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ON THE TRANSFER THEOREMS FOR OBSERVED AND UNOBSERVED RANDOM VARIABLES

BY

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Abstract. We characterize the possible weak limits of

$$\sum_{i=1}^{n} \epsilon_i X_i / k_n$$

for a sequence $\{X_n, n \geqslant 1\}$ of independent random variables and a sequence $\{\epsilon_n, n \geqslant 1\}$ of indicator random variables ($P[\epsilon_n \in \{0,1\}] = 1$ for $n \geqslant 1$) and a non-random normalizing sequence $\{k_n, n \geqslant 1\}$ of positive reals. We consider two cases: when $\{X_n, n \geqslant 1\}$ and $\{\epsilon_n, n \geqslant 1\}$ are independent or dependent. In the first case we obtain results generalizing transfer theorems, whereas in the other case, only a partial characterization was possible.

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables. We will denote by $\boldsymbol{\epsilon} = \{\epsilon_n, n \geqslant 1\}$ and $\boldsymbol{\nu} = \{\nu_n, n \geqslant 1\}$ random and non-random sequences of values from the set $\{0, 1\}$, respectively. Let $\{N_n, n \geqslant 1\}$ be a sequence of integer-valued random variables. For $n \geqslant 1$ set

$$S_n = \sum_{i=1}^{n} X_i, \quad S_n(\boldsymbol{\epsilon}) = \sum_{i=1}^{n} \epsilon_i X_i, \quad S_n(\boldsymbol{\nu}) = \sum_{i=1}^{n} \nu_i X_i, \quad S_{N_n} = \sum_{k=1}^{N_n} X_k.$$

Throughout the paper the upper case Greek letters are used to denote the distribution functions while their lower case counterparts denote the corresponding characteristic functions.

Gnedenko and Fahim [1] proved the so-called transfer theorem, establishing sufficient conditions for the weak convergence of random sums of independent identically distributed random variables under the assumption of independence of indices and summands. Gnedenko also posed the problem of describing criteria, that is, necessary and sufficient conditions for the convergence of random sums. Partial inverses of the transfer theorem and criteria under some rather loose additional assumptions were published in [7] and [8]. The final solution and some important corollaries are presented in [9]. Kruglov and Korolev [5, Section 2] gathered many of those results. Recently Kern [2] presented a widely applicable transfer theorem for random variables on a general metric space with random multiparameters. On the other hand, Klebanov and Rachev [3] constructed a general theory of summation of a random number of random variables generalizing the definition of infinite divisibility and introducing ν-infinitely divisible families of random variables.

The main theorem of [9] can be formulated as follows.

THEOREM 1.1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables, and let* $\{N_n, n \geqslant 1\}$ *be a sequence of positive integer-valued random variables independent of the sequence* $\{X_n, n \geqslant 1\}$ *and such that* $N_n \xrightarrow{\mathcal{P}} \infty$. *Let* $\{k_n, n \geqslant 1\}$ *be a sequence of integers such that*

$$k_n \to \infty \quad \text{as } n \to \infty.$$

Let $\{a_n, n \geqslant 1\}$ *be a sequence of positive reals and let* $\Phi(x)$ *and* $A(x)$ *be distribution functions. If*

$$\text{(A)} \qquad P[S_{k_n}/a_n < x] \to \Phi(x) \quad \text{as } n \to \infty,$$

$$\text{(B)} \qquad P[N_n/k_n < x] \to A(x) \quad \text{as } n \to \infty,$$

then

$$\text{(C)} \qquad P[S_{N_n}/a_n < x] \to \Psi(x) \quad \text{as } n \to \infty,$$

where

$$\psi(t) = \int_{-\infty}^{\infty} e^{itx}\, d\Psi(x) = \int_0^{\infty} [\phi(t)]^y\, dA(y),$$

and $\psi(t)$ *and* $\phi(t)$ *are the characteristic functions of the distribution functions* $\Psi(x)$ *and* $\Phi(x)$, *respectively.*

If $\Phi(x)$ *is a non-degenerate distribution function, then conditions* (A) *and* (C) *imply* (B).

