PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 41, Fasc. 2 (2021), pp. 321–345
Published online 20.9.2021
doi:10.37190/0208-4147.41.2.7

ON TAILS OF SYMMETRIC AND TOTALLY ASYMMETRIC α-STABLE DISTRIBUTIONS*

BY

WITOLD M. BEDNORZ (WARSZAWA), RAFAŁ M. ŁOCHOWSKI (WARSZAWA), AND RAFAŁ MARTYNEK (WARSZAWA)

Abstract. We estimate up to universal constants tails of symmetric and totally asymmetric 1-dimensional α-stable distributions in terms of functions of the parameters of these distributions. In particular, for values of α close to 2 we specify where exactly the tail changes from being Gaussian and starts to behave like in the Pareto distribution.

2020 Mathematics Subject Classification: Primary 60E07; Secondary 60E15, 60E10.

Key words and phrases: alpha-stable distributions, tail estimates.

1. INTRODUCTION

A random variable X is called (one-dimensional) *stable* if for any numbers $a, b > 0$ and independent copies X_1, X_2 of X there exist numbers $c(a, b)$ and $d(a, b)$ such that

$$aX_1 + bX_2 \stackrel{d}{=} c(a, b)X + d(a, b).$$

Random variables of this type constitute an important family used in stochastic modelling. Let us recall some fundamental properties of stable distributions; for a comprehensive study see e.g. [12]. It is a classical result that $c(a, b)$ is of the form $(a^\alpha + b^\alpha)^{1/\alpha}$ for $\alpha \in (0, 2]$. The number α is sometimes called the *index of stability* and a stable random variable with index α is called α-*stable*. For $\alpha \neq 1$ the characteristic function of X is given by

$$\mathbb{E}\exp(itX) = \exp\left(-\sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu t\right),$$

* The research of all authors was funded by the National Science Centre, Poland, under Grant No. 2016/21/B/ST1/01489.

while for $\alpha = 1$ the characteristic function is

$$\mathbb{E}\exp(itX) = \exp\Big(-\sigma|t|\Big(1 + i\beta\frac{2}{\pi}\operatorname{sgn}(t)\ln(|t|)\Big) + i\mu t\Big),$$

where $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$ is a shift parameter. β is a skewness (asymmetry) parameter, while σ is a scale parameter. The case $\beta = 0$ is referred to as the *symmetric case*, and $\beta = -1$ or $\beta = 1$ is the *totally asymmetric case*. When $\mu = 0$ we call X a *strictly α-stable* random variable, in which case the characteristic function can be represented as

$$\mathbb{E}\exp(itX) = \exp\Big(\int_{\mathbb{R}} \psi(t,x)\,\nu(dx)\Big), \tag{1.1}$$

where

$$\psi(t,x) = \begin{cases} e^{itx} - 1 & \text{if } \alpha \in (0,1), \\ e^{itx} - 1 - itx & \text{if } \alpha \in (1,2), \end{cases}$$

and ν, called the *Lévy measure*, is given by

$$\nu(dx) = \frac{C_1}{x^{\alpha+1}}\mathbb{1}_{(0,\infty)}(x)dx + \frac{C_2}{|x|^{\alpha+1}}\mathbb{1}_{(-\infty,0)}(x)dx,$$

where $C_1, C_2 \geqslant 0$ and $C_1 + C_2 > 0$. The relation between C_1, C_2 and β is given by the equation $\beta = \frac{C_1 - C_2}{C_1 + C_2}$. In particular, for the symmetric case we take $C_1 = C_2 = 1$ and for the totally asymmetric case $C_1 = 1$ and $C_2 = 0$. Moreover, the dependence of the scale parameter σ on the parameter α and the constants C_1 and C_2 is given by $\sigma^\alpha = \Gamma(-\alpha)\cos\big(\frac{(2-\alpha)\pi}{2}\big)(C_1 + C_2)$, where Γ denotes the gamma function.

There are usually no closed formulas for densities and distribution functions of stable distributions. The exceptions are the Gaussian distribution ($\alpha = 2, \beta = 0$), the Cauchy distribution ($\alpha = 1$, $\beta = 0$) and the Lévy distribution ($\alpha = 1/2$, $\beta = 1$). To deal with the lack of explicit densities for other cases the series expansions were established; see [17] and [6, Chapt. XVII, Sect. 6]. For $\sigma = 1, \beta = 0$ there is the following series expansion of the density function of X: for $\alpha \in (0, 1)$,

$$f_X(x) = \frac{1}{\pi}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n!}\Gamma(n\alpha + 1)\sin\Big(\frac{n\alpha}{2}\Big)\frac{1}{x^{n\alpha+1}},$$

while for $\alpha \in (1, 2]$ (see [15, Chapt. IV, Sect. 1])

$$f_X(x) = \frac{1}{\alpha\pi}\sum_{n\geqslant 1}\frac{(-1)^n}{2n!}\Gamma\Big(\frac{2n+1}{\alpha}\Big)x^{2n}.$$

The tail asymptotics of α-stable distributions are well-known [12, Property 1.2.15]. For $\alpha \in (0, 2)$ we have

$$\lim_{y \to \infty} y^{\alpha} \mathbb{P}(X \geqslant y) = C_{\alpha} \frac{1+\beta}{2} \sigma^{\alpha},$$
$$\lim_{y \to -\infty} |y|^{\alpha} \mathbb{P}(X \leqslant y) = C_{\alpha} \frac{1-\beta}{2} \sigma^{\alpha},$$

where

$$C_{\alpha} = \Big(\int_0^{\infty} x^{\alpha} \sin x \, dx \Big)^{-1} = \frac{1}{\alpha \Gamma(-\alpha) \cos\big(\frac{(2-\alpha)\pi}{2}\big)}.$$

Observe that

$$C_{\alpha} = \begin{cases} 1 + o(1) & \text{as } \alpha \to 0^{+}, \\ \frac{2}{\pi}(1 + o(1)) & \text{as } \alpha \to 1, \\ (2-\alpha)(1+o(1)) & \text{as } \alpha \to 2^{-}. \end{cases}$$

For $\beta = -1$, $\lim_{y \to \infty} y^{\alpha} \mathbb{P}(X \geqslant y) = 0$, so the rate of convergence of $\mathbb{P}(X \geqslant y)$ to 0 is faster than $1/y^{\alpha}$. It is known [12, (1.2.11)] that the right rate of convergence for $\alpha \in (1, 2)$ is given by

$$\frac{1+o(1)}{\sqrt{2\alpha\pi(\alpha-1)}} \Big(\frac{|y|}{\kappa_{\alpha}} \Big)^{-\frac{\alpha}{2(\alpha-1)}} \exp\Big(-(\alpha-1) \Big(\frac{|y|}{\kappa_{\alpha}} \Big)^{\frac{-\alpha}{\alpha-1}} \Big),$$

where $\kappa_{\alpha} = \frac{\alpha\sigma}{\cos\left(\frac{(2-\alpha)\pi}{2}\right)}$, and exactly the same for the left tails in the case of $\beta = 1$. Recall that for $\alpha < 1$ and $\beta = 1$, $\mathbb{P}(X < 0) = 0$.

There is a rich literature on numerical calculation of stable densities and distribution functions; see for example [7] and references therein. In this article we are interested in 'qualitative' behavior of tails of symmetric and totally asymmetric α-stable distributions. More precisely, we are interested in the description of these tails in terms of functions of the parameters of the distribution up to universal constants.

Let X be an α-stable random variable. As presented above, the asymptotic behavior of $\mathbb{P}(|X| > t)$ as $t \to \infty$ is fully understood, but the value of the tail $\mathbb{P}(X > t)$ for moderate values of t does not seem to be well investigated. The study of densities of α-stable distributions goes back to Pólya [8] as well as Blumenthal and Getoor [1]. Upper bounds for densities of multidimensional α-stable random variables were given by Watanabe [16]. The classical work by Pruitt [9] provides estimates for the tails of suprema of Lévy processes. The idea of truncating the characteristic function used in both [9] and [16] is also applied in this work. Some of the results presented here can be related to much more general work of Grzywny et al. [2] where estimates for densities were provided together with explicit constants [3], which are however of rather intricate form. Also, upper bounds for $\beta \neq 0$ can be found in [13], while lower bounds for $|\beta| \neq 1$ in [4].

The value of the results presented here lies mainly in the transparency of constants in the estimates, which, as we believe, have not been explicitly presented so far. Also the approach based on elementary techniques might be of independent interest especially since it outlines the nature of α-stable variables whose tails are determined by the analysis of heavy-tailed jumps. The main novelty compared to the results in [2] is that we also consider the strictly asymmetric case ($\beta = 1$). Finally, the calculations we provide for α close to 2 allow us to establish the order of the boundary value at which the tail of α-stable random variable alters from behaving like a Gaussian and starts to resemble a tail of Pareto distribution (see Remark 4.7).

2. METHODS

Our approach is based on the analysis of the series representation of α-stable random variables as well as their characteristic functions. First, we present a classical series representation (see [12, Section 1.4]).

Let $(\tau_i)_{i\geqslant 1}$ be a sequence of arrival times of a Poissonian process with parameter 1, i.e. $\tau_i = \Gamma_1 + \cdots + \Gamma_i$, where the sequence $(\Gamma_k)_{k\geqslant 1}$ is i.i.d. and for $u \geqslant 0$, $\mathbb{P}(\Gamma_k \geqslant u) = e^{-u}$. Then

- for $\alpha \in (0,1)$ and $\beta = 1$, $X \stackrel{d}{=} \sum_{i=1}^{\infty} (\alpha\tau_i)^{-1/\alpha}$,
- for $\alpha \in (0,1) \cup (1,2)$ and $\beta = 0$, $X \stackrel{d}{=} (\alpha/2)^{-1/\alpha} \sum_{i=1}^{\infty} \varepsilon_i \tau_i^{-1/\alpha}$ (where ε_i are independent Rademacher random variables),
- for $\alpha \in (1,2)$ and $\beta = 1$, $X \stackrel{d}{=} c_\alpha \sum_{i=1}^{\infty} (\tau_i^{-1/\alpha} - a_i)$, for an α-dependent constant c_α and compensating terms a_i given by $a_i = \frac{\alpha-1}{\alpha}(i^{(\alpha-1)/\alpha} - (i-1)^{(\alpha-1)/\alpha})$.

