PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 41, Fasc. 1 (2021), pp. 25–38
Published online 12.3.2021
doi:10.37190/0208-4147.41.1.3

ON THE EXACT ASYMPTOTICS OF EXIT TIME FROM A CONE OF AN ISOTROPIC α-SELF-SIMILAR MARKOV PROCESS WITH A SKEW-PRODUCT STRUCTURE*

BY

ZBIGNIEW PALMOWSKI (WROCŁAW) AND LONGMIN WANG (TIANJIN)

Abstract. In this paper we identify the asymptotic tail of the distribution of the exit time τ_C from a cone C of an isotropic α-self-similar Markov process X_t with a skew-product structure, that is, X_t is a product of its radial process and an independent time changed angular component Θ_t. Under some additional regularity assumptions, the angular process Θ_t killed on exiting the cone C has a transition density that can be expressed in terms of a complete set of orthogonal eigenfunctions with corresponding eigenvalues of an appropriate generator. Using this fact and some asymptotic properties of the exponential functional of a killed Lévy process related to the Lamperti representation of the radial process, we prove that

$$\mathbb{P}_x(\tau_C > t) \sim h(x) t^{-\kappa_1}$$

as $t \to \infty$ for h and κ_1 identified explicitly. The result extends the work of De Blassie (1988) and Bañuelos and Smits (1997) concerning the Brownian motion.

2020 Mathematics Subject Classification: Primary 31B05; Secondary 60J45.

Key words and phrases: α-self-similar process, cone, exit time, skew-product structure, Lamperti representation, exponential functional, Brownian motion.

1. INTRODUCTION

For a dimension $d \geqslant 2$ and an index $\alpha > 0$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ we consider an $\mathbb{R}^d$-valued *α-self-similar isotropic Markov process* $\{X_t, t \geqslant 0\}$, where $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid X_0 = x)$. We recall that the process X is said to be *α-self-*

* This work is partially supported by the Polish National Science Centre under the grant 2015/17/B/ST1/01102 (2016-2019). Longmin Wang kindly acknowledges partial support by the project RARE-318984, a Marie Curie IRSES Fellowship within the 7th European Community Framework Programme.

similar if for every $x \in \mathbb{R}^d$ and $\lambda > 0$,

$$\textit{the law of } (\lambda X_{\lambda^{-\alpha}t}, t \geqslant 0) \textit{ under } \mathbb{P}_x \textit{ is the same as under } \mathbb{P}_{\lambda x}.$$

Moreover, this process is said to be *isotropic* (or $O(d)$-*invariant*) if for any $x \in \mathbb{R}^d$ and $\varrho \in O(d)$,

$$\textit{the law of } (\varrho(X_t), t \geqslant 0) \textit{ under } \mathbb{P}_x \textit{ is the same as under } \mathbb{P}_{\varrho(x)},$$

where $O(d)$ is the group of orthogonal transformations on $\mathbb{R}^d$. In this paper we assume that the radial process $R_t = |X_t|$ and the angular process X_t/R_t do not jump at the same time. Then by Liao and Wang [15, Theorem 1] the process X_t observed up to its first hitting time of 0 has a *skew-product structure*:

$$X_t = R_t \Theta_{A(t)}, \tag{1.1}$$

where $A(t)$ is a strictly increasing continuous process defined by

$$A(t) = \int_0^t R_s^{-\alpha}\,\mathrm{d}s, \qquad t < T_0, \tag{1.2}$$

for

$$T_0 = \inf\{t > 0 : X_t = 0\} = \inf\{t > 0 : R_t = 0\}, \tag{1.3}$$

and Θ_t is an $O(d)$-invariant Markov process on the unit sphere S^{d-1} and is *independent* of the radial process R_t. The classical example concerns d-dimensional Brownian motion that may be expressed as a product of a Bessel process and a time changed spherical Brownian motion. Moreover, the Bessel process is independent of the spherical Brownian motion. More generally, any continuous isotropic Markov proces will have the above representation (1.1) with possibly different time change; see [9]. In particular, a self-similar diffusion will have it. Note that an isotropic self-similar Markov process might not satisfy (1.1) though. The most famous examples are the symmetric $(1/\alpha)$-stable Lévy processes for $\alpha > 1/2$. Their Lévy measures are absolutely continuous on $\mathbb{R}^d \backslash \{0\}$, so their radial and angular parts may jump together, and thus do not possess a skew product structure as defined above.

We will also consider an *open cone* C in $\mathbb{R}^d$ generated by a domain D in the unit sphere S^{d-1}, that is, $C = \bigcup_{r>0} rD$. We define the *first exit time* of X_t from the cone C by

$$\tau_C = \inf\{t > 0 : X_t \notin C\}. \tag{1.4}$$

The purpose of this paper is to study the *asymptotic behavior of the exit probability* $\mathbb{P}_x(\tau_C > t)$ as $t \to \infty$ for $x \in C$. In fact we prove that

$$\mathbb{P}_x(\tau_C > t) \sim h(x) t^{-\kappa_1} \tag{1.5}$$

as $t \to \infty$ for h and κ_1 identified explicitly, where we write $f(t) \sim g(t)$ for some positive functions f and g iff $\lim_{t\to\infty} f(t)/g(t) = 1$.