If the equality

$$\text{(1.1)} \qquad \int_0^{\infty} [\phi_1(t)]^y\, dA(y) \equiv \int_0^{\infty} [\phi_2(t)]^y\, dA(y)$$

implies $\phi_1(t) \equiv \phi_2(t)$ *for any characteristic functions* ϕ_1, ϕ_2 *of stable laws, and* $A(x)$ *is a non-degenerate distribution function, then conditions* (B) *and* (C) *imply* (A).

For $\alpha \in (0,2], \beta \in [-1,1], \lambda > 0$ let $G_{\alpha,\beta,\gamma,\lambda}$ be the stable distribution function with characteristic function

$$\int e^{itx}\, dG_{\alpha,\beta,\gamma,\lambda}(t) = \begin{cases} \exp\{it\gamma - \lambda^\alpha |t|^\alpha (1 - i\beta \mathrm{sign}(t) \tan(\pi\alpha/2))\}, & \alpha \neq 1, \\ \exp\{it\gamma - \lambda |t| \big(1 + i\beta \frac{2}{\pi} \mathrm{sign}(t) \ln(|t|)\big)\}, & \alpha = 1. \end{cases}$$

In (1.1) it is enough to consider the characteristic functions of stable laws, because, by the well known fact, the set of possible weak limits of appropriate centered and normalized sums of i.i.d. random variables is exactly equal to the class of stable laws.

Let $\boldsymbol{\epsilon} = \{\epsilon_n, n \geqslant 1\}$ be a sequence of random variables taking values 0 and 1 only. We may interpret them as indicators which random variables among $\{X_n, n \geqslant 1\}$ are observed ($\epsilon_n(\omega) = 1$) or unobserved ($\epsilon_n(\omega) = 0$). Throughout this paper it is assumed that $\boldsymbol{\epsilon}$ is arbitrarily interdependent. In Section 2 we assume that $\boldsymbol{\epsilon}$ is independent of $\{X_n, n \geqslant 1\}$, but in Section 3 we allow dependence between these random sequences.

The general aim of this paper is to obtain results analogous to the transfer theorem 1.1 for sums of observed and unobserved random variables. We will show that in some cases the sums of the sequence $\{\epsilon_n X_n, n \geqslant 1\}$ behave "similarly" to randomly indexed sums of random variables. Such sums were also investigated by Krajka and Rychlik [4]. In two special cases we may formulate the transfer theorem for sums of such observed and unobserved summands.

THEOREM 1.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables. Let the sequence $\boldsymbol{\epsilon} = \{\epsilon_n, n \geqslant 1\}$ be independent of the previous one, let $\{a_n, n \geqslant 1\}$ be a sequence of positive reals and let $\Phi(x)$ and $A(x)$ be distribution functions. If*

(A) $$P[S_{k_n}/a_n < x] \to \Phi(x) \qquad \text{as } n \to \infty,$$

(B) $$P\Big[\sum_{i=1}^{n} \epsilon_i / k_n < x\Big] \to A(x) \quad \text{as } n \to \infty,$$

then

(C) $$P[S_n(\boldsymbol{\epsilon})/a_n < x] \to \Psi(x), \quad \text{where}$$

$$\psi(t) = \int_{-\infty}^{\infty} e^{itx}\, d\Psi(x) = \int_0^{\infty} [\phi(t)]^y\, dA(y),$$

and $\psi(t)$ and $\phi(t)$ are the characteristic functions of $\Psi(x)$ and $\Phi(x)$, respectively.

If $\Phi(x)$ is a non-degenerate distribution function, then conditions (A) *and* (C) *imply* (B).

If the equality

$$\int_0^{\infty} [\phi_1(t)]^y\, dA(y) \equiv \int_0^{\infty} [\phi_2(t)]^y\, dA(y)$$

implies $\phi_1(t) \equiv \phi_2(t)$ for any characteristic functions ϕ_1, ϕ_2 of stable laws, and $A(x)$ is a non-degenerate distribution function, then conditions (B) *and* (C) *imply* (A).