Series representations are particularly useful for simulations (see e.g. [11]). Also, it is worth mentioning that a more general class of infinitely divisible processes admit a similar representation, known as Rosiński's representation [10]. Working with this representation turns out to be efficient when estimating tails of both symmetric and asymmetric α-stable random variables for $\alpha \in (0,1)$. The proof of convergence of the above series can be found in [12]. To verify the above series representations, one simply calculates the characteristic function of X in each case and checks that it obeys the definition (1.1); the following two lemmas might serve as a tool and will also be helpful in further calculations.

Lemma 2.1. *Consider a Borel function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $\int_0^\infty f(x)\,dx < \infty$. Then $\mathbb{E}\sum_{i=1}^{\infty} f(\tau_i) = \int_0^\infty f(x)\,dx$.*

Proof. This is a consequence of the fact that for each $i \geqslant 1$, τ_i has the Erlang distribution, i.e. its density function is given by $\frac{x^{i-1}e^{-x}}{(i-1)!}$, where $x \geqslant 0$. Since f

is non-negative and integrable we can put the summation outside the expectation. The result then follows easily. ■

The second lemma uses equivalence between Poissonian arrival times and Poissonian point processes; we omit the proof.

Lemma 2.2 ([14, Lemma 11.3.3]). *For any $a > 0$ and a continuous function $f : \mathbb{R}^+ \to \mathbb{C}$ we have $\mathbb{E}\prod_{\tau_i < a} f(\tau_i) = \exp(-\int_0^a (1 - f(x))\,dx)$.*

With the above properties the calculations of the characteristic function for the asymmetric case and $\alpha \in (0, 1)$ are straightforward, while in the symmetric case it suffices to notice that the characteristic function can be expressed as

$$\begin{aligned}
\mathbb{E}e^{itX} &= \exp\left(\int_0^\infty (e^{itx} - 1 - itx)\,\frac{dx}{x^{\alpha+1}} + \int_{-\infty}^0 (e^{itx} - 1 - itx)\,\frac{dx}{|x|^{\alpha+1}}\right)\\
&= \exp\left(\int_0^\infty (e^{itx} - 1 - itx + e^{-itx} - 1 + itx)\,\frac{dx}{x^{\alpha+1}}\right)\\
&= \exp\left(2\int_0^\infty (\cos(tx) - 1)\,\frac{dx}{x^{\alpha+1}}\right).
\end{aligned}$$

The main trick used when dealing with the totally asymmetric case for $\alpha \in (0, 1)$ is conditioning the series $\sum_{i=1}^\infty (\alpha\tau_i)^{-1/\alpha}$ on the first term. To this end we observe that $\sum_{i=1}^\infty (\alpha\tau_i)^{-1/\alpha}$ can be written as $(\alpha\tau_1)^{-1/\alpha} + \sum_{i=1}^\infty \alpha^{-1/\alpha}(\tau_1 + \tilde{\tau}_i)^{-1/\alpha}$, where for $i \geqslant 1$ we define

$$\tilde{\tau}_i = \tau_{i+1} - \tau_1. \tag{2.1}$$

We notice that $\tilde{\tau}_i \overset{d}{=} \tau_i$ and $\tilde{\tau}_i$, $i \geqslant 1$, are independent of τ_i. For $x > 0$ define the series

$$S(x) = \sum_{i=1}^\infty \alpha^{-1/\alpha}(x + \tilde{\tau}_i)^{-1/\alpha}.$$

It is well-defined. Notice that $S(x)$ is decreasing. With the use of Lemma 2.2 we calculate the moments of $S(x)$.

Lemma 2.3. *The moment generating function of $S(x)$ is*

$$\Lambda_{S(x)}(\lambda) = \mathbb{E}\exp(\lambda S(x)) = \exp(-f(\lambda, x)), \qquad \lambda \geqslant 0,$$

where

$$f(\lambda, x) = \int_0^\infty 1 - \exp(\alpha^{-1/\alpha}\lambda(x + y)^{-1/\alpha})\,dy. \tag{2.2}$$

Proof. Let $a > 0$. Then, by Lemma 2.2,

$$\mathbb{E}\exp\Big(\lambda\sum_{\tilde{\tau}_i<a}\alpha^{-1/\alpha}(x+\tilde{\tau}_i)^{-1/\alpha}\Big) = \mathbb{E}\prod_{\tilde{\tau}_i<a}\exp(\lambda\alpha^{-1/\alpha}(x+\tilde{\tau}_i)^{-1/\alpha})$$
$$= \exp\Big(-\int_0^a 1-\exp(\alpha^{-1/\alpha}\lambda(x+y)^{-1/\alpha})\,dy\Big).$$

Letting $a \to \infty$ on both sides is allowed since $S(x)$ is a convergent series and the integral on the right-hand side stays finite. To see this we use the inequality $1 - e^u \geqslant -2u$ for small positive u. Consider a sufficiently large constant y_0 and the quantity $I_{y_0} = \int_0^{y_0}(1-\exp(\alpha^{-1/\alpha}\lambda(x+y)^{-1/\alpha}))\,dy$, which is bounded. Then

$$\begin{aligned} f(\lambda,x) &= I_{y_0} + \int_{y_0}^{\infty}\big(1-\exp(\alpha^{-1/\alpha}\lambda(x+y)^{-1/\alpha})\big)\,dy \\ &\geqslant I_{y_0} - 2\lambda\alpha^{-1/\alpha}\int_{y_0}^{\infty}(x+y)^{-1/\alpha}\,dy \\ &= I_{y_0} - 2\lambda\frac{(\alpha(x+y_0))^{1-1/\alpha}}{1-\alpha} > -\infty. \quad \blacksquare \end{aligned}$$

Therefore we can calculate any moment of $S(x)$. In particular, we have the following result. Obviously it could also be deduced from Lemma 2.1.

Lemma 2.4. *With the above notation we have, for* $\alpha \in (0,1)$,

$$\mathbb{E}(S(x)) = \frac{(\alpha x)^{1-1/\alpha}}{1-\alpha} \quad \textit{and} \quad \mathrm{Var}(S(x)) = \frac{(\alpha x)^{1-2/\alpha}}{2-\alpha}.$$

Proof. Fix $x > 0$. Notice that $f(0,x) = 0$ and write $\frac{\partial f}{\partial\lambda} = f'$, $\frac{\partial^2 f}{\partial\lambda^2} = f''$. A simple calculation yields

$$\mathbb{E}(S(x)) = -f'|_{\lambda=0} = \int_0^{\infty}\alpha^{-1/\alpha}(x+y)^{-1/\alpha}\,dy = \frac{(\alpha x)^{1-1/\alpha}}{1-\alpha}.$$

Moreover,

$$-f''|_{\lambda=0} = \int_0^{\infty}\alpha^{-2/\alpha}(x+y)^{-2/\alpha}\,dy = \frac{(\alpha x)^{1-2/\alpha}}{2-\alpha}$$

and

$$\mathbb{E}(S(x)^2) = -f''|_{\lambda=0} + (f')^2|_{\lambda=0},$$

so $\mathrm{Var}(S(x)) = \mathbb{E}(S(x)^2) - (\mathbb{E}(S(x)))^2 = -f''|_{\lambda=0}$. $\blacksquare$

Now, we describe tools which we use to analyse tails by means of characteristic functions. For any random variable Z we denote by $\varphi_Z(t)$ its characteristic function. First, we recall an elementary but useful result which we apply in the symmetric case for all $\alpha \in (0,1)$.

Lemma 2.5 ([5, Lemma 5.1]). *For any random variable Z on $\mathbb{R}$ we have*

$$\mathbb{P}(|Z| > y) \leqslant \frac{y}{2} \int_{-2/y}^{2/y} (1 - \varphi_Z(t))\, dt.$$

Next, we introduce the idea of truncating the characteristic function which will be applied for $\alpha \in (1, 2)$. Let us start by considering a totally asymmetric random variable X with characteristic function

$$\mathbb{E}\exp(itX) = \exp\left(\int_0^\infty (e^{itx} - 1 - itx)\, \frac{dx}{x^{\alpha+1}}\right). \tag{2.3}$$

Unlike the asymmetric case when $\alpha \in (0, 1)$, the support of the distribution of a random variable X with characteristic function (2.3) is the whole real line. Thus, we need upper and lower estimates for both right and left tails. The method is to split X into the sum $X = X_1 + X^1$ such that

$$\varphi_{X_1}(t) = \exp\left(\int_0^1 (e^{itx} - 1 - itx)\, \frac{dx}{x^{\alpha+1}}\right), \tag{2.4}$$

$$\varphi_{X^1}(t) = \exp\left(\int_1^\infty (e^{itx} - 1 - itx)\, \frac{dx}{x^{\alpha+1}}\right). \tag{2.5}$$

It is easy to calculate that

$$\int_1^\infty (e^{itx} - 1 - itx)\, \frac{dx}{x^{\alpha+1}} = \int_1^\infty (e^{itx} - 1)\, \frac{dx}{x^{\alpha+1}} - \frac{it}{\alpha - 1}$$

and thus the characteristic function of X^1 can be expressed as

$$\varphi_{X^1}(t) = \exp\left(\frac{1}{\alpha}(\varphi_Y(t) - 1) - \frac{it}{\alpha - 1}\right),$$

where the random variable Y has density $\frac{\alpha}{x^{\alpha+1}} \mathbb{1}_{(1,\infty)}(x)$. This means that $X^1 + \frac{1}{\alpha-1}$ has a compound Poisson distribution, i.e.

$$X^1 + \frac{1}{\alpha - 1} = \sum_{k=1}^{N} Y_k,$$

where $N \sim \text{Poisson}(1/\alpha)$ while Y_k's are independent random variables all distributed as Y and independent of N.