The main idea of the proof is based on the following steps. In the first step we give the representation

$$q_D(t,\theta,\eta) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} m_j(\theta) m_j(\eta),$$

of the transition density for the angular process Θ_t killed upon exiting the cone C in terms of orthogonal eigenfunctions m_j with corresponding eigenvalues λ_j of $-\mathcal{S}|_D$ for the generator $\mathcal{S}$ of Θ_t restricted to D with the Dirichlet boundary condition. Then

$$\mathbb{P}_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|)\Big(\int_D m_j \,\mathrm{d}\sigma\Big)\mathbb{E}_{|x|}[\mathrm{e}^{-\lambda_j A(t)},\, t < T_0]$$

for σ being the normalized surface measure on S^{d-1}. Using the Lamperti [12] transformation we can express the process $\{R_t, t < T_0\}$ as a time change of the exponential of an $\mathbb{R}\cup\{-\infty\}$-valued Lévy process, that is, there exists an $\mathbb{R}\cup\{-\infty\}$-valued Lévy process ξ_t starting from 0 and with lifetime ζ, whose law does not depend on $|x|$, such that

$$R_t = |x| \exp(\xi_{A(t)}), \quad 0 \leqslant t < T_0. \tag{1.6}$$

This gives the following representation of the tail exit probability:

$$\mathbb{P}_x(\tau_C > t) = \sum_{j=1}^{\infty} m_j(x/|x|)\Big(\int_D m_j \,\mathrm{d}\sigma\Big)\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha} t),$$

where

$$I_t(\alpha\xi) = \int_0^t \exp(\alpha\xi_s)\,\mathrm{d}s, \tag{1.7}$$

e_{λ_j} is an independent exponential random variable with intensity λ_j and $I(\alpha\xi) = \lim_{t\to\zeta} I_t(\alpha\xi)$ is an exponential functional. The final result (1.5) follows from Rivero [19, Lemma 4] and Maulik and Zwart [17, Theorem 3.1] concerning the tail asymptotics of the exponential functional $I(\alpha\xi)$. In this case κ_1 solves the equation

$$\phi(\alpha\kappa_1) = \lambda_1 \tag{1.8}$$

for the Laplace exponent of the process ξ.

Our main result (1.5) extends the work of De Blassie [6] and Bañuelos and Smits [1] concerning the Brownian motion (see also [18] for the case of α-stable processes).

The asymptotic (1.5) also determines the critical exponents of integrability of the exit time τ_C. In this sense it generalizes the result of a series of papers concerning α-stable processes: see Kulczycki [11] and Bañuelos and Bogdan [2] and references therein.

The paper is organized as follows. In Preliminaries we state and prove the main facts used later. In Section 3 we give the main result and its proof.

2. PRELIMINARIES

2.1. Skew-product structure. Let X_t be an α-self-similar isotropic Markov process with skew-product representation (1.1). The process Θ_t is an $O(d)$-invariant Markov process on S^{d-1} with transition semigroup Q_t and infinitesimal generator $\mathcal{S}$.

Throughout this paper, we assume the following.

ASSUMPTION 2.1. *Θ_t possesses a bounded transition density $q(t,\theta,\eta)$ with respect to σ, the normalized surface measure on S^{d-1}, and there exist positive constants C and β such that*

$$q(t,\theta,\eta) \leqslant Ct^{-\beta} \tag{2.1}$$

for all $(t,\theta,\eta) \in (0,1) \times S^{d-1} \times S^{d-1}$.

EXAMPLE 2.1 (Brownian motion). In the case when Θ_t is a Brownian motion, Assumption 2.1 is satisfied. Indeed, the generator $\mathcal{S}$ of Θ_t is a multiple of the Laplace–Beltrami operator $\Delta_{S^{d-1}}$ on S^{d-1}. Moreover, it is known that the transition density $h(t,\theta,\eta)$ of Θ_t has a Gaussian upper bound:

$$h(t,\theta,\eta) \leqslant c_1 t^{-(d-1)/2} \mathrm{e}^{-c_2 d(\theta,\eta)^2/t}, \quad t>0,\ \theta,\eta \in S^{d-1}, \tag{2.2}$$

for some positive constants c_1 and c_2.

EXAMPLE 2.2 (Subordinate Brownian motion). Fix $\gamma \in (0,1)$. Let W_t be a Brownian motion on S^{d-1} with transition density $h(t,\theta,\eta)$. Let S_t be a γ-stable subordinator, i.e., a Lévy process in $\mathbb{R}$, supported by $[0,\infty)$, with Laplace transform

$$\mathbb{E}[\mathrm{e}^{-\vartheta S_t}] = \exp(-t\vartheta^\gamma), \quad \vartheta > 0.$$

Assumption 2.1 is also satisfied when $\Theta_t = W_{S_t}$. Indeed, in this case Θ_t is an $O(d)$-invariant pure jump Markov process on S^{d-1} with transition density

$$q(t,\theta,\eta) = \int_0^\infty h(u,\theta,\eta) p_t(u)\,\mathrm{d}u,$$

where $p_t(u)$ is the probability density of S_t. By Theorem 37.1 of Doetsch [8],

$$\lim_{u\to\infty} p_1(u) u^{1+\gamma} = \frac{\gamma}{\Gamma(1-\gamma)}. \tag{2.3}$$

The limit (2.3) together with the scaling property

$$p_t(u) = t^{-1/\gamma} p_1(t^{-1/\gamma} u)$$

gives the following upper bound:

$$p_t(u) \leqslant c_1 t u^{-1-\gamma}, \quad t, u > 0.$$

Using (2.2) one can now observe that

$$q(t, \theta, \eta) \leqslant c_3 t^{-\frac{d-1}{2\gamma}} \wedge \frac{t}{d(\theta, \eta)^{d-1+2\gamma}}$$

for $(t, \theta, \eta) \in (0, \infty) \times S^{d-1} \times S^{d-1}$.

