{\Gamma_{1n}}}\,, \tag{2.3}$$

$$|F_n - \mathrm{G}|_K \geqslant \frac{C_2}{\alpha^3}\left(\frac{|\Gamma_{3n}|}{\Gamma_{1n}\sqrt{\Gamma_{1n}}} - \frac{R_1^2}{\alpha\Gamma_{1n}^3} - \frac{R_2}{\alpha\Gamma_{1n}^2}\right). \tag{2.4}$$

Also, when $\Gamma_{2n} > 0$,

$$|F_n - \mathrm{NB}|_K \geqslant \frac{C_3}{\alpha^3}\left\{\frac{1}{\Gamma_{1n}\sqrt{\Gamma_{1n}}}\left|\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right| - \frac{R_1^2}{\alpha\Gamma_{1n}^3} - \frac{R_2}{\alpha\Gamma_{1n}^2} - \frac{\Gamma_{2n}^3}{\alpha\Gamma_{1n}^4}\right\}, \tag{2.5}$$

and when $\Gamma_{2n} < 0$,

(2.6)

$$|F_n - \mathrm{Bi}|_K \geqslant \frac{C_4}{\alpha^3}\left\{\frac{1}{\Gamma_{1n}\sqrt{\Gamma_{1n}}}\left|\Gamma_{3n} - \frac{N\overline{p}^3}{3}\right| - \frac{R_1^2}{\alpha\Gamma_{1n}^3} - \frac{R_2}{\alpha\Gamma_{1n}^2} - \frac{|\Gamma_{2n}|^3}{\alpha\Gamma_{1n}^4}\right\} - \frac{C_5\Gamma_{2n}^2\epsilon}{\alpha^2\Gamma_{1n}^3}.$$

Observe that all the estimates contain the quantity α which can be chosen arbitrarily. If all remainder terms are of the same magnitude, then we can get rid of the negative terms by taking α sufficiently large. Note also that the assumptions of Theorem 2.1 do not ensure nontriviality of estimates: in some cases, the right-hand side estimates may be negative.

If all variables are bounded by some positive constant, some estimates can be simplified.

COROLLARY 2.1. *Suppose that* $\Gamma_{1n} \geqslant 1$, $X_k \leqslant C_0$ *for* $1 \leqslant k \leqslant n$, *and conditions* (2.1) *and* (2.2) *are satisfied. Then, for each* $n \geqslant 1$ *and any* $\alpha \geqslant 1$,

$$|F_n - \mathrm{G}|_K \geqslant \frac{C_6}{\alpha^3\Gamma_{1n}\sqrt{\Gamma_{1n}}}\left(|\Gamma_{3n}| - \frac{R_1}{\alpha}\right).$$

If in addition $\Gamma_{2n} > 0$, *then*

$$|F_n - \mathrm{NB}|_K \geqslant \frac{C_7}{\alpha^3\Gamma_{1n}\sqrt{\Gamma_{1n}}}\left(\left|\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right| - \frac{R_1}{\alpha\sqrt{\Gamma_{1n}}} - \frac{\Gamma_{2n}^3}{\alpha\Gamma_{1n}^{5/2}}\right).$$

The corollary immediately follows from the fact that for X_i bounded, we have $R_2 \leqslant CR_1$ and $R_1 \leqslant C\Gamma_{1n}$. As shown in [5], under the assumptions of Corollary 2.1, we have

$$\|F_n - \Pi - \Pi_1\| \leqslant C(R_1 + R_0^2\Gamma_{1n}^{-1/2})\Gamma_{1n}^{-3/2}, \qquad \|F_n - \mathrm{G}\| \leqslant CR_1\Gamma_{1n}^{-3/2}. \tag{2.7}$$

If $\Gamma_{2n} > 0$ or $\Gamma_{2n} < 0$ then, respectively,

$$\|F_n - \mathrm{NB}\| \leqslant C(R_1 + \Gamma_{2n}^2\Gamma_{1n}^{-1})\Gamma_{1n}^{-3/2}, \tag{2.8}$$

$$\|F_n - \mathrm{Bi}\| \leqslant C(R_1 + \Gamma_{2n}^2\Gamma_{1n}^{-1})\Gamma_{1n}^{-3/2}. \tag{2.9}$$

In general, comparison of (2.7)–(2.9) to Corollary 2.1 is somewhat complicated. On the other hand, if we do not deal with triangular arrays, that is, if no X_i depends on n, then $\Gamma_{in} = R_i = O(n)$, and therefore the upper and lower bounds for the G, NB and Bi approximations are of the same order $Cn^{-1/2}$. As can be expected, in this case, the second order Poisson approximation is bounded from below by a factor some C. In the next section, we demonstrate that lower bounds can be of the correct order even if $X_1, X_2, \dots$ form a triangular array.

For the local metric, all the estimates are smaller by a factor $\sqrt{\Gamma_{1n}}$, as seen in the next result.