Similarly, for the symmetric α-stable random variable X with $\alpha \in (1, 2)$ with characteristic function

$$\varphi_X(t) = \exp\left(\int_{-\infty}^{\infty} (e^{itx} - 1)\, \frac{dx}{|x|^{\alpha+1}}\right) \tag{2.6}$$

we use the split $X = \tilde{X}_1 + \tilde{X}^1$, where

$$\varphi_{\tilde{X}_1}(t) = \exp\Big(\int_{-1}^{1} (e^{itx} - 1)\,\frac{dx}{|x|^{\alpha+1}}\Big), \tag{2.7}$$

$$\varphi_{\tilde{X}^1}(t) = \exp\Big(\int_{\mathbb{R}\setminus[-1,1]} (e^{itx} - 1)\,\frac{dx}{|x|^{\alpha+1}}\Big). \tag{2.8}$$

Analogously to the asymmetric case we observe $\varphi_{\tilde{X}^1}(t) = \exp\big(\frac{2}{\alpha}(\varphi_{\tilde{Y}}(t) - 1)\big)$, where the random variable $\tilde{Y}$ has density $\frac{\alpha}{2|x|^{\alpha+1}}\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)$. So, again $\tilde{X}^1$ is compound Poisson given by $\tilde{X}^1 = \sum_{k=1}^{\tilde{N}} \tilde{Y}_k$, where $\tilde{N} \sim \mathrm{Poisson}(2/\alpha)$ and $\tilde{Y}_k$'s are independent, all distributed as $\tilde{Y}$ and independent of $\tilde{N}$.

3. RESULTS FOR $\alpha \in (0, 1)$

3.1. Totally asymmetric case. We now present results for a totally asymmetric α-stable random variable X with characteristic function $\mathbb{E}\exp(itX) = \exp\big(\int_0^\infty (e^{itx} - 1)\frac{1}{x^{\alpha+1}}\,dx\big)$ and with series representation $X \overset{d}{=} \sum_{i=1}^{\infty} (\alpha\tau_i)^{-1/\alpha}$.

THEOREM 3.1. *Let $\alpha \in (0, 1)$ and $y \geqslant 1$. For a totally asymmetric α-stable random variable X we have the tail estimate*

$$\mathbb{P}\Big(X \geqslant \frac{1}{1-\alpha} + 3y\Big) \leqslant \frac{2}{\alpha y^{\alpha}}. \tag{3.1}$$

Moreover, for $y \geqslant 1$ and $\theta \in (0, 1)$ we have

$$\mathbb{P}\Big(X \geqslant \frac{\theta}{1-\alpha} + y\Big) \geqslant \frac{2}{3}(1-\theta)^2\frac{1}{1+\alpha y^{\alpha}}. \tag{3.2}$$

Proof. From Lemma 2.4 it follows that

$$\mathbb{E}S\Big(\frac{1}{\alpha y^{\alpha}}\Big) = \frac{y^{1-\alpha}}{1-\alpha} \quad \text{and} \quad \mathrm{Var}\Big(S\Big(\frac{1}{\alpha y^{\alpha}}\Big)\Big) = \frac{y^{2-\alpha}}{2-\alpha}.$$

Now,

$$\begin{aligned}
\mathbb{P}\Big(X \geqslant \frac{1}{1-\alpha} + 3y\Big) &= \int_0^\infty e^{-x}\mathbb{P}\Big((\alpha x)^{-1/\alpha} + S(x) > 3y + \frac{1}{1-\alpha}\Big)\,dx \\
&\leqslant \int_0^{1/(\alpha y^\alpha)} e^{-x}\,dx + \int_{1/(\alpha y^\alpha)}^\infty e^{-x}\mathbb{P}\Big(S(x) \geqslant 3y - (\alpha x)^{-1/\alpha} + \frac{1}{1-\alpha}\Big)\,dx \\
&\leqslant 1 - e^{-1/(\alpha y^\alpha)} + \int_{1/(\alpha y^\alpha)}^\infty e^{-x}\mathbb{P}\Big(S(x) \geqslant 2y + \frac{1}{1-\alpha}\Big)\,dx \\
&\leqslant \frac{1}{(\alpha y^\alpha)} + \mathbb{P}\Big(S\Big(\frac{1}{\alpha y^\alpha}\Big) \geqslant 2y + \frac{1}{1-\alpha}\Big)\int_{1/(\alpha y^\alpha)}^\infty e^{-x}\,dx \\
&\leqslant \frac{1}{(\alpha y^\alpha)} + \mathbb{P}\Big(S\Big(\frac{1}{\alpha y^\alpha}\Big) \geqslant y + \mathbb{E}S\Big(\frac{1}{\alpha y^\alpha}\Big)\Big)\int_{1/(\alpha y^\alpha)}^\infty e^{-x}\,dx \\
&\leqslant \frac{1}{\alpha y^\alpha} + \frac{\operatorname{Var}\big(S\big(\frac{1}{\alpha y^\alpha}\big)\big)}{y^2}e^{-\frac{1}{\alpha y^\alpha}} = \frac{1}{\alpha y^\alpha} + \frac{1}{(2-\alpha)y^\alpha}e^{-\frac{1}{\alpha y^\alpha}} \leqslant \frac{2}{\alpha y^\alpha},
\end{aligned}$$

where in the third inequality we have used the elementary inequality $1 - e^{-u} \leqslant u$. Also, if $x > 1/(\alpha y^\alpha)$, then $(\alpha x)^{-1/\alpha} < y$. Next, we have used the fact that for $y \geqslant 1$ we have $y + \frac{y^{1-\alpha}}{1-\alpha} \leqslant 2y + \frac{1}{1-\alpha}$ and then we have applied Chebyshev's inequality.

We now turn to the lower bound. We use the decomposition $X \overset{d}{=} \big(\frac{1}{\alpha\tau_1}\big)^{1/\alpha} + S(\tau_1)$. Note that if $x \leqslant 1/(\alpha y^\alpha)$ then $(\alpha x)^{-1/\alpha} \geqslant y$ and hence

$$\mathbb{P}\Big((\alpha\tau_1)^{-1/\alpha} + S(\tau_1) \geqslant y + \frac{\theta}{1-\alpha}\Big) \geqslant \int_0^{1/(\alpha y^\alpha)} e^{-x}\mathbb{P}\Big(S(x) \geqslant \frac{\theta}{1-\alpha}\Big)\,dx.$$

Moreover, since $y \geqslant 1$ and $x \leqslant 1/(\alpha y^\alpha)$ we have $S(x) \geqslant S(1/\alpha)$, so by the Paley–Zygmund inequality and Lemma 2.4 we obtain

$$\begin{aligned}
&\int_0^{1/(\alpha y^\alpha)} e^{-x}\mathbb{P}\Big(S(x) \geqslant \frac{\theta}{1-\alpha}\Big)\,dx \geqslant \int_0^{1/(\alpha y^\alpha)} e^{-x}\mathbb{P}\Big(S\Big(\frac{1}{\alpha}\Big) \geqslant \frac{\theta}{1-\alpha}\Big)\,dx \\
&= \int_0^{1/(\alpha y^\alpha)} e^{-x}\mathbb{P}\Big(S\Big(\frac{1}{\alpha}\Big) \geqslant \theta\mathbb{E}S\Big(\frac{1}{\alpha}\Big)\Big)\,dx \geqslant \int_0^{1/(\alpha y^\alpha)} e^{-x}(1-\theta)^2\frac{1}{1+\frac{(1-\alpha)^2}{2-\alpha}}\,dx \\
&\geqslant \frac{2}{3}(1-\theta)^2\Big(1 - \exp\Big(-\frac{1}{\alpha y^\alpha}\Big)\Big),
\end{aligned}$$

where in the last line we have used $\frac{1}{1+\frac{(1-\alpha)^2}{2-\alpha}} \geqslant \frac{2}{3}$. The conclusion follows from the inequality $1 - e^{-1/u} > \frac{1}{1+u}$ for $u > 0$. ■

3.2. Symmetric case. The lower bound for the symmetric case coincides for $\alpha \in (0,1)$ and $\alpha \in (1,2)$. However, as explained in the last section, a further analysis is provided to reveal the Gaussian nature of tails in the latter case.

THEOREM 3.2. *Let X be a symmetric α-stable random variable. Let $y > 0$. For $\alpha \in (0,1)$,*

$$\mathbb{P}(X \geqslant y) \leqslant \frac{4}{\alpha y^{\alpha}}, \tag{3.3}$$

and for $\alpha \in (0,1) \cup (1,2)$,

$$\mathbb{P}(X \geqslant y) \geqslant \frac{1}{2}\,\frac{1}{2+\alpha y^{\alpha}}. \tag{3.4}$$

Proof. In order to apply Lemma 2.5 we need a lower estimate for the characteristic function. To this end notice that

$$\begin{aligned}\varphi_X(t) &= \exp\Big(\int_{-\infty}^{\infty}(e^{itx}-1)\,\frac{dx}{|x|^{\alpha+1}}\Big) = \exp\Big(2\int_0^{\infty}(\cos(tx)-1)\,\frac{dx}{x^{\alpha+1}}\Big)\\ &= \exp\Big(-2|t|^{\alpha}\int_0^{\infty}(1-\cos z)\,\frac{dz}{z^{\alpha+1}}\Big) \geqslant \exp\Big(-|t|^{\alpha}\cdot 2^{-\alpha}\frac{8}{\alpha(2-\alpha)}\Big),\end{aligned}$$

where in the last inequality we have used

$$\int_0^{\infty}(1-\cos z)\,\frac{dz}{z^{\alpha+1}} \leqslant \int_0^2 \frac{z^2}{2}\,\frac{dz}{z^{\alpha+1}} + \int_2^{\infty} 2\,\frac{dz}{z^{\alpha+1}} = 2^{-\alpha}\cdot\frac{4}{\alpha(2-\alpha)}.$$

Denote $C_\alpha = \frac{8}{\alpha(2-\alpha)}$. Then

$$\begin{aligned}\mathbb{P}(|X| > y) &\leqslant \frac{y}{2}\int_{-2/y}^{2/y}\Big(1-\exp\Big(-\Big(\frac{|t|}{2}\Big)^{\alpha}\frac{8}{\alpha(2-\alpha)}\Big)\Big)\,dt\\ &= 2y\int_0^{1/y}(1-\exp(-C_\alpha s^{\alpha}))\,ds\\ &\leqslant \frac{2C_\alpha}{1+\alpha}\,\frac{1}{y^{\alpha}} \leqslant \frac{1}{\alpha y^{\alpha}}\,\frac{16}{(1+\alpha)(2-\alpha)} \leqslant \frac{8}{\alpha y^{\alpha}},\end{aligned}$$

where in the second inequality we have used $1-e^{-u} \leqslant u$ and in the last the fact that $\alpha < 1$.