We will now give sufficient conditions for Assumption 2.1 to be satisfied for a general $O(d)$-invariant Markov process Θ_t on S^{d-1}. If we identity S^{d-1} with $O(d)/O(d-1)$ then Θ_t may be viewed as a Lévy process on the compact homogeneous space $O(d)/O(d-1)$. Furthermore, the generator $\mathcal{S}$ of Θ_t was given by Hunt [10] (see also Liao [13]) explicitly. We state this result as follows. Let $C^\infty(S^{d-1})$ be the space of smooth functions on S^{d-1} and let $\pi : O(d) \to S^{d-1}$ be the map $g \mapsto g\overline{o}$ for $\overline{o} = (0, \ldots, 0, 1) \in \mathbb{R}^d$. Restricted to a sufficiently small neighborhood V of $\overline{o}$, the map

$$\varphi : (y_1, \ldots, y_d) \mapsto \pi(e^{\sum_{j=1}^d y_j O_j})$$

is a diffeomorphism and $(y_1, \ldots, y_d)$ may be used as local coordinates on $\varphi(V)$, where $(O_1, \ldots, O_d)$ is a basis of the Lie algebra of $O(d)$. Then by [13, Theorem 2.2] the domain of $\mathcal{S}$ contains $C^\infty(S^{d-1})$ and for $f \in C^\infty(S^{d-1})$,

$$\mathcal{S}f(o) = Tf(o) + \int_{S^{d-1}} \left(f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j} \right) \nu(\mathrm{d}\theta), \tag{2.4}$$

where o is the origin in S^{d-1}, T is an $O(d)$-invariant second order differential operator on S^{d-1} and ν is an $O(d-1)$-invariant measure on S^{d-1}, called the *Lévy measure* of Θ_t, that satisfies $\nu(\{o\}) = 0$ and

$$\int_{S^{d-1}} [\mathrm{dist}(\theta, o)]^2 \, \nu(\mathrm{d}\theta) < \infty. \tag{2.5}$$

Since $O(d)/O(d-1)$ is irreducible, all the $O(d)$-invariant second order differential operators are multiples of $\Delta_{S^{d-1}}$. Therefore we may rewrite (2.4) as

$$\mathcal{S}f(o) = a\Delta_{S^{d-1}} f(o) + \int_{S^{d-1}} \left(f(\theta) - f(o) - \sum_{j=1}^d y_j(\theta) \frac{\partial f(o)}{\partial y_j} \right) \nu(\mathrm{d}\theta) \tag{2.6}$$

for some $a \geqslant 0$.

Note that when Θ_t is the subordinate Brownian motion defined in Example 2.2, we have $a = 0$ and the Lévy measure satisfies

$$\nu \asymp d(\theta, o)^{-d+1-2\gamma} \quad \text{near } o.$$

As already observed, in this case Assumption 2.1 holds true.

This phenomenon holds for more general Θ_t. Using [14, Theorems 3 and 6] we can state the following proposition giving sufficient conditions for Assumption 2.1 to hold.

PROPOSITION 2.1. *Θ_t is a Lévy process on S^{d-1} with infinitesimal generator $\mathcal{S}$ given by* (2.6). *Assume that either $a > 0$ or the Lévy measure ν is asymptotically larger than $d(\theta, o)^{-\gamma}$ near $\theta = o$ for some $\gamma \in (d-1, d-1+2)$. Then Θ_t has a bounded transition density $q(t, \theta, \eta)$ and it satisfies Assumption* 2.1.

For any open subset $D \subset S^{d-1}$ we define the first exit time of Θ_t from D by

$$\tau_D^{\Theta} = \inf\{t > 0 : \Theta_t \notin D\}. \tag{2.7}$$

Let Θ^D be the killed process of Θ upon exiting D, that is, $\Theta_t^D = \Theta_t$ if $t < \tau_D^{\Theta}$ and $\Theta_t^D = \partial$ if $t \geqslant \tau_D^{\Theta}$, where ∂ is a cemetery state. Its infinitesimal generator is $\mathcal{S}|_D$, the restriction of $\mathcal{S}$ to D with the Dirichlet boundary condition. Then

$$q_D(t, \theta, \eta) = q(t, \theta, \eta) - \mathbb{E}_\theta[q(t - \tau_D^{\Theta}, \Theta_{\tau_D^{\Theta}}, \eta); \tau_D^{\Theta} < t] \tag{2.8}$$

is the transition density of Θ^D. Clearly, $q_D(t, \theta, \eta) \leqslant q(t, \theta, \eta)$ for all $t > 0$ and $\theta, \eta \in S^{d-1}$. As a consequence, the transition semigroup Q_t^D associated to the subprocess Θ^D is compact on L^2. Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of $-\mathcal{S}|_D$ written in increasing order and repeated according to their multiplicity, and m_j the corresponding eigenfunctions normalized by $\|m_j\|_2 = 1$. Then by [5, Theorem 2.1.4], $m_j \in L^\infty$ for all j and Q_t^D has a transition density $q_D(t, \theta, \eta)$, which can be represented as the series

$$q_D(t, \theta, \eta) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} m_j(\theta) m_j(\eta) \tag{2.9}$$

that converges uniformly on $[\delta, \infty) \times D \times D$ for all $\delta > 0$.