THEOREM 2.2. *Suppose that* $\Gamma_{1n} \geqslant 1$ *and conditions* (2.1) *and* (2.2) *are satisfied. Then, for each* $n \geqslant 1$ *and any* $\alpha \geqslant 1$,

$$\|F_n - \Pi - \Pi_1\|_\infty \geqslant \frac{C_8\Gamma_{2n}^2}{\alpha^4\Gamma_{1n}^2\sqrt{\Gamma_{1n}}} - \frac{R_1}{\alpha^3\Gamma_{1n}^2}, \tag{2.10}$$

$$\|F_n - \mathrm{G}\|_\infty \geqslant \frac{C_9}{\alpha^3}\left(\frac{|\Gamma_{3n}|}{\Gamma_{1n}^2} - \frac{R_1^2}{\alpha\Gamma_{1n}^3\sqrt{\Gamma_{1n}}} - \frac{R_2}{\alpha\Gamma_{1n}^2\sqrt{\Gamma_{1n}}}\right). \tag{2.11}$$

In addition, when $\Gamma_{2n} > 0$,

(2.12)
$$\|F_n - \mathrm{NB}\|_\infty \geqslant \frac{C_{10}}{\alpha^3}\left\{\frac{1}{\Gamma_{1n}^2}\left|\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right| - \frac{R_1^2}{\alpha\Gamma_{1n}^3\sqrt{\Gamma_{1n}}} - \frac{R_2}{\alpha\Gamma_{1n}^2\sqrt{\Gamma_{1n}}} - \frac{\Gamma_{2n}^3}{\alpha\Gamma_{1n}^{9/2}}\right\},$$

and when $\Gamma_{2n} < 0$,

$$\begin{aligned}\|F_n - \mathrm{Bi}\|_\infty \geqslant \frac{C_{11}}{\alpha^3}\Bigg\{&\frac{1}{\Gamma_{1n}^2}\left|\Gamma_{3n} - \frac{N\overline{p}^3}{3}\right| - \frac{R_1^2}{\alpha\Gamma_{1n}^3\sqrt{\Gamma_{1n}}}\\ &- \frac{R_2}{\alpha\Gamma_{1n}^2\sqrt{\Gamma_{1n}}} - \frac{|\Gamma_{2n}|^3}{\alpha\Gamma_{1n}^4\sqrt{\Gamma_{1n}}}\Bigg\} - \frac{C_{12}\Gamma_{2n}^2\epsilon}{\alpha^2\Gamma_{1n}^3\sqrt{\Gamma_{1n}}}.\end{aligned} \tag{2.13}$$

It should be noted that our method of proof can also be applied to prove estimates without the 'magic factor', that is, for small Γ_1. As an illustration we formulate one result for $\Pi + \Pi_1$ and G.

THEOREM 2.3. *Suppose that* $\Gamma_{1n} < 1$ *and conditions* (2.1) *and* (2.2) *hold. Then, for each* $n \geqslant 1$ *and any* $\alpha \geqslant 1$,

$$|F_n - \Pi - \Pi_1|_K \geqslant \frac{C_{13}}{\alpha^3}\left(|\Gamma_{3n}| - \frac{R_1^2 + R_2 + \Gamma_{2n}^2}{\alpha}\right),$$

$$|F_n - \mathrm{G}|_K \geqslant \frac{C_{14}}{\alpha^3}\left(|\Gamma_{3n}| - \frac{R_1^2 + R_2}{\alpha}\right).$$

The same accuracy is achieved for the local norm.

3. SOME APPLICATIONS

In this section, we discuss some applications of the results stated in the previous section.

3.1. Approximations to the Poisson-binomial distribution. The Poisson-binomial distribution, which has been comprehensively investigated in numerous papers, provides a good possibility for checking the accuracy of our estimates. We consider Theorems 2.1 and 2.3 only. Let $W_n = \xi_1 + \cdots + \xi_n$, where the ξ_i are independent Bernoulli variables with $P(\xi_i = 1) = 1 - P(\xi_i = 0) = p_i \leqslant 0.01$. Further, let $\lambda := \Gamma_{1n} = \sum_{i=1}^n p_i$ and $\lambda_j := \sum_{i=1}^n p_i^j$. If $\lambda \geqslant 1$, then $R_1 \leqslant C\lambda_3$, $R_2 \leqslant \lambda_4$, $\Gamma_{2n} = -\lambda_2/2$ and $\Gamma_{3n} = \lambda_3/3$. From Hölder's inequality it follows that $\lambda^2 \leqslant \lambda\lambda_3$. If $\lambda \geqslant 1$, then by choosing a sufficiently large absolute constant α, we obtain

$$|\mathcal{L}(W_n) - \mathrm{G}|_K \geqslant \frac{C_{15}\lambda_3}{\lambda\sqrt{\lambda}}.$$