For the lower bound we again condition on the first arrival time and use $(\tilde{\tau}_i)_{i\geqslant 1}$ defined in (2.1). By symmetry we have

$$\begin{aligned}
\mathbb{P}(X \geqslant y) &= \mathbb{P}\bigg(\varepsilon_1\Big(\frac{\alpha}{2}\tau_1\Big)^{-1/\alpha} + \Big(\frac{\alpha}{2}\Big)^{-1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i(\tilde{\tau}_i+\tau_1)^{-1/\alpha} \geqslant y\bigg)\\
&\geqslant \mathbb{P}\bigg(\varepsilon_1\Big(\frac{\alpha}{2}\tau_1\Big)^{-1/\alpha} \geqslant y \text{ and } \Big(\frac{\alpha}{2}\Big)^{-1/\alpha}\sum_{i=1}^{\infty}\varepsilon_i(\tilde{\tau}_i+\tau_1)^{-1/\alpha} \geqslant 0\bigg)\\
&= \frac{1}{2}\mathbb{P}\bigg(\varepsilon_1\Big(\frac{\alpha}{2}\tau_1\Big)^{-1/\alpha} \geqslant y\bigg) = \frac{1}{4}\int_0^{2/(\alpha y^\alpha)} e^{-x}\,dx \geqslant \frac{1}{2}\,\frac{1}{2+\alpha y^\alpha},
\end{aligned}$$

where we have used the inequality $1-e^{-1/u} > \frac{1}{1+u}$ for $u > 0$. ■

4. RESULTS FOR $\alpha \in (1,2)$

4.1. Totally asymmetric case. We now consider a random variable X with characteristic function (2.3) and we use the split $X = X_1 + X^1$, where the characteristic functions of X_1 and X^1 are given by (2.4) and (2.5).

Lemma 4.1. *For $y \geqslant 1$ one has the lower bound*

$$\mathbb{P}\Big(X^1 \geqslant y - \frac{1}{\alpha-1}\Big) \geqslant e^{-1/\alpha}\frac{1}{\alpha}\,\frac{1}{y^\alpha} \geqslant \frac{1}{2\sqrt{e}}\,\frac{1}{y^\alpha} \tag{4.1}$$

and the upper bound

$$\mathbb{P}\Big(X^1 \geqslant y - \frac{1}{\alpha-1}\Big) \leqslant \Big(e^{-1/\alpha}\sum_{k=1}^{\infty}\frac{k^{\alpha+1}}{\alpha^k k!}\Big)\frac{1}{y^\alpha} \leqslant \frac{2}{y^\alpha}. \tag{4.2}$$

Proof. We notice that

$$\mathbb{P}\Big(X^1 \geqslant y - \frac{1}{\alpha-1}\Big) \geqslant \mathbb{P}(N=1)\mathbb{P}(Y_1 \geqslant y) = e^{-1/\alpha}\frac{1}{\alpha}\,\frac{1}{y^\alpha}$$

and $e^{-1/\alpha}/\alpha \geqslant 1/(2\sqrt{e})$, since, by simple calculus, the function $\alpha \mapsto e^{-1/\alpha}/\alpha$ is decreasing on the interval $[1,2]$. On the other hand, whenever $\sum_{n=1}^{N} Y_n \geqslant t$ and $N = k$ then at least for one $i = 1,\ldots,k$ we have $Y_i \geqslant y/k$, which occurs with probability no greater than $\sum_{i=1}^{k}\mathbb{P}(Y_i \geqslant y/k)$; thus

$$\begin{aligned}
\mathbb{P}\Big(X^1 \geqslant t - \frac{1}{\alpha-1}\Big) &\leqslant \sum_{k=1}^{\infty}\mathbb{P}(N=k)\Big(\sum_{i=1}^{k}\mathbb{P}(Y_i \geqslant y/k)\Big)\\
&\leqslant e^{-1/\alpha}\sum_{k=1}^{\infty}\frac{1}{\alpha^k k!}k\Big(\frac{y}{k}\Big)^{-\alpha}\\
&= \Big(e^{-1/\alpha}\sum_{k=1}^{\infty}\frac{k^{\alpha+1}}{\alpha^k k!}\Big)\frac{1}{y^\alpha} \leqslant \frac{2}{y^\alpha},
\end{aligned}$$

where we have used the fact that for each k the function $e^{-1/\alpha}\frac{k^{\alpha+1}}{\alpha^k k!}$ is decreasing in α, so we plug in $\alpha = 1$ and notice that $\sum_{k=1}^{\infty}\frac{k^2}{k!} = 2e$. ∎

Now we proceed to analyse X_1, which is a more delicate task.

Lemma 4.2. *For* $0 \leqslant y \leqslant \frac{1}{2-\alpha}$ *one has*

$$\mathbb{P}(X_1 \geqslant y) \leqslant e^{\frac{1}{4}} e^{-\frac{1}{2}(2-\alpha)y^2},$$

and for $0 \leqslant y \leqslant \frac{2}{2-\alpha}$,

$$\mathbb{P}(X_1 \leqslant -y) \leqslant e^{4/3} e^{-\frac{1}{2}(2-\alpha)y^2}.$$

Proof. We calculate

$$\begin{aligned}\mathbb{E}\exp(tX_1) &= \exp\left(\int_0^1 (e^{tx} - 1 - tx)\,\frac{dx}{x^{\alpha+1}}\right)\\ &= \exp\left(\frac{1}{2}\,\frac{t^2}{2-\alpha} + \int_0^1 \left(e^{tx} - 1 - tx - \frac{1}{2}t^2x^2\right)\frac{dx}{x^{\alpha+1}}\right).\end{aligned}$$

We estimate the integrand using the following observation. Since $tx \geqslant 0$, we have

$$\begin{aligned}e^{tx} - 1 - tx - \frac{1}{2}t^2x^2 &= \frac{1}{3!}t^3x^3\sum_{k=0}^{\infty}\frac{3!}{(k+3)!}t^kx^k = \frac{1}{3!}t^3x^3\sum_{k=0}^{\infty}\frac{3!k!}{(k+3)!}\,\frac{1}{k!}t^kx^k\\ &\leqslant \frac{1}{6}t^3x^3\left(\frac{3}{4} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{k!}t^kx^k\right) = \frac{1}{6}t^3x^3\left(\frac{3}{4} + \frac{1}{4}e^{tx}\right).\end{aligned}$$

For $t \geqslant 0$ we estimate

$$\begin{aligned}\mathbb{E}\exp(tX_1) &\leqslant \exp\left(\frac{1}{2}\,\frac{t^2}{2-\alpha} + \left(\frac{1}{8}t^3 + \frac{1}{24}t^3e^t\right)\int_0^1 x^3\,\frac{dx}{x^{\alpha+1}}\right)\\ &= \exp\left(\frac{1}{2}\,\frac{t^2}{2-\alpha} + \frac{1}{3-\alpha}\left(\frac{1}{8}t^3 + \frac{1}{24}t^3e^t\right)\right).\end{aligned}\tag{4.3}$$

Now, for $0 \leqslant y \leqslant \frac{1}{2-\alpha}$, taking $t_y = (2-\alpha)y$ we get $t_y \leqslant 1$. By Chebyshev's inequality and (4.3) we get

$$\begin{aligned}\mathbb{P}(X_1 \geqslant y) &\leqslant \mathbb{E}\exp(t_yX_1)e^{-t_yy} \leqslant \exp\left(-\frac{1}{2}(2-\alpha)y^2 + \frac{1}{8} + \frac{1}{24}e\right)\\ &\leqslant e^{1/4}e^{-1/2(2-\alpha)y^2}.\end{aligned}$$

Similarly, since for $tx \leqslant 0$ we have $|e^{tx} - 1 - tx - \frac{1}{2}t^2x^2| \leqslant \frac{1}{3!}|t^3x^3|$, for $t \leqslant 0$ we get

$$\mathbb{E}\exp(tX_1) \leqslant \exp\biggl(\frac{1}{2}\,\frac{t^2}{2-\alpha} + \frac{|t^3|}{6}\int_0^1 x^3\,\frac{dx}{x^{\alpha+1}}\biggr) = \exp\biggl(\frac{1}{2}\,\frac{t^2}{2-\alpha} + \frac{1}{6}\frac{|t^3|}{3-\alpha}\biggr). \tag{4.4}$$

Again, for $0 \leqslant y \leqslant \frac{2}{2-\alpha}$, taking $t_y = -(2-\alpha)y$, by Chebyshev's inequality we get

$$\mathbb{P}(X_1 \leqslant -y) \leqslant \mathbb{E}\exp(-t_yX_1)e^{t_y y} \leqslant \exp\biggl(-\frac{1}{2}(2-\alpha)y^2 + \frac{8}{6}\biggr). \quad \blacksquare$$

For lower bounds we use the Paley–Zygmund inequality.