REMARK 2.1. Note that we do not assume any regularity condition on the boundary ∂D of D. Thus $q_D(t, \theta, \eta)$ (or $m_j(\theta)$) need not vanish continuously on ∂D.

REMARK 2.2. If $S^{d-1} \setminus \overline{D}$ is not empty, then from the monotonicity of Dirichlet eigenvalues we have $\lambda_1 > 0$; see [3, Section I.5] for more details (cf. also Lemma 2.2 below).

REMARK 2.3. Assume that $q_D(t,\theta,\eta)$ is strictly positive for $t > 0$ and $\theta, \eta \in D$. Then we knoww from Jentzsch's theorem [20, Theorem V.6.6] that λ_1 is a simple eigenvalue for $-\mathcal{S}|_D$. Using standard arguments, like the ones in [4, proof of Theorem 2.4], one can show that $q_D(t,\theta,\eta)$ is strictly positive when Θ_t is a Brownian motion on S^{d-1} and D is connected or Θ_t is a subordinate Brownian motion on S^{d-1} satisfying the conditions of Example 2.2.

In our analysis the crucial fact is the following lower bound for the eigenvalues λ_j ($j \geqslant 1$).

LEMMA 2.2. *Assume* (2.1) *holds true. Then for every* $j \geqslant 1$, *we have*

$$\lambda_j \geqslant [C\sigma(D)]^{-1/\beta} j^{1/\beta} \tag{2.10}$$

and

$$\|m_j\|_\infty \leqslant \mathrm{e}C[\sigma(D)]^{1/2}\lambda_j^\beta. \tag{2.11}$$

Proof. We will follow the same idea as in [7, proof of Lemma 2.7]. In particular, since the λ_j are ordered increasingly, we see from (2.9) and (2.1) that

$$j\mathrm{e}^{-\lambda_j t} \leqslant \sum_{i=1}^{\infty} \mathrm{e}^{-\lambda_i t} = \int_D q_D(t,\theta,\theta)\,\sigma(\mathrm{d}\theta) \leqslant C\sigma(D)t^{-\beta}.$$

Taking $t = \lambda_j^{-1}$ we obtain $j \leqslant C\sigma(D)\lambda_j^\beta$ and (2.10) follows immediately.

Note that

$$m_j(\theta) = \mathrm{e}^{\lambda_j t} Q_t^D m_j(\theta) = \mathrm{e}^{\lambda_j t} \int_D q_D(t,\theta,\eta) m_j(\eta)\,\sigma(\mathrm{d}\eta).$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\|m_j\|_\infty &\leqslant \mathrm{e}^{\lambda_j t} \sup_\theta \Big(\int_D q_D(t,\theta,\eta)^2\,\sigma(\mathrm{d}\eta)\Big)^{1/2} \Big(\int_D m_j(\eta)^2\,\sigma(\mathrm{d}\eta)\Big)^{1/2} \\ &\leqslant C[\sigma(D)]^{1/2} \mathrm{e}^{\lambda_j t} t^{-\beta}.\end{aligned}$$

The proof is completed by setting $t = \lambda_j^{-1}$. ■

2.2. Positive self-similar Markov processes. Recall that $R_t = |X_t|$ is a positive ($\mathbb{R}_+$-valued) α-self-similar Markov process starting at $|x|$. According to Lamperti [12], up to its first hitting time of 0, R_t may be expressed as a time change of the exponential of an $\mathbb{R} \cup \{-\infty\}$-valued Lévy process. More formally, there exists an $\mathbb{R} \cup \{-\infty\}$-valued Lévy process ξ_t starting from 0 and with lifetime ζ, whose law does not depend on $|x|$, such that

$$R_t = |x| \exp(\xi_{A(t)}), \quad 0 \leqslant t < T_0, \tag{2.12}$$

where T_0 is the first hitting time of 0 by R defined formally in (1.3) and $A(t)$ is the positive continuous functional given by $A(t) = \int_0^t R_s^\alpha \, \mathrm{d}s$. The law of ξ is characterized completely by its Lévy–Khinchin exponent

$$\Psi(z) = \log \mathbb{E}[\mathrm{e}^{\mathrm{i}z\xi_1}] = -q + \mathrm{i}bz - \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{+\infty} (\mathrm{e}^{\mathrm{i}zy} - 1 - \mathrm{i}zy\mathbf{1}_{\{|y|<1\}})\,\Pi(\mathrm{d}y), \tag{2.13}$$

where $q \geqslant 0$, $\sigma \geqslant 0$, $b \in \mathbb{R}$ and Π is a Lévy measure satisfying the condition $\int_{\mathbb{R}} (1 \wedge |y|^2)\,\Pi(\mathrm{d}y) < \infty$. The lifetime ζ of ξ is an exponential random variable with parameter q, with the convention that $\zeta = \infty$ when $q = 0$. Observe that the process ξ does not depend on the starting point of X. Hence we will denote the law of ξ by $\mathbb{P}$.