This estimate is of the correct order and was already obtained in [7]. The lower bound in (2.3) for the second order Poisson approximation compares well with the upper bound estimate in the following way (see [5]):

$$\frac{C_{16}\lambda_2^2}{\lambda^2} - \frac{C_{17}\lambda_3}{\lambda\sqrt{\lambda}} \leqslant \|\mathcal{L}(W_n) - \Pi - \Pi_1\| \leqslant \frac{C_{18}\lambda_2^2}{\lambda^2} + \frac{C_{19}\lambda_3}{\lambda\sqrt{\lambda}}.$$

If $p_i = p$, then the upper and lower bounds are of the same order Cp^2. Since $\Gamma_{2n} < 0$, we next obtain a lower bound for the binomial approximation, which has not been investigated in the literature. By definition,

$$\overline{p} = \frac{\lambda_2}{\lambda} + \theta\frac{\lambda_2}{N\lambda}, \quad N\overline{p}^3 = \frac{\lambda_2^2}{\lambda} + C\theta\frac{\lambda_2^2}{N\lambda}, \quad \frac{1}{N} \leqslant \frac{C}{\sqrt{\lambda}}.$$

Putting all the estimates in (2.6) and choosing α sufficiently large, we obtain

$$|\mathcal{L}(W_n) - \mathrm{Bi}|_K \geqslant \frac{C_{20}}{\sqrt{\lambda}}\left(\frac{\lambda_3}{\lambda} - \frac{\lambda_2^2}{\lambda^2} - \frac{C_{21}\lambda_3}{\lambda\sqrt{\lambda}}\right), \tag{3.1}$$

where Bi is defined in (1.1).

The upper bound in total variation is of the order

$$\|\mathcal{L}(W_n) - \mathrm{Bi}\| \leqslant \frac{C_{22}}{\sqrt{\lambda}}\left(\frac{\lambda_3}{\lambda} - \frac{\lambda_2^2}{\lambda^2} + \frac{C_{23}\lambda_2^2}{\lambda^2\sqrt{\lambda}}\right) + \exp\{-C_{24}\lambda\}$$

(see [4] for more precise estimates). If all the p_i are different and uniformly bounded away from zero, we get

$$\frac{C_{25}}{\sqrt{n}} \leqslant |\mathcal{L}(W_n) - \mathrm{Bi}|_K \leqslant \|\mathcal{L}(W) - \mathrm{Bi}\| \leqslant \frac{C_{26}}{\sqrt{n}}.$$

On the other hand, if $p_i = p$, then $\mathcal{L}(W_n) = \mathrm{Bi}$ and one can expect the lower bound to be zero, which is not the case in (3.1), as it becomes negative.

3.2. Approximations to (k_1,k_2)-events. Let η_i be independent Bernoulli $Be(p)$ $(0<p<1)$ variables and let $Y_j=(1-\eta_{j-m+1})\cdots(1-\eta_{j-k_2})\eta_{j-k_2+1}\cdots\eta_{j-1}\eta_j$, $j=m,m+1,\dots,n$, $k_1+k_2=m$. Then $N(n;k_1,k_2)=Y_m+Y_{m+1}+\cdots+Y_n$ denotes the number of (k_1,k_2)-events in n Bernoulli trials. We denote the distribution of $N(n;k_1,k_2)$ by H. Let $a(p)=(1-p)^{k_1}p^{k_2}$. It is well known that $N(n;k_1,k_2)$ has limiting Poisson distribution if $na(p)\to\tilde{\lambda}$ (see [6] or [10]). Note also that $N(n;k_1,k_2)$ is the sum of $m(\geqslant 2)$-dependent random variables $Y_1,Y_2,\dots$. However, as already mentioned in the introduction, we can switch to 1-dependent random variables by grouping consecutive summands:

$$\begin{aligned}N(n;k_1,k_2)&=(Y_m+Y_{m+1}+\cdots+Y_{2m-1})\\&\quad+(Y_{2m}+Y_{2m+1}+\cdots+Y_{3m-1})+\cdots\\&=\sum_{j=1}^{K+1}X_j,\end{aligned}$$

where

$$K=\left\lfloor\frac{n-m+1}{m}\right\rfloor=\left\lfloor\frac{n+1}{m}\right\rfloor-1.$$

Further, we assume that $k_1>0$, $k_2>0$, $m\leqslant n/2$, $ma(p)\leqslant 0.01$ and $(n-m+1)a(p)\geqslant 1$. Then, as shown in [5], all the X_j are 1-dependent Bernoulli variables and

$$\Gamma_{1n}=(n-m+1)a(p),\qquad \Gamma_{2n}=-\frac{a^2(p)}{2}[(n-m+1)(2m-1)-m(m-1)],$$

$$\Gamma_{3n}=\frac{a^3(p)}{6}[(n-m+1)(3m-1)(3m-2)-4m(2m-1)(m-1)],$$

$$R_1\leqslant C(n-m+1)m^2a^3(p),\qquad R_2\leqslant C(n-m+1)m^3a^4(p),$$

$$\Gamma_{3n}=\frac{N\overline{p}^3}{3}+\frac{a^3(m)}{6}(n-m+1)m(m-1)+\theta Cm^3a^3(m).$$