Theorem 1.2 immediately follows from Theorem 1.1 applied to $N_n = \sum_{i=1}^n \epsilon_i$ and the fact that for exchangeable random variables $\{X_n, n \geqslant 1\}$ we have

$$\{\epsilon_1 X_1, \epsilon_2 X_2, \ldots, \epsilon_n X_n\} \stackrel{\mathcal{D}}{=} \{X_1, X_2, \ldots, X_{N_n}, 0, 0, \ldots, 0\},$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in law.

Furthermore, we may formulate the following characterization theorem:

THEOREM 1.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables belonging to the area of attraction of the stable (or normal) limit law $G_{\alpha,\beta,\gamma,\lambda}$, $0 < \alpha \leqslant 2$, $-1 \leqslant \beta \leqslant 1$, $\lambda > 0$. Then the set of all possible weak limits of the sum $\{\sum_{i=1}^n \epsilon_i X_i / a_n, n \geqslant 1\}$, where $\{\epsilon_n, n \geqslant 1\}$ are independent of $\{X_n, n \geqslant 1\}$, and $\{a_n, n \geqslant 1\}$ is a sequence of positive reals divergent to infinity, is equal to the family of random variables Y such that $P[Y < 0] = 0$ with law Φ_Y satisfying*

$$\int\limits_{-\infty}^{\infty} e^{itx}\, d\Phi_Y(x) = \int\limits_0^{\infty} \Big[\int\limits_{-\infty}^{\infty} e^{itx}\, dG_{\alpha,\beta,\gamma,\lambda}(t) \Big]^y \, dP[Y < y].$$

REMARK 1.1. Theorems 1.2 and 1.3 may be generalized to the case of more observers. For example, if $\{\epsilon_n, n \geqslant 1\}$ and $\{\epsilon'_n, n \geqslant 1\}$ are two sequences describing two observers and if

$$\text{(A)} \qquad P[S_{k_n}/a_n < x] \to \Phi(x) \quad \text{as } n \to \infty,$$

$$\text{(B)} \qquad P\Big[\sum_{i=1}^n \epsilon_i (1 - \epsilon'_i)/k_n < x, \sum_{i=1}^n \epsilon_i \epsilon'_i / k_n < y, \sum_{i=1}^n \epsilon'_i (1 - \epsilon_i)/k_n < z\Big] \to A(x, y, z)$$

as $n \to \infty$, then

$$\text{(C)} \qquad P[S_n(\boldsymbol{\epsilon})/a_n < x, S_n(\boldsymbol{\epsilon}')/a_n < y] \to \Psi(x, y),$$

where

$$\begin{aligned} \psi(t_1, t_2) &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{i(t_1 x_1 + t_2 x_2)} d\Psi(x_1, x_2) \\ &= \int\limits_0^{\infty} \int\limits_0^{\infty} \int\limits_0^{\infty} [\phi(t_1)]^{x+y} [\phi(t_2)]^{y+z}\, dA(x, y, z), \end{aligned}$$

and $\psi(t_1, t_2)$ and $\phi(t)$ are the characteristic functions of the distribution functions $\Psi(x, y)$ and $\Phi(x)$, respectively.

In general, we cannot prove this theorem for sequences $\{X_n, n \geqslant 1\}$ that are not identically distributed. The proof of Theorem 1.2 cannot be generalized to this case, and the arising difficulties are explained in the next section.