For fixed $\alpha > 0$, we define the exponential functional $I_t(\alpha\xi)$ by (1.7). Then by a change of variable $s = A(u)$,

$$I_t(\alpha\xi) = \int_0^{A^{-1}(t)} \exp(\alpha\xi_{A(u)}) R_u^{-\alpha} \, \mathrm{d}u = |x|^{-\alpha} A^{-1}(t).$$

Hence $A(|x|^\alpha t)$ is the right inverse of the strictly increasing continuous process $I_t(\alpha\xi)$ and we can recover the law of $(R_t, t < T_0)$ from the law of ξ_t for fixed $|x|$ and $\alpha > 0$. In particular, we have

$$(T_0, \mathbb{P}_x) \stackrel{d}{=} (|x|^\alpha I(\alpha\xi), \mathbb{P}). \tag{2.14}$$

As mentioned in [12], the probabilities $\mathbb{P}_x(T_0 = +\infty)$, $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} = 0)$ and $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} > 0)$ are 0 or 1 independently of x. Moreover, we have

(1) if $\mathbb{P}_x(T_0 = +\infty) = 1$, then $\zeta = +\infty$, $\limsup_{t\to\infty} \xi_t = +\infty$, and $\lim_{t\to\infty} A(t) = +\infty$;

(2) if $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} = 0) = 1$, then $\zeta = +\infty$, $\lim_{t\to\infty} \xi_t = -\infty$, and $\lim_{t\to T_0-} A(t) = +\infty$;

(3) if $\mathbb{P}_x(T_0 < +\infty, R_{T_0-} > 0) = 1$, then ζ is an exponentially distributed random time with parameter $q > 0$. Moreover, $A(T_0-)$ has the same distribution as that of ζ, thus the functional $A(t)$ always jumps from a finite value to $+\infty$, that is, $\mathbb{P}_x(A(T_0-) < +\infty, A(T_0) = +\infty) = 1$.

Let e_λ be an independent exponential random variable with parameter λ. Then

$$\mathbb{E}_{|x|}[\mathrm{e}^{-\lambda A(t)},\, t < T_0] = \mathbb{P}_{|x|}(A(t) < e_\lambda,\, t < T_0).$$

Note that by the construction above we find that $t < T_0$ is equivalent to $A(t) < \zeta$. Thus

$$\mathbb{E}_{|x|}[\mathrm{e}^{-\lambda A(t)},\, t < T_0] = \mathbb{P}_{|x|}(A(t) < e_\lambda \wedge \zeta) = \mathbb{P}\Big(\int_0^{e_\lambda} \exp(\alpha\xi_s)\,\mathrm{d}s > |x|^{-\alpha} t\Big), \tag{2.15}$$

where in the last equality we have used the fact that $|x|^\alpha I_t(\alpha\xi)$ is the right inverse of $A(t)$. The equation (2.15) will give (along with the representation (2.9)) another main ingredient of the proof of the main result. In the last step we will need the tail asymptotic behaviour of the exponential function $I_{e_\lambda}(\alpha\xi)$ described below.

2.3. Exponential functional of a killed Lévy process. Let ξ^λ be a Lévy process with Lévy–Khinchin exponent Ψ given by (2.13) killed by the independent exponential time e_λ with parameter $\lambda > 0$. Thus the resulting process has lifetime $\zeta' = e_\lambda \wedge \zeta$, an exponential random variable with parameter $\lambda + q$.

We define the Laplace exponent of ξ via

$$\mathbb{E}(\exp(\vartheta\xi_t)) = \exp(t\phi(\vartheta)), \qquad t \geqslant 0,\ \vartheta \in \Xi, \tag{2.16}$$

where $\Xi = \{\vartheta : \phi(\vartheta) < \infty\}$. By (2.13), for $\vartheta \in \Xi$ we have $\phi(\vartheta) = \Psi(-\mathrm{i}\vartheta)$. It is easy to see from the Hölder inequality that $\phi(\vartheta)$ is a convex function. From now we assume that ξ satisfies the following conditions.

ASSUMPTION 2.2. *ξ is not arithmetic, that is, there is no d such that the support of ξ_1 is $d\mathbb{N}$.*

ASSUMPTION 2.3. *There exists a constant $\vartheta^* > \alpha(1 \vee (2\beta))$ such that $\phi(\vartheta) < \infty$ for $0 < \vartheta < \vartheta^*$ and $\lim_{\vartheta\to\vartheta^*} \phi(\vartheta) = \infty$, where β is the constant in Assumption* 2.1.

Under Assumption 2.3, for every $\lambda > 0$, there exists a unique $0 < \kappa < \vartheta^*$ such that

$$\phi(\alpha\kappa) = \lambda. \tag{2.17}$$

Moreover,

$$\mathbb{E}[\xi_1^\lambda \mathrm{e}^{\alpha\kappa\xi_1^\lambda}] = \phi'(\alpha\kappa) < \infty.$$

In the proof we will use the following crucial result giving the tail asymptotics of the distribution of the exponential functional.

THEOREM 2.3 ([19, Lemma 4], [17, Theorem 3.1]). *Suppose that Assumptions* 2.2 *and* 2.3 *are satisfied. Then, as $t \to \infty$,*

$$t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha\xi) > t) \sim \frac{1}{\alpha\phi'(\alpha\kappa)} \mathbb{E}[I_{e_\lambda}(\alpha\xi)^{\kappa-1}]. \tag{2.18}$$