Using the theorems of the previous section, we obtain

$$|\mathrm{H}-\mathrm{G}|_K\geqslant\frac{C_{27}m^2a^{3/2}(p)}{\sqrt{n-m+1}},\qquad \|\mathrm{H}-\mathrm{G}\|_\infty\geqslant\frac{C_{28}m^2a(p)}{n-m+1},$$

$$|\mathrm{H}-\mathrm{Bi}|_K\geqslant\frac{C_{29}m^2a^{3/2}(p)}{\sqrt{n-m+1}}\left(1-\frac{C_{30}}{\sqrt{(n-m+1)a(p)}}\right),$$

$$\|\mathrm{H}-\mathrm{Bi}\|_\infty\geqslant\frac{C_{31}m^2a(p)}{n-m+1}\left(1-\frac{C_{32}}{\sqrt{(n-m+1)a(p)}}\right).$$

These estimates match the following upper bounds for the total variation:

$$\|\mathrm{H}-\mathrm{G}\|\leqslant\frac{C_{33}m^2a^{3/2}(p)}{\sqrt{n-m+1}},\qquad \|\mathrm{H}-\mathrm{G}\|_\infty\leqslant\frac{C_{34}m^2a(p)}{n-m+1},$$

$$\|\mathrm{H}-\mathrm{Bi}\| \leqslant \frac{C_{35}m^2a^{3/2}(p)}{\sqrt{n-m+1}}, \quad \|\mathrm{H}-\mathrm{Bi}\|_\infty \leqslant \frac{C_{36}m^2a(p)}{n-m+1}$$

(see Theorem 4.3 and [5, remark at the end of Section 3].

4. AUXILIARY RESULTS

We now present some useful auxiliary results. Let

$$V_1(t)=\frac{N}{3}\overline{p}^3z^3(t), \qquad V_2(t)=\frac{5}{12}N\overline{p}^4|z(t)|^4+\frac{50}{51}\,\frac{\Gamma_{2n}^3\epsilon|z(t)|^2}{\Gamma_{1n}^2},$$

$$\widehat{M_1}(t)=\frac{4}{3}\,\frac{\Gamma_{2n}^2z^3(t)}{\Gamma_{1n}}, \quad \widehat{M_2}(t)=\frac{20}{7}\,\frac{\Gamma_{2n}^3|z(t)|^4}{\Gamma_{1n}^2}.$$

LEMMA 4.1. *If conditions* (2.1) *and* (2.2) *are satisfied, then*

$$|\Gamma_{2n}|\leqslant 0.08\Gamma_{1n}, \quad N\overline{p}^3\leqslant C\frac{\Gamma_{2n}^2}{\Gamma_{1n}}, \quad \overline{p}\leqslant\frac{50|\Gamma_{2n}|}{21\Gamma_{1n}}<\frac{1}{5}, \tag{4.1}$$

$$|\widehat{F}_n(t)-\widehat{\mathrm{G}}(t)|\leqslant CR_1|z(t)|^3, \tag{4.2}$$

$$|\widehat{F}_n(t)-\widehat{\mathrm{G}}(t)(1+\Gamma_{3n}z^3(t))|\leqslant C(R_1^2|z(t)|^6+R_2|z(t)|^4), \tag{4.3}$$

$$|\widehat{G}(t)|\leqslant|\Pi(t)|\exp\{|\Gamma_{2n}|\,|z(t)|^2/2\}\leqslant\exp\{-0.42\Gamma_{1n}|z(t)|^2\}, \tag{4.4}$$

$$\widehat{\mathrm{Bi}}(t)=\widehat{\mathrm{G}}(t)\exp\{\widehat{V}_1(t)+\theta\widehat{V}_2(t)\}, \tag{4.5}$$

$$|\widehat{\mathrm{G}}(t)|\exp\{|\widehat{V}_1(t)|+\widehat{V}_2(t)\}\leqslant 1, \tag{4.6}$$

$$\widehat{\mathrm{NB}}(t)=\exp\{\Gamma_{1n}z(t)+3\theta\Gamma_{1n}|z(t)|^2/28\}, \tag{4.7}$$

$$\widehat{\mathrm{NB}}(t)=\widehat{\mathrm{G}}(t)(1+\widehat{M_1}(t))+C\theta(\widehat{M_2}(t)+|\widehat{M_1}(t)|^2). \tag{4.8}$$