Lemma 4.3. *For $\alpha \in (7/4, 2)$ and $y \in \bigl[\frac{2}{\sqrt{2-\alpha}}, \frac{1}{2-\alpha}\bigr]$ one has*

$$\mathbb{P}\biggl(X_1 \geqslant \frac{1}{4}y\biggr) \geqslant 10^{-2}e^{-(2-\alpha)y^2}, \tag{4.5}$$

while for $\alpha \in (1,2)$ and $y \in \bigl[\frac{2}{\sqrt{2-\alpha}}, \frac{2}{2-\alpha}\bigr]$ one has

$$\mathbb{P}\biggl(X_1 \leqslant -\frac{1}{24}y\biggr) \geqslant e\cdot 10^{-3}e^{-(2-\alpha)y^2}. \tag{4.6}$$

Proof. Since for $tx \geqslant 0$ we have $e^{tx} - 1 - tx - \frac{1}{2}t^2x^2 \geqslant 0$, for $t \geqslant 0$ we get

$$\mathbb{E}\exp(tX_1) \geqslant \exp\biggl(\frac{1}{2}\,\frac{t^2}{2-\alpha}\biggr). \tag{4.7}$$

Next, notice that for $y \geqslant \frac{2}{\sqrt{2-\alpha}}$ we have $\frac{1}{y} \leqslant \frac{2-\alpha}{4}y$ so

$$\frac{1}{2}y - \frac{1}{2-\alpha}\,\frac{1}{y} \geqslant \frac{1}{2}y - \frac{1}{2-\alpha}\,\frac{2-\alpha}{4}y = \frac{1}{4}y,$$

and for $t_y = (2-\alpha)y$ and $\lambda = \frac{1}{e}$ by (4.7) we have

$$\frac{1}{t_y}\ln(\lambda\mathbb{E}\exp(t_yX_1)) \geqslant \frac{1}{t_y}\ln\biggl(\lambda\exp\biggl(\frac{1}{2}\,\frac{t_y^2}{2-\alpha}\biggr)\biggr) = \frac{1}{2}\,\frac{t_y}{2-\alpha} + \frac{\ln\lambda}{t_y} = \frac{1}{2}y - \frac{1}{2-\alpha}\,\frac{1}{y} \geqslant \frac{1}{4}y.$$

This together with the Paley–Zygmund inequality, (4.7) and (4.3) (notice that $t_y \leqslant 1$ for $y \leqslant \frac{1}{2-\alpha}$) yields (4.5):

$$\begin{aligned}
\mathbb{P}\Big(X_1 \geqslant \frac{1}{4}y\Big) &\geqslant \mathbb{P}\Big(X_1 \geqslant \frac{1}{t_y}\ln(\lambda\mathbb{E}\exp(t_yX_1))\Big)\\
&= \mathbb{P}(\exp(t_yX_1) \geqslant \lambda\mathbb{E}\exp(t_yX_1))\\
&\geqslant \Big(1-\frac{1}{e}\Big)^2\frac{(\mathbb{E}\exp(t_yX_1))^2}{\mathbb{E}\exp(2t_yX_1)}\\
&\geqslant \Big(1-\frac{1}{e}\Big)^2\frac{\exp\big(\frac{t_y^2}{2-\alpha}\big)}{\exp\big(\frac{2t_y^2}{2-\alpha}+t_y^3+\frac{1}{3}t_y^3e^{2t_y}\big)}\\
&\geqslant \Big(1-\frac{1}{e}\Big)^2 e^{-(1+\frac{1}{3}e^2)}e^{-(2-\alpha)y^2} \geqslant 10^{-2}e^{-(2-\alpha)y^2}.
\end{aligned}$$

For negative tails we use the estimate $e^{tx}-1-tx-\frac{1}{2}t^2x^2 \geqslant \frac{1}{3!}t^3x^3$ for $tx \leqslant 0$, which for $t \leqslant 0$ yields

$$\mathbb{E}\exp(tX_1) \geqslant \exp\Big(\frac{1}{2}\frac{t^2}{2-\alpha}-\frac{1}{6}|t^3|\Big). \tag{4.8}$$

Next, notice that for $\frac{2}{\sqrt{2-\alpha}} \leqslant y \leqslant \frac{2}{2-\alpha}$ we have $\frac{1}{y} \leqslant \frac{2-\alpha}{4}y$ and $(2-\alpha)^2y^2 \leqslant 4 \leqslant 2y$, so

$$\frac{1}{2}y-\frac{1}{2-\alpha}\frac{1}{2y}-\frac{1}{6}(2-\alpha)^2y^2 \geqslant \frac{1}{2}y-\frac{1}{2-\alpha}\frac{2-\alpha}{8}y-\frac{1}{3}y=\frac{1}{24}y.$$

From this for $t_y=-(2-\alpha)y$ and $\lambda = 1/\sqrt{e}$, by (4.8) we have

$$\begin{aligned}
\frac{1}{|t_y|}\ln(\lambda\mathbb{E}\exp(t_yX_1)) &\geqslant \frac{1}{|t_y|}\ln\Big(\lambda\exp\Big(\frac{1}{2}\frac{t_y^2}{2-\alpha}-\frac{1}{6}|t_y^3|\Big)\Big)\\
&= \frac{1}{2}\frac{|t_y|}{2-\alpha}+\frac{\ln\lambda}{|t_y|}-\frac{1}{6}|t_y^2|\\
&= \frac{1}{2}y-\frac{1}{2-\alpha}\frac{1}{2y}-\frac{1}{6}(2-\alpha)^2y^2 \geqslant \frac{1}{24}y.
\end{aligned}$$

This together with the Paley–Zygmund inequality yields

$$\begin{aligned}
\mathbb{P}\Big(X_1 \leqslant -\frac{1}{24}y\Big) &\geqslant \mathbb{P}\Big(X_1 \leqslant \frac{1}{t_y}\ln(\lambda\mathbb{E}\exp(t_yX_1))\Big) \\
&= \mathbb{P}(\exp(t_yX_1) \geqslant \lambda\mathbb{E}\exp(t_yX_1)) \\
&\geqslant \Big(1-\frac{1}{\sqrt{e}}\Big)^2 \frac{(\mathbb{E}\exp(t_yX_1))^2}{\mathbb{E}\exp(2t_yX_1)} \\
&\geqslant \Big(1-\frac{1}{\sqrt{e}}\Big)^2 \frac{\exp\big(\frac{t_y^2}{2-\alpha}-\frac{8}{3}\big)}{\exp\big(\frac{2t_y^2}{2-\alpha}+\frac{8}{6}\big)} \\
&= e^{2\ln(\sqrt{e}-1)-5}e^{-(2-\alpha)y^2} \geqslant e\cdot 10^{-3}e^{-(2-\alpha)y^2}. \quad \blacksquare
\end{aligned}$$

To complete the picture we estimate $\mathbb{P}(X_1 \leqslant -y)$ in the case $y \geqslant \frac{2}{2-\alpha}$.

Lemma 4.4. *For $y \geqslant \frac{2}{2-\alpha}$ one has*

$$\mathbb{P}(X_1 \leqslant -y) \leqslant \exp\Big(-\frac{\big(\frac{1}{2}\big(y+\frac{1}{\alpha-1}\big)\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\Big)$$

and

$$\mathbb{P}\Big(X_1 \leqslant -\Big(\frac{1}{e}-\frac{1}{4}\Big)y\Big) \geqslant \Big(1-\frac{1}{\sqrt{e}}\Big)^2 \exp\Big(-\frac{\big(\sqrt{4-\frac{2}{e}}\big(y+\frac{1}{\alpha-1}\big)\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\Big).$$

Proof. For $t < -1$ we first split

$$\int_0^1 (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} = \int_0^{1/|t|} (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} + \int_{1/|t|}^1 (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}}.$$

For $t < -1$ and $0 \leqslant x \leqslant 1/|t|$ we calculate $e^{tx}-1-tx \leqslant \frac{1}{2}t^2x^2 \leqslant t^2x^2$ and get

$$\int_0^{1/|t|} (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} \leqslant \int_0^{1/|t|} t^2x^2\,\frac{dx}{x^{\alpha+1}} = \frac{1}{2-\alpha}|t|^\alpha.$$

Next, for $x > 1/|t|$ we bound $e^{tx}-1-tx \leqslant -tx = |t|x$ and get

$$\int_{1/|t|}^1 (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} \leqslant |t|\int_{1/|t|}^1 x\,\frac{dx}{x^{\alpha+1}} = \frac{1}{\alpha-1}(|t|^\alpha-|t|).$$

Finally, we arrive at

$$\int_0^1 (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} \leqslant \Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t|^\alpha - \frac{1}{\alpha-1}|t|,$$

which yields, for $t < -1$,

$$\mathbb{E}\exp(tX_1) \leqslant \exp\bigg(\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t|^\alpha - \frac{1}{\alpha-1}|t|\bigg). \tag{4.9}$$

Let $y \geqslant \frac{2}{2-\alpha}$ and $t_y < -1$ be such that

$$\alpha\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t_y|^{\alpha-1} = y + \frac{1}{\alpha-1}. \tag{4.10}$$

We estimate

$$\begin{aligned}
\mathbb{P}(X_1 < -y) &\leqslant \mathbb{E}\exp(-t_y X)e^{t_y y}\\
&\leqslant \exp\bigg(\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t_y|^\alpha - \frac{1}{\alpha-1}|t_y| - y|t_y|\bigg)\\
&= \exp\bigg(-(\alpha-1)\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t_y|^\alpha\bigg)\\
&= \exp\bigg(-\frac{\big((y+\frac{1}{\alpha-1})(\alpha-1)^{(\alpha-1)/\alpha}/\alpha\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\bigg)\\
&\leqslant \exp\bigg(-\frac{\big(\frac{1}{2}(y+\frac{1}{\alpha-1})\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\bigg),
\end{aligned}$$

where we have used $\inf_{\alpha\in(1,2)} \frac{(\alpha-1)^{(\alpha-1)/\alpha}}{\alpha} = \frac{1}{2}$. On the other hand, for $t < -1$ and $0 \leqslant x \leqslant 1/|t|$ we have $e^{tx} - 1 - tx \geqslant \frac{1}{e}t^2x^2$ and we get

$$\int_0^{1/|t|} (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} \geqslant \int_0^{1/|t|} \frac{t^2x^2}{e}\,\frac{dx}{x^{\alpha+1}} = \frac{1}{e}\,\frac{1}{2-\alpha}|t|^\alpha.$$

Similarly, for $x > 1/|t|$ we bound $e^{tx} - 1 - tx \geqslant -\frac{1}{e}tx = \frac{1}{e}|t|x$. So,

$$\int_{1/|t|}^1 (e^{tx}-1-tx)\,\frac{dx}{x^{\alpha+1}} \geqslant \frac{1}{e}|t|\int_{1/|t|}^1 x\,\frac{dx}{x^{\alpha+1}} = \frac{1}{e}\,\frac{1}{\alpha-1}(|t|^\alpha - |t|).$$