2. THE CASE WHEN ϵ IS INDEPENDENT OF $\{X_n, n \geqslant 1\}$

EXAMPLE 2.1. Let $\{Y_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with stable distribution function $G_{\alpha,0,0,\lambda}$, $\alpha \neq 1$, and let $\{Z_n, n \geqslant 1\}$ be a sequence of independent random variables such that $Z_n \sim N(0, n^{2/\alpha} - (n-1)^{2/\alpha})$, $n \geqslant 1$. Putting

$$X_n = \begin{cases} Y_{n/2} & \text{for } n \text{ even,} \\ Z_{(n+1)/2} & \text{for } n \text{ odd,} \end{cases}$$
$$\epsilon_n = \begin{cases} 1 & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases}$$
$$\epsilon'_n = 1 - \epsilon_n, \quad n \geqslant 1.$$

we see that

$$\begin{aligned} & S_n/n^{1/\alpha} \xrightarrow{\mathcal{D}} G_{\alpha,0,0,(1/2)^{1/\alpha^2}\lambda} + N(0, (1/2)^{2/\alpha}), \\ & \sum_{i=1}^{n} \epsilon_i/n \xrightarrow{\mathcal{D}} E_{1/2}, \\ & \sum_{i=1}^{n} \epsilon'_i/n \xrightarrow{\mathcal{D}} E_{1/2}, \\ & S_n(\boldsymbol{\epsilon}')/n^{1/\alpha} \xrightarrow{\mathcal{D}} N(0, (1/2)^{2/\alpha}), \\ & S_n(\boldsymbol{\epsilon})/n^{1/\alpha} \xrightarrow{\mathcal{D}} G_{\alpha,0,0,(1/2)^{1/\alpha^2}\lambda} \quad \text{as } n \to \infty; \end{aligned} \tag{2.1}$$

here and in what follows, $E_x(u) = I[u < x]$, $u, x \in \mathbb{R}$. Taking into account the first three convergences in (2.1) and Theorem 1.2 it seems that the appropriate normalized limits of $S_n(\boldsymbol{\epsilon})$ and $S_n(\boldsymbol{\epsilon}')$ should be the same, but in (2.1) they are different. We conclude that conditions (A) and (B) in Theorem 1.2 are not sufficient for (C) to hold. In this example $\{X_n, n \geqslant 1\}$ are not i.i.d. Because every sequence of numbers is independent of each random sequence, $\{X_n, n \geqslant 1\}$ and $\{\epsilon_n, n \geqslant 1\}$ are independent.

This example shows that the expression $\sum_{i=1}^{n} \epsilon_i$ is not suitable to consider the convergence of $\{S_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$; it should rather be considered using some random sequences $\{a_n(\boldsymbol{\epsilon}), n \geqslant 1\}$ of positive reals. Furthermore, we should have some information about the possible weak limits of appropriate normalized subsequences of $\{S_n, n \geqslant 1\}$. The normalizing sequences may be dependent on possible limits. For example, if we change in Example 2.1 the law $\{Z_n, n \geqslant 1\}$ into $Z_n \sim N(0,1)$, then for the subsequence $\{S_{2n}, n \geqslant 1\}$ the appropriate normalizing is $n^{1/\alpha}$, whereas for $\{S_{2n-1}, n \geqslant 1\}$ we should normalize by $n^{1/2}$.

We see that the main problem is the possibility of obtaining different weak limits for the sums of different subsequences with different normalizing sequences.

Therefore, instead of condition (A), we should consider the condition:

$$S_n(\boldsymbol{\nu})/a_n(\boldsymbol{\nu}) \xrightarrow{\mathcal{D}} \Psi_\nu \quad \text{as } n \to \infty,$$

and instead of (B),

$$a_n(\boldsymbol{\epsilon})/k_n \xrightarrow{\mathcal{D}} A \quad \text{as } n \to \infty,$$

for some subsequence $\{k_n, n \geqslant 1\}$ and distribution functions $A, \Psi_{\boldsymbol{\nu}}, \boldsymbol{\nu} \in \mathcal{M}$, for some family of subsequences $\mathcal{M} \subset \{0,1\}^{\mathbb{N}}$.