Proof. The estimates of Lemma 4.1 are proved in [5] or easily follow from the estimates obtained there. Indeed, the estimates (4.1) are proved in [5, Lemma 6.10], (4.3) is given in [5, proof of Theorem 3.2] and (4.2) is proved similarly. For the proof of (4.4), check that $|\widehat{\Pi}(t)|=\exp\{-2\Gamma_{1n}\sin^2(t/2)\}=\exp\{-0.5\Gamma_{1n}|z(t)|^2\}$ and apply (4.1). Estimates (4.5) and (4.7) are given in [5, Lemmas 6.9 and 6.10]. For (4.6), apply (4.4), (4.1) and the trivial inequality $|z(t)|\leqslant 2$. To prove (4.8), note that $|\widehat{\mathrm{G}}(t)|\exp\{|\widehat{M_1}(t)|+\widehat{M_2}(t)\}\leqslant 1$ and $\widehat{\mathrm{NB}}(t)=\widehat{\mathrm{G}}(t)\exp\{\widehat{M_1}(t)+\theta\widehat{M_2}(t)\}$. Then

$$\begin{aligned}
\widehat{\mathrm{NB}}(t)&=\widehat{\mathrm{G}}(t)\mathrm{e}^{\widehat{M_1}(t)+\theta\widehat{M_2}(t)}=\widehat{\mathrm{G}}(t)\mathrm{e}^{\widehat{M_1}(t)}\big(1+\theta\widehat{M_2}(t)\mathrm{e}^{\widehat{M_2}(t)}\big)\\
&=\widehat{\mathrm{G}}(t)\mathrm{e}^{\widehat{M_1}(t)}+\theta\widehat{M_2}(t)|\widehat{\mathrm{G}}(t)|\mathrm{e}^{|\widehat{M_1}(t)|}\mathrm{e}^{\widehat{M_2}(t)}\\
&=\widehat{\mathrm{G}}(t)\big(1+\widehat{M_1}(t)+0.5\theta|\widehat{M_1}|^2\mathrm{e}^{|\widehat{M_1}(t)|}\big)+\theta\widehat{M_2}(t)\\
&=\widehat{\mathrm{G}}(t)(1+\widehat{M_1}(t))+0.5\theta|\widehat{M_1}(t)|^2|\widehat{\mathrm{G}}(t)|\mathrm{e}^{|\widehat{M_1}(t)|}+\theta\widehat{M_2}(t)\\
&=\widehat{\mathrm{G}}(t)(1+\widehat{M_1}(t))+\theta(\widehat{M_2}(t)+|\widehat{M_1}(t)|^2). \blacksquare
\end{aligned}$$

The following lemma is the main tool for our proofs.

LEMMA 4.2. *Let M be concentrated on* $\mathbb{Z}$, $u \in \mathbb{R}$ *and* $v \geqslant 1$. *Then*

$$|M|_K \geqslant C_{37}\bigg|\int\limits_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\Big(\frac{t}{v}\Big)\mathrm{e}^{-\mathrm{i}tu}\,\mathrm{d}t\bigg|, \tag{4.9}$$

$$\|M\|_\infty \geqslant \frac{C_{38}}{v}\bigg|\int\limits_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\Big(\frac{t}{v}\Big)\mathrm{e}^{-\mathrm{i}tu}\,\mathrm{d}t\bigg|. \tag{4.10}$$

The estimates in (4.9) *and* (4.10) *remain valid if* $\mathrm{e}^{-t^2/2}$ *is replaced by* $t\mathrm{e}^{-t^2/2}$.

A proof of the above lemma can be found in [3, Lemmas 10.3 and 10.4].

5. PROOFS

Proof of Theorems 2.1 and 2.2. We will often use the trivial estimate $|z(t)| \leqslant |t|$ and the inequality $|\mathrm{e}^a - 1| \leqslant |a|$ for a complex number a with nonpositive real part (see [3, (1.34)]). Then applying Lemma 4.1, we have

$$\begin{aligned}
&|\widehat{\Pi}(t)[\exp\{\Gamma_{2n}z^2(t)\} - 1 - \Gamma_{2n}z^2(t) - (\Gamma_{2n}z^2(t))^2/2]| \\
&\qquad \leqslant \bigg|\widehat{\Pi}(t)\frac{\Gamma_{2n}^3 z^6(t)}{2}\int\limits_0^1 (1-\tau)^2\exp\{\tau\Gamma_{2n}z^2(t)\}\,\mathrm{d}\tau\bigg| \leqslant \frac{|\Gamma_{2n}|^3|z(t)|^6}{2},
\end{aligned}$$