Finally, we arrive at the estimate

$$\mathbb{E}\exp(tX_1) \geqslant \exp\bigg(\frac{1}{e}\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|t|^\alpha - \frac{1}{e}\,\frac{1}{\alpha-1}|t|\bigg), \tag{4.11}$$

which for $\tilde{t}_y < -1$ satisfying

$$\Big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\Big)|\tilde{t}_y|^{\alpha-1} = \frac{1}{\alpha-1} + y, \tag{4.12}$$

which is equivalent to

$$\left(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\right)|\tilde{t}_y|^\alpha-\frac{1}{\alpha-1}|\tilde{t}_y|=|\tilde{t}_y|y,$$

and for $\lambda=\frac{1}{\sqrt{e}}$ yields

$$\begin{aligned}(4.13)\qquad &\frac{1}{|\tilde{t}_y|}\ln(\lambda\mathbb{E}\exp(\tilde{t}_yX))\\ &\geqslant\frac{1}{|\tilde{t}_y|}\left(\ln(\lambda)+\frac{1}{e}\left(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\right)|\tilde{t}_y|^\alpha-\frac{1}{e}\,\frac{1}{\alpha-1}|\tilde{t}_y|\right)\\ &=\frac{1}{|\tilde{t}_y|}\left(\ln(\lambda)+\frac{1}{e}|\tilde{t}_y|y\right)=\frac{1}{e}y-\frac{1}{2|\tilde{t}_y|}.\end{aligned}$$

To estimate $1/|\tilde{t}_y|$ notice that from (4.12) for $y\geqslant\frac{2}{2-\alpha}\geqslant 2$ we have

$$|\tilde{t}_y|\geqslant|\tilde{t}_y|^{\alpha-1}=\frac{\frac{1}{\alpha-1}+y}{\frac{1}{2-\alpha}+\frac{1}{\alpha-1}}\geqslant 1\geqslant\frac{2}{y},$$

which together with (4.13) yields

$$\frac{1}{|\tilde{t}_y|}\ln(\lambda\mathbb{E}\exp(\tilde{t}_yX))\geqslant\frac{1}{e}y-\frac{1}{2|\tilde{t}_y|}\geqslant\left(\frac{1}{e}-\frac{1}{4}\right)y.$$

Finally, using the estimate just obtained, the Paley–Zygmund inequality, (4.9) and (4.11) we arrive at

$$\begin{aligned}\mathbb{P}\left(X_1\leqslant-\left(\frac{1}{e}-\frac{1}{4}\right)y\right)&\geqslant\mathbb{P}\left(X_1\leqslant-\frac{1}{|\tilde{t}_y|}\ln(\lambda\mathbb{E}\exp(\tilde{t}_yX_1))\right)\\ &=\mathbb{P}\big(\exp(\tilde{t}_yX)\geqslant\lambda\mathbb{E}\exp(\tilde{t}_yX_1)\big)\geqslant\left(1-\frac{1}{\sqrt{e}}\right)^2\frac{(\mathbb{E}\exp(\tilde{t}_yX_1))^2}{\exp(2\tilde{t}_yX_1)}\\ &\geqslant\left(1-\frac{1}{\sqrt{e}}\right)^2\frac{\exp\big(\frac{2}{e}\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)|\tilde{t}_y|^\alpha-\frac{2}{e}\frac{1}{\alpha-1}|\tilde{t}_y|\big)}{\exp\big(\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)|2\tilde{t}_y|^\alpha-\frac{1}{\alpha-1}|2\tilde{t}_y|\big)}\\ &\geqslant\left(1-\frac{1}{\sqrt{e}}\right)^2\exp\left(-\left(4-\frac{2}{e}\right)\left(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\right)|\tilde{t}_y|^\alpha\right)\\ &=\left(1-\frac{1}{\sqrt{e}}\right)^2\exp\left(-\frac{\big(4-\frac{2}{e}\big)\big(y+\frac{1}{\alpha-1}\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\right)\\ &\geqslant\left(1-\frac{1}{\sqrt{e}}\right)^2\exp\left(-\frac{\big(\sqrt{4-\frac{2}{e}}\big(y+\frac{1}{\alpha-1}\big)\big)^{\alpha/(\alpha-1)}}{\big(\frac{1}{2-\alpha}+\frac{1}{\alpha-1}\big)^{1/(\alpha-1)}}\right).\qquad\blacksquare\end{aligned}$$

As an easy consequence of Lemmas 4.1–4.3 we have the following theorem.

THEOREM 4.5. *Let X be a strictly asymmetric α-stable random variable with characteristic function* (2.3). *For any $\alpha \in (7/4, 2)$ and $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{1}{2-\alpha}\right]$ one has*

$$\mathbb{P}\left(X \geqslant 2y - \frac{1}{\alpha-1}\right) \leqslant \frac{2}{e}\,\frac{1}{y^\alpha} + e^{1/4} e^{-1/2(2-\alpha)y^2}, \tag{4.14}$$

$$\mathbb{P}\left(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\right) \geqslant \frac{1}{400\sqrt{e}}\left(30\frac{1}{y^\alpha} + e^{-(2-\alpha)y^2}\right); \tag{4.15}$$

while for $\alpha \in (1,2)$ and $y \geqslant \frac{1}{2-\alpha}$ one has

$$\mathbb{P}\left(X \geqslant 2y - \frac{1}{\alpha-1}\right) \leqslant \frac{8}{y^\alpha}, \tag{4.16}$$

$$\mathbb{P}\left(X \geqslant y - \frac{1}{\alpha-1}\right) \geqslant 16 \cdot 10^{-3}\frac{1}{y^\alpha}. \tag{4.17}$$

REMARK 4.6. Notice that from (4.15) it follows that for α close to 2 (in fact for $\alpha > 7/4$) and $y = \frac{2}{\sqrt{2-\alpha}}$ the probability $\mathbb{P}\big(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\big)$ is of order $O(1)$. We seemingly lack estimates for $\alpha \in (1, 7/4)$ but in this case $\frac{1}{2-\alpha} = O(1)$ and from (4.17) it follows that for $\alpha \in (1, 7/4)$ the probability $\mathbb{P}\big(X \geqslant y - \frac{1}{\alpha-1}\big)$ is of order $O(1)$ even for $y = \frac{1}{2-\alpha}$.

Proof of Theorem 4.5. To prove (4.14) we estimate

$$\mathbb{P}\left(X \geqslant 2y - \frac{1}{\alpha-1}\right) \leqslant \mathbb{P}\left(X^1 \geqslant y - \frac{1}{\alpha-1}\right) + \mathbb{P}(X_1 \geqslant y)$$

and then use (4.2) and the upper bound for $\mathbb{P}(X_1 \geqslant y)$ from Lemma 4.2.

To prove (4.15) we write, for $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{1}{2-\alpha}\right]$,

$$\mathbb{P}\left(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\right) \geqslant \mathbb{P}\left(X^1 \geqslant \frac{5}{4}y - \frac{1}{\alpha-1}\right)\mathbb{P}(X_1 \geqslant -y)$$

and then use (4.1) and Lemma 4.2 to obtain

$$\begin{aligned}\mathbb{P}\left(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\right) &\geqslant \mathbb{P}\left(X^1 \geqslant \frac{5}{4}y - \frac{1}{\alpha-1}\right)\mathbb{P}(X_1 \geqslant -y)\\ &\geqslant \frac{1}{2\sqrt{e}}\,\frac{4^\alpha}{5^\alpha y^\alpha}\big(1 - e^{4/3}e^{-\frac{1}{2}(2-\alpha)y^2}\big)\\ &\geqslant \frac{1}{2\sqrt{e}}\,\frac{16}{25y^\alpha}\big(1 - e^{4/3}e^{-\frac{1}{2}(2-\alpha)\frac{4}{2-\alpha}}\big)\\ &\geqslant \frac{0.3}{2\sqrt{e}}\,\frac{1}{y^\alpha}.\end{aligned} \tag{4.18}$$

Next, for $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{1}{2-\alpha}\right]$ we also have

$$\mathbb{P}\Big(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\Big) \geqslant \mathbb{P}\Big(X^1 \geqslant 1 - \frac{1}{\alpha-1}\Big)\mathbb{P}\Big(X_1 \geqslant \frac{1}{4}y\Big),$$

which together with (4.1) and (4.5) gives

$$\mathbb{P}\Big(X \geqslant \frac{1}{4}y - \frac{1}{\alpha-1}\Big) \geqslant \frac{1}{2\sqrt{e}}10^{-2}e^{-(2-\alpha)y^2}. \tag{4.19}$$

Summing (4.18) and (4.19) we get (4.15).

To prove (4.16), we differentiate (2.4) and get

$$\mathbb{E}X_1 = 0, \qquad \mathbb{E}X_1^2 = \int_0^1 x^2 \,\frac{dx}{x^{\alpha+1}} = \frac{1}{2-\alpha}$$

and

$$\mathbb{E}X_1^4 = 3(\mathbb{E}X_1^2)^2 + \int_0^1 x^4 \,\frac{dx}{x^{\alpha+1}} = \frac{3}{(2-\alpha)^2} + \frac{1}{4-\alpha}.$$

From this we easily get for any $y > 0$ the estimate

$$\mathbb{P}(X_1 \geqslant y) \leqslant \mathbb{P}(|X_1| \geqslant y) \leqslant \frac{\mathbb{E}X_1^4}{y^4} = \frac{3}{(2-\alpha)^2 y^4} + \frac{1}{(4-\alpha)y^4}, \tag{4.20}$$

and since for $y \geqslant \frac{1}{2-\alpha}$,

$$\frac{1}{(4-\alpha)y^4} \leqslant \frac{1}{(2-\alpha)^2 y^4} \leqslant \frac{1}{y^2} \leqslant \frac{1}{y^\alpha}, \tag{4.21}$$

using also (4.2) we obtain (4.16):

$$\begin{aligned}\mathbb{P}\Big(X \geqslant 2y - \frac{1}{\alpha-1}\Big) &\leqslant \mathbb{P}(X_1 \geqslant y) + \mathbb{P}\Big(X^1 \geqslant y - \frac{1}{\alpha-1}\Big) \\ &\leqslant \frac{3}{y^\alpha} + \frac{1}{y^\alpha} + \frac{5}{\sqrt{e}}\,\frac{1}{y^\alpha} \leqslant \frac{8}{y^\alpha}.\end{aligned}$$