THEOREM 2.1. *Consider five conditions:*

(a) *For every $\boldsymbol{\nu} \in \mathcal{M}$,*

$$S_n(\boldsymbol{\nu})/a_n(\boldsymbol{\nu}) \xrightarrow{\mathcal{D}} \Psi_{\boldsymbol{\nu}} \quad \text{as } n \to \infty.$$

(b) *For a random sequence $\boldsymbol{\epsilon} \in \mathcal{M}$, a.s., a sequence $\{k_n, n \geqslant 1\}$ diverging to infinity, and a distribution function A we have*

$$a_n(\boldsymbol{\epsilon})/k_n \xrightarrow{\mathcal{D}} A \quad \text{as } n \to \infty.$$

(c) *For a random sequence $\boldsymbol{\epsilon} \in \mathcal{M}$, a.s., a sequence $\{k_n, n \geqslant 1\}$ diverging to infinity, and distribution functions $\Psi_{\boldsymbol{\nu}}, \Psi$ we have*

$$\lim_{n\to\infty} E\psi_{\boldsymbol{\epsilon}}(ta_n(\boldsymbol{\epsilon})/k_n) = \psi(t),$$

where $\psi_{\boldsymbol{\nu}}(t) = \int e^{itx}\, d\Psi_{\boldsymbol{\nu}}(x)$ and $\psi(t) = \int e^{itx}\, d\Psi(x)$.

(d) *For a random sequence $\boldsymbol{\epsilon} \in \mathcal{M}$, a.s., and a distribution function Ψ we have*

$$S_n(\boldsymbol{\epsilon})/k_n \xrightarrow{\mathcal{D}} \Psi \quad \text{as } n \to \infty.$$

(e) *For a random sequence $\boldsymbol{\epsilon} \in \mathcal{M}$, a.s., and a sequence $\{k_n, n \geqslant 1\}$ diverging to infinity, the sequence $\{a_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ is tight.*

We have:

(i) (a) *and* (c) *and* (e) $\Longrightarrow$ (d),

(ii) (a) *and* (d) *and $E\Psi_{\boldsymbol{\epsilon}}$ is continuous at* 0 $\Longrightarrow$ $\{a_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ *is relatively compact,*

(iii) (b) *and* (d) *and A is continuous at* 0 $\Longrightarrow$ $\{S_n(\boldsymbol{\epsilon})/a_n(\boldsymbol{\epsilon}), n \geqslant 1\}$ *is relatively compact.*

Proof. If $S_n(\boldsymbol{\nu})/a_n(\boldsymbol{\nu}) \xrightarrow{\mathcal{D}} \Psi_{\boldsymbol{\nu}}$ then for every continuous and bounded function f we have $Ef(S_n(\boldsymbol{\nu})/a_n(\boldsymbol{\nu})) \to \int f(x)\, d\Psi_{\boldsymbol{\nu}}(x)$, thus from the independence of $\boldsymbol{\epsilon}$ from $\{X_n, n \geqslant 1\}$ we have $Ef(S_n(\boldsymbol{\epsilon})/a_n(\boldsymbol{\epsilon})) \to \iint f(x)\, d\Psi_{\boldsymbol{\nu}}(x)\, dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu}) = \int f(x)\, dE\Psi_{\boldsymbol{\epsilon}}(x)$, where $P_{\boldsymbol{\epsilon}}$ denotes the law of $\boldsymbol{\epsilon}$. Therefore $S_n(\boldsymbol{\epsilon})/a_n(\boldsymbol{\epsilon}) \xrightarrow{\mathcal{D}} E\Psi_{\boldsymbol{\epsilon}}$.

Proof of (i). Let $\phi_j(t) = Ee^{itX_j}$, $j \geqslant 1$, and let $P_{\boldsymbol{\epsilon}}$ denote the law of $\boldsymbol{\epsilon}$. Because $\phi_j(0) = 1$, we have

$$\begin{aligned}\lim_{n\to\infty} Ee^{itS_n(\boldsymbol{\epsilon})/k_n} &= \lim_{n\to\infty} \int_{\boldsymbol{\nu}\in\mathcal{M}} \prod_{1\leqslant j\leqslant n,\, \nu_j=1} \phi_j(t/k_n)\, dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu})\\ &= \lim_{n\to\infty} \int_{\boldsymbol{\nu}\in\mathcal{M}} \prod_{j=1}^{n} \phi_j(t\nu_j/k_n)\, dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu}).\end{aligned}$$