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}[\widehat{F}_n(t) - \widehat{\Pi}(t) - \widehat{\Pi}_1(t)] &= \mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}\Big\{[\widehat{F}_n(t) - \widehat{\mathrm{G}}(t)] \\
&\quad + \widehat{\Pi}(t)[\exp\{\Gamma_{2n}z^2(t)\} - 1 - \Gamma_{2n}z^2(t) - (\Gamma_{2n}z^2(t))^2/2] \\
&\quad + \widehat{\Pi}(t)\frac{\Gamma_{2n}^2}{2}[z^4(t) - (\mathrm{i}t)^4]\Big\} + \frac{\Gamma_{2n}^2(\mathrm{i}t)^4}{2}[\mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}\widehat{\Pi}(t) - 1] + \frac{\Gamma_{2n}^2(\mathrm{i}t)^4}{2} \\
&= \frac{\Gamma_{2n}^2(\mathrm{i}t)^4}{2} + \theta C\big\{R_1|z(t)|^3 + |\Gamma_{2n}|^3|z(t)|^6 + \Gamma_{2n}^2|t|^5 + \Gamma_{2n}^2|t|^6\Gamma_{1n}\big\}.
\end{aligned}$$

Apply Lemma 4.2 with $u = \Gamma_{1n}$ and $v = b\sqrt{\Gamma_{1n}}$, where $b \geqslant 1$ is a constant, and (4.1), to get

$$\begin{aligned}
|F_n - \Pi - \Pi_1|_K &\geqslant C_{39}\frac{\Gamma_{2n}^2}{b^4\Gamma_{1n}^2} - C_{40}\bigg(\frac{R_1}{b^3\Gamma_{1n}^{3/2}} + \frac{|\Gamma_{2n}|^3}{b^6\Gamma_{1n}^3} + \frac{\Gamma_{2n}^2}{b^5\Gamma_{1n}^{5/2}} + \frac{\Gamma_{2n}^2}{b^6\Gamma_{1n}^2}\bigg) \\
&\geqslant C_{41}\frac{\Gamma_{2n}^2}{b^4\Gamma_{1n}^2} - \frac{C_{40}R_1}{b^3\Gamma_{1n}^{3/2}},
\end{aligned}$$

for sufficiently large b. Note that we can assume $b^3 \geqslant C_{40} \geqslant 1$. The proof of (2.3) is completed by choosing $\alpha^3 = b^3/C_{40}$. For the local estimate in (2.10), one has to use the second inequality in Lemma 4.2. The lower bounds for G are proved in a similar way. Using Lemma 4.1, we get

$$
\begin{aligned}
(5.1)\qquad & \mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}\Gamma_{3n}\widehat{\mathrm{G}}(t)z^3(t)\\
&= \Gamma_{3n}(\mathrm{i}t)^3 + \Gamma_{3n}(z^3(t)-(\mathrm{i}t)^3) + \Gamma_{3n}z^3(t)(\mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}\widehat{\mathrm{G}}(t)-1)\\
&= \Gamma_{3n}(\mathrm{i}t)^3 + \theta C|\Gamma_{3n}|\,|t|^3(|t|+\Gamma_{1n}t^2),\\
(5.2)\qquad & \mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}(\widehat{F}_n(t)-\widehat{\mathrm{G}}(t))\\
&= \mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}(\Gamma_{3n}\widehat{\mathrm{G}}(t)z^3(t) + \widehat{F}_n(t) - \widehat{\mathrm{G}}(t)(1+\Gamma_{3n}z^3(t)))\\
&= \mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}\Gamma_{3n}\widehat{\mathrm{G}}(t)z^3(t) + \theta C(R_1^2|z(t)|^6 + R_2|z(t)|^4).
\end{aligned}
$$

Next we apply Lemma 4.2 with $u=\Gamma_{1n}$, $v=b\sqrt{\Gamma_{1n}}\,t\exp\{-t^2/2\}$ and $b\geqslant 1$ to obtain

$$
\begin{aligned}
|F_n-\mathrm{G}|_K &\geqslant \frac{C_{42}|\Gamma_{3n}|}{b^3\Gamma_{1n}^{3/2}} - \frac{C_{43}|\Gamma_{3n}|}{b^4\Gamma_{1n}^{3/2}} - \frac{C_{44}|\Gamma_{3n}|}{b^5\Gamma_{1n}^{3/2}} - \frac{C_{45}R_1^2}{b^6\Gamma_{1n}^3} - \frac{C_{46}R_2}{b^4\Gamma_{1n}^2}\\
&\geqslant \frac{C_{47}}{b^3}\left(\frac{|\Gamma_{3n}|}{\Gamma_{1n}^{3/2}}\left(1-\frac{C_{48}}{b}-\frac{C_{49}}{b^2}\right) - \frac{C_{50}}{b}\left(\frac{R_1^2}{\Gamma_{1n}^3}+\frac{R_2}{\Gamma_{1n}^2}\right)\right)\\
&\geqslant \frac{C_{51}}{b^3}\left(\frac{|\Gamma_{3n}|}{\Gamma_{1n}^{3/2}} - \frac{C_{52}}{b}\left(\frac{R_1^2}{\Gamma_{1n}^3}+\frac{R_2}{\Gamma_{1n}^2}\right)\right),
\end{aligned}
$$

when b is sufficiently large. We can always assume that $C_{59}/b\leqslant 1$. If we choose $\alpha=b/C_{52}$, the proof of (2.4) follows. The local estimate in (2.11) follows from Lemma 4.2 immediately.