To prove (4.17), for $y \geqslant \frac{1}{2-\alpha}$ we write

$$\mathbb{P}\Big(X \geqslant y - \frac{1}{\alpha-1}\Big) \geqslant \mathbb{P}\Big(X^1 \geqslant 3y - \frac{1}{\alpha-1}\Big)\mathbb{P}(X_1 \geqslant -2y)$$

and then use (4.1) and Lemma 4.2 to obtain

$$\begin{aligned}\mathbb{P}\Big(X \geqslant y - \frac{1}{\alpha-1}\Big) &\geqslant \mathbb{P}\Big(X^1 \geqslant 3y - \frac{1}{\alpha-1}\Big)\mathbb{P}(X_1 \geqslant -2y) \\ &\geqslant \frac{1}{2\sqrt{e}}\,\frac{1}{3^\alpha y^\alpha}\mathbb{P}\Big(X_1 \geqslant -\frac{2}{2-\alpha}\Big) \\ &\geqslant \frac{1}{2\sqrt{e}}\,\frac{1}{9y^\alpha}\big(1 - e^{4/3}e^{-1/2(2-\alpha)\frac{4}{(2-\alpha)^2}}\big) \geqslant 16\cdot 10^{-3}\frac{1}{y^\alpha}. \quad \blacksquare\end{aligned}$$

REMARK 4.7. For $\delta \in (0, 1/e)$ the equation $\delta \cdot y = \ln y$ has exactly two solutions $1 < y_1 < e < y_2$, and the larger one satisfies

$$\frac{1}{\delta}\ln\left(\frac{1}{\delta}\right) < y_2 < \frac{2}{\delta}\ln\left(\frac{1}{\delta}\right).$$

From this we see that for $\alpha \approx 2$, the term containing $\frac{1}{y^\alpha}$ in (4.14) and (4.15) starts to dominate the term containing $\exp(-\kappa(2-\alpha)y^2)$, $\kappa \in \{1/2, 1\}$, already for

$$y = O\left(\sqrt{\frac{1}{2-\alpha}\ln\left(\frac{1}{2-\alpha}\right)}\right).$$

Finally, to complete the picture, we analyse the decay of left tails of X.

THEOREM 4.8. *Let X be a strictly asymmetric α-stable random variable, $\alpha \in (1, 2)$, with characteristic function* (2.3). *For any $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{2}{(2-\alpha)}\right]$ one has*

$$\mathbb{P}\left(X \leqslant -y - \frac{1}{\alpha-1}\right) \leqslant e^{4/3}e^{-1/2(2-\alpha)y^2}, \tag{4.22}$$

$$\mathbb{P}\left(X \leqslant -\frac{1}{24}y - \frac{1}{\alpha-1}\right) \geqslant 10^{-3}e^{-(2-\alpha)y^2}; \tag{4.23}$$

while for $y \geqslant \frac{2}{2-\alpha}$ one has

$$\mathbb{P}\left(X \leqslant -y - \frac{1}{\alpha-1}\right) \leqslant \exp\left(-\frac{\left(\frac{1}{2}\left(y + \frac{1}{\alpha-1}\right)\right)^{\alpha/(\alpha-1)}}{\left(\frac{1}{2-\alpha} + \frac{1}{\alpha-1}\right)^{1/(\alpha-1)}}\right) \tag{4.24}$$

and

$$\mathbb{P}\left(X \leqslant -\left(\frac{1}{e} - \frac{1}{4}\right)y - \frac{1}{\alpha-1}\right) \geqslant e^{-1}\exp\left(-\frac{\left(\sqrt{4-\frac{2}{e}}\left(y + \frac{1}{\alpha-1}\right)\right)^{\alpha/(\alpha-1)}}{\left(\frac{1}{2-\alpha} + \frac{1}{\alpha-1}\right)^{1/(\alpha-1)}}\right). \tag{4.25}$$

Proof. Estimate (4.22) follows from Lemma 4.2 and the fact that $X^1 \geqslant \frac{-1}{\alpha-1}$. Estimate (4.23) follows from Lemma 4.3 and the fact that $\mathbb{P}\left(X^1 = \frac{-1}{\alpha-1}\right) = e^{-1/\alpha} \geqslant 1/e$.

Similarly, estimate (4.24) follows from Lemma 4.4 and the fact that

$$X^1 \geqslant \frac{-1}{\alpha-1},$$

while (4.25) follows from the estimate

$$\mathbb{P}\left(X^1 = \frac{-1}{\alpha-1}\right) = e^{-1/\alpha} \geqslant \frac{1}{e}$$

and Lemma 4.4. ■

4.2. Symmetric case. In this section we provide tail estimates for symmetric α-stable random variables in the case when $\alpha \in (1,2)$. We follow two different approaches and as a consequence we obtain two types of bounds. The first method was already presented in Theorem 3.2. Estimates obtained in this way hold on the whole real line, but do not capture an important property one might expect for α close to 2, namely the Gaussian behavior of the tail which has already been presented in the asymmetric case. For this reason we also give a reasoning analogous to that in the previous section, i.e. we estimate $\tilde{X}^1$ and $\tilde{X}_1$ with characteristic functions (2.7) and (2.8) respectively. To simplify the notation we denote $\tilde{X}^1$ by X^1 and $\tilde{X}_1$ by X_1.

Lemma 4.9. *Let $y \geqslant 1$. Then*

$$\mathbb{P}(X^1 \geqslant y) \geqslant \frac{1}{e}\,\frac{1}{y^\alpha}, \tag{4.26}$$

$$\mathbb{P}(X^1 \geqslant y) \leqslant \frac{1}{y^\alpha}\,\frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{-2/\alpha}(2/\alpha)^k k^{\alpha+1}}{k!} \leqslant \frac{10}{3}\,\frac{1}{y^\alpha}. \tag{4.27}$$

Proof. Recall that $X^1 = \sum_{k=1}^{N} Y_k$, where $\mathbb{P}(N=k) = \frac{e^{-2/\alpha}(2/\alpha)^k}{k!}$ and each Y_k has density $\frac{\alpha}{2|x|^{\alpha+1}}\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)$. Arguing in the same manner as in Lemma 4.1 we obtain

$$\mathbb{P}(X^1 \geqslant y) \geqslant \mathbb{P}(N=1)\mathbb{P}(Y_1 \geqslant y) = \frac{2}{\alpha}e^{-2/\alpha}\frac{1}{y^\alpha} \geqslant \frac{1}{e}\,\frac{1}{y^\alpha},$$

since $\frac{2}{\alpha}e^{-2/\alpha}$ is increasing for $\alpha \in (1,2)$. For the upper bound we have

$$\begin{aligned}\mathbb{P}(X^1 > y) &\leqslant \sum_{k=1}^{\infty}\mathbb{P}(N=k)\mathbb{P}\left(Y > \frac{y}{k}\right)\\ &\leqslant \sum_{k=1}^{\infty}\frac{e^{-2/\alpha}(2/\alpha)^k}{k!}\,\frac{k}{2}\left(\frac{y}{k}\right)^{-\alpha}\\ &= \frac{1}{y^\alpha}\,\frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{-2/\alpha}(2/\alpha)^k k^{\alpha+1}}{k!} \leqslant \frac{10}{3}\,\frac{1}{y^\alpha},\end{aligned}$$

where we estimate the function $\frac{e^{-2/\alpha}(2/\alpha)^k k^{\alpha+1}}{k!}$ for $k = 1,2,3$ by its value at $\alpha = 2$ and for $k = 4,5,\dots$ by the value at $\alpha = 1$. ■

For both upper and lower bounds of tails of X_1 we need an estimate for its Laplace transform.

Lemma 4.10. *Let X_1 be a random variable with characteristic function* (2.8). *Then for $t \in \mathbb{R}$,*

$$\mathbb{E}(\exp(tX_1)) \leqslant \exp\left(\frac{1}{24}t^4\left(\frac{14}{15} + \frac{1}{15}\cosh(t)\right)\right)\exp\left(\frac{1}{2-\alpha}t^2\right), \tag{4.28}$$

$$\mathbb{E}(\exp(tX_1)) \geqslant \exp\left(\frac{1}{2-\alpha}t^2\right). \tag{4.29}$$

Proof. We simply calculate

$$\begin{aligned}
\mathbb{E}(\exp(tX_1)) &= \exp\left(\int_{-1}^{1}(e^{tx} - 1 - tx)\,\frac{dx}{|x|^{\alpha+1}}\right)\\
&= \exp\left(2\int_{0}^{1}(\cosh(tx) - 1)\,\frac{dx}{x^{\alpha+1}}\right)\\
&= \exp\left(2\int_{0}^{1}\sum_{k=1}^{\infty}\frac{(t^2x^2)^k}{(2k)!}\,\frac{dx}{x^{\alpha+1}}\right)\\
&= \exp\left(\int_{0}^{1}t^2x^2\,\frac{dx}{x^{\alpha+1}} + 2\int_{0}^{1}\sum_{k=2}^{\infty}\frac{(t^2x^2)^k}{(2k)!}\,\frac{dx}{x^{\alpha+1}}\right)\\
&= \exp\left(\frac{t^2}{2-\alpha} + \frac{2}{4!}\int_{0}^{1}t^4x^4\sum_{k=0}^{\infty}\frac{4!(2k)!(x^2t^2)^k}{(2k+4)!(2k)!}\,\frac{dx}{x^{\alpha+1}}\right)\\
&\leqslant \exp\left(\frac{t^2}{2-\alpha} + \frac{2}{4!}\int_{0}^{1}t^4x^4\left(\frac{14}{15} + \frac{1}{15}\cosh(tx)\right)\frac{dx}{x^{\alpha+1}}\right)\\
&\leqslant \exp\left(\frac{t^2}{2-\alpha} + \frac{2}{4!}t^4\left(\frac{14}{15} + \frac{1}{15}\cosh(t)\right)\frac{1}{4-\alpha}\right)\\
&\leqslant \exp\left(\frac{1}{24}t^4\left(\frac{14}{15} + \frac{1}{15}\cosh(t)\right)\right)\exp\left(\frac{1}{2-\alpha}t^2\right).
\end{aligned}$$

The lower bound is obvious from the fourth line above. ■

Lemma 4.11. *For $0 \leqslant y \leqslant \frac{2}{2-\alpha}$,*

$$\mathbb{P}(X_1 \geqslant y) \leqslant e^{2/45}e^{-1/4(2-\alpha)y^2}, \tag{4.30}$$

and for $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{2}{2-\alpha}\right]$,

$$\mathbb{P}\left(X_1 \geqslant \frac{\sqrt{2}}{4}y\right) \geqslant \frac{1}{137}e^{-(2-\alpha)y^2}. \tag{4.31}$$

Proof. Denote $C(t) = \frac{1}{24}t^4\left(\frac{14}{15} + \frac{1}{15}\cosh(t)\right)$. By Chebyshev's inequality and (4.28) we get

$$\mathbb{P}(X_1 > y) \leqslant \frac{\mathbb{E}(tX_1)}{\exp(ty)} \leqslant \exp(C(t))\exp\left(\frac{t^2}{2-\alpha} - ty\right).$$

Choose $t = \frac{2-\alpha}{2}y$, so $t \leqslant 1$. Then, since $\cosh(1) \leqslant 2$, we have $C(t) \leqslant \frac{2}{45}$ and we conclude that $\mathbb{P}(X_1 \geqslant y) \leqslant e^{\frac{2}{45}} e^{-\frac{1}{4}(2-\alpha)y^2}$.