EXAMPLE 2.3 (Linear Brownian motion with drift). Let $\sigma > 0$, $b \in \mathbb{R}$ and $\xi_t = \sigma B_t + bt$, where B_t is a standard linear Brownian motion. Then $\phi(\vartheta) = \frac{\sigma^2\vartheta^2}{2} + b\vartheta$ and for

$$\kappa = \frac{1}{\alpha\sigma^2}(\sqrt{2\sigma^2\lambda + b^2} - b),$$

we have $\mathbb{E}[\mathrm{e}^{\alpha\kappa\sigma(B_1+b)},\ 1 < e_\lambda] = 1$. By Theorem 2.3,

$$\lim_{t\to\infty} t^\kappa \mathbb{P}(I_{e_\lambda}(\alpha\xi) > t) = C_\kappa,$$

where

$$C_\kappa = \frac{1}{\alpha\sqrt{2\sigma^2\lambda + b^2}}\mathbb{E}\Big\{\Big[\int_0^{e_\lambda} \exp(\alpha\sigma B_s + \alpha b s)\,\mathrm{d}s\Big]^{\kappa-1}\Big\}.$$

By the scaling property of Brownian motion, the random variable

$$\int_0^{e_\lambda} \exp(\alpha\sigma B_s + \alpha b s)\,\mathrm{d}s$$

has the same distribution as the integral

$$\int_0^{\frac{\alpha^2\sigma^2}{4}e_\lambda} \exp\Big(2B_s + \frac{4b}{\alpha\sigma^2}s\Big)\,\mathrm{d}s.$$

Note that $\frac{\alpha^2\sigma^2}{4}e_\lambda$ is an exponentially distributed random variable with parameter $\frac{4}{\alpha^2\sigma^2}\lambda$ independent of B_t. Yor [21] (see also [16, Theorem 4.12]) proved the following identity in law:

$$I_{e_\lambda}(\alpha\xi) \stackrel{\mathrm{d}}{=} \frac{Z_{1,a}}{2\gamma_\kappa}, \tag{2.19}$$

where $a = \kappa + \frac{2b}{\alpha\sigma^2}$, $Z_{1,a}$ is a beta variable with parameters $(1, a)$, and γ_κ is a gamma variable with parameter κ, which is independent of $Z_{1,a}$. Since

$$\mathbb{E}[Z_{1,a}^{\kappa-1}] = \int_0^1 t^{\kappa-1}a(1-t)^{a-1}\,\mathrm{d}t = \frac{\Gamma(\kappa)\Gamma(a+1)}{\Gamma(a+\kappa)}$$

and

$$\mathbb{E}[\gamma_\kappa^{1-\kappa}] = \frac{1}{\Gamma(\kappa)}\int_0^\infty t^{1-\kappa}t^{\kappa-1}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{1}{\Gamma(\kappa)},$$

we have

$$\mathbb{E}[(I_{e_\lambda}(\alpha\xi))^{\kappa-1}] = \frac{2^{1-\kappa}\Gamma(a+1)}{\Gamma(\kappa+a)} = \frac{2^{1-\kappa}\Gamma\big(\kappa + \frac{2b}{\alpha\sigma^2} + 1\big)}{\Gamma\big(2\kappa + \frac{2b}{\alpha\sigma^2}\big)}.$$

Therefore,

$$C_\kappa = \frac{4}{\alpha^2\sigma^2 2^\kappa}\,\frac{\Gamma\big(\kappa + \frac{2b}{\alpha\sigma^2} + 1\big)}{\Gamma\big(2\kappa + \frac{2b}{\alpha\sigma^2} + 1\big)}.$$

3. MAIN RESULT

Let

$$M(x) = \sum_{j:\,\lambda_j=\lambda_1} \Big(\int_D m_j \, \mathrm{d}\sigma\Big) m_j(x/|x|) \tag{3.1}$$

be an eigenfunction corresponding to the eigenvalue λ_1 of the operator $\mathcal{S}$ in D with the Dirichlet boundary condition. Moreover, let κ_1 solve $\phi(\alpha\kappa_1) = \lambda_1$; that is, (1.8) is satisfied.

Recall that τ_C is the exit time from the cone C of the α-self-similar Markov process X_t with a skew-product structure (1.1). The main result of this paper is the following asymptotic.

THEOREM 3.1. *Under Assumptions* 2.1–2.3*, we have*

$$\mathbb{P}_x(\tau_C > t) \sim \frac{1}{\alpha\phi'(\alpha\kappa_1)} \mathbb{E}[I_{e_{\lambda_1}}(\alpha\xi)^{\kappa_1-1}] M(x)(|x|^{-\alpha}t)^{-\kappa_1} \tag{3.2}$$

as $t \to \infty$.

REMARK 3.1. $M(x)$ does not depend on the choice of the eigenfunction m_j with $\lambda_j = \lambda_1$. Indeed, if we have another choice m'_j, then there exists an orthogonal matrix (a_{ik}) such that $m'_i = \sum_k a_{ik}m_k$, which is equivalent to $m_j = \sum_i a_{ij}m'_i$. Thus,

$$\begin{aligned}\sum_i \Big(\int_D m'_i \, \mathrm{d}\sigma\Big) m'_i &= \sum_i \Big(\sum_j \int_D a_{ij}m_j \, \mathrm{d}\sigma\Big) m'_i \\ &= \sum_j \Big(\int_D m_j \, \mathrm{d}\sigma\Big) \sum_i a_{ij}m'_i = \sum_j \Big(\int_D m_j \, \mathrm{d}\sigma\Big) m_j.\end{aligned}$$