For the proofs of (2.5) and (2.12), let $\widehat{M_3}(t)=\Gamma_{3n}z^3(t)-\widehat{M_1}(t)$. Note that

$$
\begin{aligned}
|1+\Gamma_{3n}z^3(t)-(1+\widehat{M_1}(t))(1+\widehat{M_3}(t))| &\leqslant \frac{C\Gamma_{2n}^2}{\Gamma_{1n}}\left|\Gamma_{3n}-\frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right||z(t)|^6,\\
\mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}\widehat{\mathrm{NB}}(t)z^3(t) &= (\mathrm{i}t)^3 + (z^3(t)-(\mathrm{i}t)^3) + (\mathrm{e}^{-\Gamma_{1n}\mathrm{i}t}\widehat{\mathrm{NB}}(t)-1)z^3(t)\\
&= (\mathrm{i}t)^3 + \theta C|t|^3(|t|+\Gamma_{1n}t^2).
\end{aligned}
$$

Also, it follows from Lemma 4.1 that

$$
\begin{aligned}
\widehat{F}_n(t)-\widehat{\mathrm{NB}}(t) &= \widehat{\mathrm{NB}}(t)M_3(t) + [\widehat{F}_n(t)-\widehat{\mathrm{G}}(t)(1+\Gamma_{3n}z^3(t))]\\
&\quad + [\widehat{\mathrm{G}}(t)(1+\Gamma_{3n}z^3(t)) - \widehat{\mathrm{G}}(t)(1+\widehat{M_1}(t))(1+\widehat{M_3}(t))]\\
&\quad + (1+\widehat{M_3}(t))[\widehat{\mathrm{G}}(t)(1+\widehat{M_1}(t)) - \widehat{\mathrm{NB}}(t)]\\
&= \widehat{\mathrm{NB}}(t)\widehat{M_3}(t) + \theta C\Bigg\{R_1^2|z(t)|^6 + R_2|z(t)|^4\\
&\quad + \frac{\Gamma_{2n}^2}{\Gamma_{1n}}\left|\Gamma_{3n}-\frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right||z(t)|^6\\
&\quad + \left(1+\left|\Gamma_{3n}-\frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\right||z(t)|^3\right)\frac{\Gamma_{2n}^3}{\Gamma_{1n}^2}|z(t)|^4(1+\Gamma_{2n}|z(t)|^2)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \widehat{\mathrm{NB}}(t)\widehat{M}_3(t) + \theta C\bigg\{ R_1^2|z(t)|^6 + R_2|z(t)|^4 + \frac{\Gamma_{2n}^3|z(t)|^4}{\Gamma_{1n}^2}(1+\Gamma_{2n}|z|^2) \\
&\quad + \bigg|\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\bigg|\frac{\Gamma_{2n}^2}{\Gamma_{1n}}|z(t)|^6\bigg(\frac{|z(t)|(1+\Gamma_{2n}|z(t)|^2)}{\Gamma_{1n}} + 1\bigg)\bigg\} \\
&= \widehat{\mathrm{NB}}(t)z^3(t)\bigg(\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\bigg) + \theta C\bigg\{ R_1^2|z(t)|^6 + R_2|z(t)|^4 \\
&\quad + \frac{\Gamma_{2n}^3|z(t)|^4}{\Gamma_{1n}^2}(1+\Gamma_{2n}|z|^2) + \bigg|\Gamma_{3n} - \frac{4\Gamma_{2n}^2}{3\Gamma_{1n}}\bigg|\frac{\Gamma_{2n}^2}{\Gamma_{1n}}|z(t)|^6\bigg\}.
\end{aligned}
$$

The proofs of (2.5) and (2.12) are completed by an application of Lemma 4.2 with $u = \Gamma_{1n}$, $v = b\sqrt{\Gamma_{1n}}\,t\exp\{-t^2/2\}$ and for sufficiently large constant b.