To prove the lower bound we use the Paley–Zygmund inequality in the following way. Let $\lambda \in (0,1)$. Then

$$\begin{aligned}\mathbb{P}(\exp(tX_1) \geqslant \lambda \mathbb{E}\exp(tX_1)) &\geqslant (1-\lambda)^2 \frac{(\mathbb{E}(tX_1))^2}{\mathbb{E}(2tX_1)} \\ &\geqslant (1-\lambda)^2 \frac{\exp\left(\frac{2t^2}{2-\alpha}\right)}{\exp(C(2t))\exp\left(\frac{4t^2}{2-\alpha}\right)} \\ &= (1-\lambda)^2 \exp(-C(2t))\exp\left(-\frac{2t^2}{2-\alpha}\right).\end{aligned}$$

Choose $t = \frac{y(2-\alpha)}{\sqrt{2}}$, so $t \leqslant \sqrt{2}$ and $C(2t) \leqslant C(2\sqrt{2})$. Moreover, since $y \geqslant \frac{2}{\sqrt{2-\alpha}}$, we have, for $\lambda = 1/e$,

$$\frac{1}{t}\ln(\lambda \mathbb{E}\exp(tX_1)) \geqslant \frac{1}{t}\ln\left(\lambda \exp\left(\frac{t^2}{2-\alpha}\right)\right) = \frac{y}{\sqrt{2}} - \frac{\sqrt{2}}{(2-\alpha)y} \geqslant \frac{\sqrt{2}}{4}y,$$

so finally

$$\begin{aligned}\mathbb{P}\left(X_1 \geqslant \frac{\sqrt{2}}{4}y\right) &\geqslant \mathbb{P}(\exp(tX_1) \geqslant \lambda \mathbb{E}\exp(tX_1)) \\ &\geqslant (1-e^{-1})^2 e^{-C(2\sqrt{2})} e^{-(2-\alpha)y^2} \geqslant \frac{1}{137} e^{-(2-\alpha)y^2}. \quad \blacksquare\end{aligned}$$

We summarize the above results in the following.

THEOREM 4.12. *Let X be a symmetric α-stable random variable, $\alpha \in (1,2)$, with characteristic function* (2.6). *For any $y \in \left[\frac{2}{\sqrt{2-\alpha}}, \frac{2}{2-\alpha}\right]$ one has*

$$\mathbb{P}(X \geqslant 2y) \leqslant \frac{10}{3}\frac{1}{y^\alpha} + e^{2/45}e^{-1/4(2-\alpha)y^2}, \tag{4.32}$$

$$\mathbb{P}\left(X \geqslant \frac{\sqrt{2}}{4}y\right) \geqslant \frac{1}{4e}\frac{1}{y^\alpha} + \frac{1}{548}e^{-(2-\alpha)y^2}; \tag{4.33}$$

while for $y \geqslant \frac{2}{2-\alpha}$ one has

$$\mathbb{P}(X \geqslant 2y) \leqslant \frac{16}{3}\frac{1}{y^\alpha}, \tag{4.34}$$

$$\mathbb{P}(X \geqslant y) \geqslant \frac{1}{2}\frac{1}{2+\alpha y^\alpha}. \tag{4.35}$$

Proof. We argue as in the proof of Theorem 4.5. For the upper bound we simply apply (4.27) and (4.30) to get

$$\mathbb{P}(X \geqslant 2y) \leqslant \mathbb{P}(X_1 \geqslant y) + \mathbb{P}(X^1 \geqslant y) \leqslant \frac{10}{3}\frac{1}{y^\alpha} + e^{2/45}e^{-1/4(2-\alpha)y^2}.$$

For the lower bound we use (4.26), (4.31) and symmetry of X^1 and X_1 to get

$$\mathbb{P}\left(X \geqslant \frac{\sqrt{2}}{4}y\right) \geqslant \mathbb{P}(X \geqslant y) \geqslant \mathbb{P}(X^1 \geqslant y)\mathbb{P}(X_1 \geqslant 0) \geqslant \frac{1}{e}\frac{1}{y^\alpha}\frac{1}{2}$$

and on the other hand

$$\mathbb{P}\left(X \geqslant \frac{\sqrt{2}}{4}y\right) \geqslant \mathbb{P}\left(X_1 \geqslant \frac{\sqrt{2}}{4}y\right)\mathbb{P}(X^1 \geqslant 0) \geqslant \frac{1}{137}e^{-(2-\alpha)y^2}\frac{1}{2}.$$

Summing the above inequalities yields (4.33).

To prove (4.34) we again proceed as in the proof of Theorem 4.5, namely we differentiate (2.7) and get

$$\mathbb{E}X_1 = 0, \qquad \mathbb{E}X_1^2 = 2\int_0^1 x^2\,\frac{dx}{x^{\alpha+1}} = \frac{2}{2-\alpha}$$

and

$$\mathbb{E}X_1^4 = 3(\mathbb{E}X_1^2)^2 + 2\int_0^1 x^4\,\frac{dx}{x^{\alpha+1}} = \frac{12}{(2-\alpha)^2} + \frac{2}{4-\alpha}.$$

By the same argument as for (4.20) and since $y \geqslant \frac{2}{2-\alpha}$ we get

$$\begin{aligned}\mathbb{P}(X_1 \geqslant y) &= \frac{1}{2}\mathbb{P}(|X_1| \geqslant y)\\ &\leqslant \frac{\mathbb{E}X_1^4}{2y^4} = \frac{1}{2}\left(\frac{12}{(2-\alpha)^2y^4} + \frac{2}{(4-\alpha)y^4}\right) \leqslant \frac{2}{y^\alpha}.\end{aligned} \tag{4.36}$$

Combining this with (4.27) yields (4.34):

$$\mathbb{P}(X \geqslant 2y) \leqslant \mathbb{P}(X_1 \geqslant y) + \mathbb{P}(X^1 \geqslant y) \leqslant \frac{2}{y^\alpha} + \frac{10}{3}\frac{1}{y^\alpha} \leqslant \frac{16}{3}\frac{1}{y^\alpha}.$$

The estimate (4.35) was presented in the proof of Theorem 3.2. ■

REMARK 4.13. Both remarks following Theorem 4.5 apply also in this case. For $y = \frac{2}{\sqrt{2-\alpha}}$ the tail probability is of order $O(1)$, and for y of order $O\Big(\sqrt{\frac{1}{2-\alpha}\ln\frac{1}{2-\alpha}}\Big)$ the term $\frac{1}{y^\alpha}$ (Pareto-like behavior) starts to dominate the $\exp(-\kappa(2-\alpha)y^2)$, $\kappa \in \{1/2, 1\}$, term (Gaussian tail).

Acknowledgments. The authors would like thank the anonymous referee for useful comments which allowed us to fix several computational mistakes in the paper.

REFERENCES

[1] R. M. Blumenthal and R. K. Getoor, *Some theorems on stable processes*, Trans. Amer. Math. Soc. 95 (1960), 263–273.
[2] K. Bogdan, T. Grzywny and M. Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. 266 (2014), 3543–3571.
[3] K. Bogdan, T. Grzywny and M. Ryznar, *Density and tails of unimodal convolution semigroups*, arXiv:1305.0976v1 (2013).
[4] T. Grzywny and K. Szczypkowski, *Estimates of heat kernels of non-symmetric Lévy processes*, arXiv:1710.07793v2 (2020).
[5] O. Kallengerg, *Foundations of Modern Probability*, Springer, New York, 2002.
[6] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed., Wiley, New York, 1971.
[7] J. P. Nolan, *Numerical calculation of stable densities and distribution functions*, Comm. Statist. Stoch. Models 13 (1997), 759–774.
[8] G. Pólya, *On the zeros of an integral function represented by Fourier's integral*, Messenger Math. 52 (1923), 185–188.
[9] W. E. Pruitt, *The growth of random walks and Lévy processes*, Ann. Probab. 9 (1981), 948–956.
[10] J. Rosiński, *On the series representation of infinitely divisible random vectors*, Ann. Probab. 18 (1990), 405–430.
[11] J. Rosiński, *Simulations of Lévy processes*, in: Encyclopedia of Statistics in Quality and Reliability: Computationally Intensive Methods and Simulation, Wiley, 2008.
[12] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York, 1994.
[13] P. Sztonyk, *Transition density estimates for jump Lévy processes*, Stochastic Process. Appl. 121 (2011), 1245–1265.
[14] M. Talagrand, *Upper and Lower Bounds for Stochastic Processes: Modern Methods and Classical Problems*, Ergeb. Math. Grenzgeb. 60, Springer, New York, 2014.
[15] V. Uchaikin and V. Zolotarev, *Chance and Stability: Stable Distributions and their Applications*, De Gruyter, Berlin, 2011.
[16] T. Watanabe, *Asymptotic estimates of multi-dimensional stable densities and their applications*, Trans. Amer. Math. Soc. 359 (2007), 2851–2879.
[17] V. M. Zolotarev, *One-Dimensional Stable Distributions*, Transl. Math. Monogr. 65, Amer. Math. Soc., Providence, RI, 1986.

Witold M. Bednorz, Rafał Martynek
University of Warsaw
Banacha 2
Warszawa, Poland
E-mail: wbednorz@mimuw.edu.pl
r.martynek@mimuw.edu.pl

Rafał M. Łochowski
Warsaw School of Economics
Al. Niepodległości 162
Warszawa, Poland
E-mail: rlocho@sgh.waw.pl

Received 21.9.2019;
revised version 8.12.2020