EXAMPLE 3.1. Assume X_t is an isotropic α-self-similar diffusion process on $\mathbb{R}^d$. Then the radial process $R_t = |X_t|$ is a positive α-self-similar diffusion process and Θ_t is a (possibly nonstandard) Brownian motion on S^{d-1} with infinitesimal generator $a\Delta_{S^{d-1}}$ for some $a > 0$. Using the Lamperti relation, we have $\xi_t = \sigma B_t + bt$ for some $\sigma > 0$ and $b \in \mathbb{R}$, where B_t is a standard Brownian motion. Clearly, Assumptions 2.1–2.3 are satisfied. It follows from Example 2.3 that

$$\lim_{t\to\infty} t^{-\kappa_1}\mathbb{P}_x(\tau_C > t) = \frac{4}{\alpha^2\sigma^2} \frac{\Gamma\big(\kappa_1 + \frac{2b}{\alpha\sigma^2} + 1\big)}{\Gamma\big(2\kappa_1 + \frac{2b}{\alpha\sigma^2} + 1\big)} \Big(\frac{|x|^2}{2}\Big)^{\kappa_1} M(x),$$

where $M(x)$ is defined by (3.1). In particular, when X_t is a d-dimensional Brownian motion, we have $\alpha = 2$, $\sigma = 1$, $b = d/2 - 1$, and $a = 1/2$. Thus

$$\lim_{t\to\infty} t^{-\kappa_1}\mathbb{P}_x(\tau_C > t) = \frac{\Gamma(\kappa_1 + d/2)}{\Gamma(2\kappa_1 + d/2)} \Big(\frac{|x|^2}{2}\Big)^{\kappa_1} M(x), \tag{3.3}$$

where

$$\kappa_1 = \frac{1}{2}(\sqrt{2\lambda_1 + (d/2-1)^2} - (d/2-1)).$$

This recovers the seminal result of De Blassie [6] (see also [1, Corollary 1]).

Proof of Theorem 3.1. We start the proof from the observation that τ_C is just the first time t that $R_t = 0$ or the angular process $\Theta_{A(t)} \notin D$, that is,

$$\mathbb{P}_x(\tau_C > t) = \mathbb{P}_x(T_0 > t,\, \tau_D^\Theta > A(t)). \tag{3.4}$$

By the assumed independence of R_t and Θ_t in (1.1) we have

$$\mathbb{P}_x(\tau_C > t) = \int_0^\infty \mathbb{P}_{x/|x|}(\tau_D^\Theta > u)\,\mathrm{d}_u\mathbb{P}_{|x|}(A(t) \leqslant u,\, t < T_0). \tag{3.5}$$

By (2.9) the exit probability $\mathbb{P}_\theta(\tau_D^\Theta > t)$ can be represented as

$$\mathbb{P}_\theta(\tau_D^\Theta > t) = \int_D q_D(t,\theta,\eta)\,\mathrm{d}\sigma(\eta) = \sum_{j=1}^\infty \mathrm{e}^{-\lambda_j t} m_j(\theta) \int_D m_j\,\mathrm{d}\sigma. \tag{3.6}$$

Thus by (2.15),

$$\begin{aligned} \mathbb{P}_x(\tau_C > t) &= \sum_{j=1}^\infty m_j(x/|x|) \int_D m_j\,\mathrm{d}\sigma \int_0^\infty \mathrm{e}^{-\lambda_j u}\,\mathrm{d}_u\mathbb{P}_{|x|}(A(t) \leqslant u,\, t < T_0) \\ &= \sum_{j=1}^\infty m_j(x/|x|)\Big(\int_D m_j\,\mathrm{d}\sigma\Big)\mathbb{E}_{|x|}[\mathrm{e}^{-\lambda_j A(t)},\, t < T_0] \\ &= \sum_{j=1}^\infty m_j(x/|x|)\Big(\int_D m_j\,\mathrm{d}\sigma\Big)\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t). \end{aligned} \tag{3.7}$$

For $j \geqslant 1$ let κ_j ($j \geqslant 1$) be the solutions of

$$\phi(\alpha\kappa_j) = \lambda_j. \tag{3.8}$$

Since $\lambda_j \to \infty$, we have $\liminf_{j\to\infty} \kappa_j \geqslant \vartheta^*/\alpha$. Fix a κ_0 with

$$1 \vee \kappa_1 \vee (2\beta) < \kappa_0 < \vartheta^*/\alpha. \tag{3.9}$$

Then there are only a finite number of j's (with $j \geqslant 1$), forming a set A say, such that $\kappa_1 < \kappa_j \leqslant \kappa_0$. Applying Theorem 2.3 for $j \in A$ we obtain

$$\lim_{t\to\infty} t^{\kappa_1}\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t) = 0.$$

Hence

$$\begin{aligned} \lim_{t\to\infty} t^{\kappa_1} \sum_{j:\,\kappa_j \leqslant \kappa_0} & m_j(x/|x|)\Big(\int_D m_j\,\mathrm{d}\sigma\Big)\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t) \\ &= \frac{1}{\alpha\phi'(\alpha\kappa_1)}\mathbb{E}[I_{e_{\lambda_1}}(\alpha\xi)^{\kappa_1-1}]M(x)|x|^{\alpha\kappa_1}. \end{aligned} \tag{3.10}$$

Now we consider the summation over $j \in A^c$, that is, for $\kappa_j > \kappa_0$. By the Markov and Hölder inequalities,

$$\begin{aligned} t^{\kappa_0}\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t) &\leqslant |x|^{\alpha\kappa_0}\mathbb{E}\Big[\Big(\int_0^{e_{\lambda_j}} \exp(\alpha\xi_s)\,\mathrm{d}s\Big)^{\kappa_0}\Big] \\ &\leqslant |x|^{\alpha\kappa_0}\mathbb{E}\Big[(e_{\lambda_j})^{\kappa_0-1}\int_0^{e_{\lambda_j}} \exp(\alpha\kappa_0\xi_s)\,\mathrm{d}s\Big]. \end{aligned}$$