The proofs of (2.6) and (2.13) are similar to that for the negative binomial approximation. It suffices to replace $\widehat{M}_1(t)$, $\widehat{M}_2(t)$, $\widehat{M}_3(t)$ by $\widehat{V}_1(t)$, $\widehat{V}_2(t)$ and $\widehat{V}_3(t) = \Gamma_{3n}z^3(t) - \widehat{V}_1(t)$, respectively, and note that

$$
\begin{aligned}
\widehat{\mathrm{G}}(t)(1+\widehat{V}_1(t)) - \widehat{\mathrm{Bi}}(t) &= \widehat{\mathrm{G}}(t)(1+\widehat{V}_1(t) - \exp\{\widehat{V}_1(t)\}) \\
&\quad + \widehat{\mathrm{G}}(t)\exp\{\widehat{V}_1(t)\}(1-\exp\{\widehat{V}_2(t)\}) \\
&= \theta C(|\widehat{V}_1(t)|^2 + \widehat{V}_2(t)) \\
&= \theta C\bigg(\frac{\Gamma_{2n}^4|z(t)|^6}{\Gamma_{1n}^2} + \frac{|\Gamma_{2n}|^3|z(t)|^4}{\Gamma_{1n}^2} + \frac{\Gamma_{2n}^2\epsilon|z(t)|^2}{\Gamma_{1n}^2}\bigg), \\
\widehat{V}_3(t)\widehat{\mathrm{Bi}}(t)\mathrm{e}^{-\mathrm{i}\Gamma_{1n}t} &= \bigg(\Gamma_{3n} - \frac{N\overline{p}^3}{3}\bigg)(\mathrm{i}t)^3 \\
&\quad + \theta C\bigg|\Gamma_{3n} - \frac{N\overline{p}^3}{3}\bigg||t|^3(|t| + \Gamma_{1n}t^2). \ \blacksquare
\end{aligned}
$$

Proof of Theorem 2.3. Using (5.1) and (5.2), we obtain

$$
\begin{aligned}
&\mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}(\widehat{F}_n(t) - \widehat{\Pi}(t) - \widehat{\Pi}_1(t)) \\
&\qquad = \mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}(\widehat{F}_n(t) - \widehat{\mathrm{G}}(t)) + \mathrm{e}^{-\mathrm{i}t\Gamma_{1n}}(\widehat{\mathrm{G}}(t) - \widehat{\Pi}(t) - \widehat{\Pi}_1(t)) \\
&\qquad = \Gamma_{3n}(\mathrm{i}t)^3 + \theta C[|\Gamma_{3n}|\Gamma_{1n}|t|^5 + |\Gamma_{3n}|t^4 + R_1^2t^6 + R_2t^4] \\
&\qquad\quad + \theta C|\widehat{\Pi}(t)(\exp\{\Gamma_{2n}z^2(t)\} - 1 - \Gamma_{2n}z^2(t))|.
\end{aligned}
$$

From Lemma 4.1, it follows that

$$
\begin{aligned}
&\big|\widehat{\Pi}(t)\big(\exp\{\Gamma_{2n}z^2(t)\} - 1 - \Gamma_{2n}z^2(t)\big)\big| \\
&\qquad \leqslant \bigg|\widehat{\Pi}(t)\Gamma_{2n}^2z^4(t)\int_0^1(1-\tau)\exp\{|\Gamma_{2n}z^2(t)|/2\}\,\mathrm{d}\tau\bigg| \leqslant C\Gamma_{2n}^2t^4.
\end{aligned}
$$

It remains to apply Lemma 4.2 with $u = \Gamma_{1n}$, $v = b$ and choose b sufficiently large. The proof of Theorem 2.3 is almost the same as that of (2.4), the main difference being in taking $v = b$ in Lemma 4.2. The details are omitted. $\blacksquare$

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REFERENCES

[1] A. D. Barbour and V. Čekanavičius, *Total variation asymptotics for sums of independent integer random variables*, Ann. Probab. 30 (2002), 509–545.

[2] A. D. Barbour, L. Holst and S. Janson, *Poisson Approximation*, Clarendon Press, Oxford, 1992.

[3] V. Čekanavičius, *Approximation Methods in Probability Theory*, Universitext, Springer, 2016.

[4] V. Čekanavičius and B. Roos, *Two-parametric compound binomial approximations*, Lithuanian Math. J. 44 (2004), 354–373.

[5] V. Čekanavičius and P. Vellaisamy, *Discrete approximations for sums of m-dependent random variables*, ALEA Latin Amer. J. Probab. Math. Statist. 12 (2015), 765–792.

[6] W. T. Huang and C. S. Tsai, *On a modified binomial distribution of order* k, Statist. Probab. Lett. 11 (1991), 125-131.

[7] J. Kruopis, *Precision of approximation of the generalized binomial distribution by convolutions of Poisson measures*, Lithuanian Math. J. 26 (1986), 37–49.

[8] B. Roos, *Poisson approximation via the convolution with Kornya–Presman signed measures*, Theory Probab. Appl. 48 (2004), 555–560.

[9] S. Y. T. Soon, *Binomial approximation for dependent indicators*, Statist. Sinica 6 (1996), 703–714.

[10] P. Vellaisamy, *Poisson approximation for* (k_1, k_2) *events via the Stein–Chen method*, J. Appl. Probab. 41 (2004), 1081–1092.

[11] X. Wang and A. Xia, *On negative binomial approximation to k-runs*, J. Appl. Probab. 45 (2008), 456–471.

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