Using the independence of e_{λ_j} and ξ_s, we have

$$\begin{aligned} t^{\kappa_0}\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t) &\leqslant |x|^{\alpha\kappa_0}\mathbb{E}\Big[(e_{\lambda_j})^{\kappa_0}\int_0^{e_{\lambda_j}} \mathrm{e}^{s\phi(\alpha\kappa_0)}\,\mathrm{d}s\Big] \\ &\leqslant |x|^{\alpha\kappa_0}\mathbb{E}[(e_{\lambda_j})^{\kappa_0}\mathrm{e}^{\phi(\alpha\kappa_0)e_{\lambda_j}}] \\ &= |x|^{\alpha\kappa_0}\int_0^{\infty} u^{\kappa_0}\mathrm{e}^{\phi(\alpha\kappa_0)u}\lambda_j\mathrm{e}^{-\lambda_j u}\,\mathrm{d}u \\ &= |x|^{\alpha\kappa_0}\frac{\lambda_j\Gamma(\kappa_0+1)}{(\lambda_j-\phi(\alpha\kappa_0))^{\kappa_0+1}} \\ &\leqslant c|x|^{\alpha\kappa_0}\lambda_j^{-\kappa_0} \end{aligned}$$

for some constant $c > 0$. By (2.10), (2.11) and the fact $\kappa_0 > 2\beta$, we have

$$\begin{aligned} \text{(3.11)} \qquad \sum_{j\colon \kappa_j>\kappa_0} |m_j(x/|x|)| \cdot \Big|\int_D m_j\,\mathrm{d}\sigma\Big|\mathbb{P}(I_{e_{\lambda_j}}(\alpha\xi) > |x|^{-\alpha}t) \\ \leqslant c|x|^{\alpha\kappa_0}\Big(\sum_j j^{-\frac{\kappa_0-\beta}{\beta}}\Big)t^{-\kappa_0}. \end{aligned}$$

Combining (3.10) and (3.11) completes the proof. ■

REFERENCES

[1] R. Bañuelos and R. G. Smits, *Brownian motion in cones*, Probab. Theory Related Fields 108 (1997), 299–319.

[2] R. Bañuelos and K. Bogdan, *Symmetric stable processes in cones*, Potential Anal. 21 (2004), 263–288.

[3] I. Chavel, *Eigenvalues in Riemannian Geometry*, Pure Appl. Math. 115, Academic Press, Orlando, FL, 1984.

[4] K. L. Chung and Z. X. Zhao, *From Brownian Motion to Schrödinger's Equation*, Grundlehren Math. Wiss. 312. Springer, Berlin, 1995.

[5] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge Univ. Press, Cambridge, 1990.

[6] R. D. De Blassie, *Remark on exit times from cones in $\mathbb{R}^n$ of Brownian motion*, Probab. Theory Related Fields 79 (1988), 95–97.

[7] H. Donnelly and P. Li, *Lower bounds for the eigenvalues of Riemannian manifolds*, Michigan Math. J. 29 (1982), 149–161.

[8] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer, Berlin, 1974.

[9] A. Galmarino, *Representation of an isotropic diffusion as a skew product*, Z. Wahrsch. Verw. Gebiete 1 (1963), 359–378.

[10] G. A. Hunt, *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. 81 (1956), 264–293.

[11] T. Kulczycki, *Exit time and Green function of cone for symmetric stable processes*, Probab. Math. Statist. 19 (1999), 337–374.

[12] J. Lamperti, *Semi-stable Markov processes. I*, Z. Wahrsch. Verw. Gebiete 22 (1972), 205–225.

[13] M. Liao, *Lévy Processes in Lie Groups*, Cambridge Tracts in Math. 162, Cambridge Univ. Press, Cambridge, 2004.

[14] M. Liao and L. Wang, *Lévy–Khinchin formula and existence of densities for convolution semigroups on symmetric spaces*, Potential Anal. 27 (2007), 133–150.

[15] M. Liao and L. Wang, *Isotropic self-similar Markov processes*, Stochastic Process. Appl. 121 (2011), 2064–2071.

[16] H. Matsumoto and M. Yor, *Exponential functionals of Brownian motion. I. Probability laws at fixed time*, Probab. Surv. 2 (2005), 312–347.

[17] K. Maulik and B. Zwart. *Tail asymptotics for exponential functionals of Lévy processes*, Stochastic Process. Appl. 116 (2006), 156–177.

[18] P. J. Méndez-Hernández, *Exit times from cones in* $\mathbb{R}^d$ *of symmetric stable processes*, Illinois J. Math. 46 (2002), 155–163.

[19] V. Rivero, *Recurrent extensions of self-similar Markov processes and Cramér's condition*, Bernoulli 11 (2005), 471–509.

[20] H. H. Schaefer, *Banach Lattices and Positive Operators*, Grundlehren Math. Wiss. 215, Springer, New York, 1974.

[21] M. Yor, *Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires*, C. R. Acad. Sci. Paris Sér. I Math. 314 (1992), 951–956.

Zbigniew Palmowski
Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: zbigniew.palmowski@pwr.edu.pl

Longmin Wang
School of Mathematical Sciences
Nankai University
Tianjin 300071, P.R. China
E-mail: wanglm@nankai.edu.cn

Received 22.4.2019;
revised version 23.12.2019