On the other hand, from (a) we have, for every $T > 0$ and $\boldsymbol{\nu} \in \mathcal{M}$,

$$\lim_{n\to\infty} \sup_{|t|<T} \left| \prod_{j=1}^{n} \phi_j(\nu_j t/a_n(\boldsymbol{\nu})) - \psi_{\boldsymbol{\nu}}(t) \right| = 0. \tag{2.2}$$

Choose any $\epsilon > 0$ and a positive number K_ϵ such that $P[|a_n(\boldsymbol{\epsilon})/k_n| > K_\epsilon] < \epsilon$ from (e). Then

$$\begin{aligned}&\sup_{|t|<T} \left| Ee^{itS_n(\boldsymbol{\epsilon})/k_n} - E\psi_{\boldsymbol{\epsilon}}\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right) \right|\\ &\leqslant \int_{\substack{\boldsymbol{\nu}\in\mathcal{M}\\ |a_n(\boldsymbol{\nu})/k_n|<K_\epsilon}} \sup_{|t|<T} \left| \prod_{j=1}^{n} \phi_j\left(\frac{\nu_j t}{a_n(\boldsymbol{\nu})}\frac{a_n(\boldsymbol{\nu})}{k_n}\right) - \psi_{\boldsymbol{\nu}}\left(t\frac{a_n(\boldsymbol{\nu})}{k_n}\right) \right| dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu})\\ &\quad + P[|a_n(\boldsymbol{\epsilon})/k_n| > K_\epsilon]\\ &\leqslant \int_{\boldsymbol{\nu}\in\mathcal{M}} \sup_{|t|<TK_\epsilon} \left| \prod_{j=1}^{n} \phi_j\left(\frac{\nu_j t}{a_n(\boldsymbol{\nu})}\right) - \psi_{\boldsymbol{\nu}}(t) \right| dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu}) + \epsilon.\end{aligned}$$

Now the conclusion follows from (2.2) and (c). ∎

Proof of (ii). From our assumptions we conclude that $\{S_n(\boldsymbol{\epsilon})/a_n(\boldsymbol{\epsilon}), n \geqslant 1\}$ and $\{S_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ are tight, and $E\Psi_{\boldsymbol{\epsilon}}$ is continuous at 0. We prove that $\{a_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ is tight. Assume the contrary; then there exists a subsequence $\{n_r, r \geqslant 1\}$ such that

$$P\left[\left|\frac{a_{n_r}(\boldsymbol{\epsilon})}{k_{n_r}}\right| \geqslant r\right] > 2\delta, \quad \text{say,} \tag{2.3}$$

for $r \in \mathbb{N}$. But since $\{S_{n_r}(\boldsymbol{\epsilon})/k_{n_r}, r \geqslant 1\}$ is tight (being a subsequence of a tight sequence), for any $\delta > 0$ and some K_δ we have

$$P\left[\left|\frac{S_{n_r}(\boldsymbol{\epsilon})}{k_{n_r}}\right| < K_\delta\right] > 1 - \delta.$$

Furthermore,

$$P\left[\left|\frac{S_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta\right] = P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < K_\delta / \frac{a_{n_r}(\epsilon)}{k_{n_r}}\right]$$
$$\leqslant P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < K_\delta/r, \left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| \geqslant r\right] + P\left[\left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| < r\right]$$
$$\leqslant P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < K_\delta/r\right] + P\left[\left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| < r\right],$$

thus

$$1-\delta < P\left[\left|\frac{S_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta\right] < P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < K_\delta/r\right] + 1 - 2\delta,$$

therefore

$$\delta < P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < K_\delta/r\right],$$

and letting $r \to \infty$, we see that the weak limit of $\{S_{n_r}(\epsilon)/a_{n_r}(\epsilon), r \geqslant 1\}$ is 0 with non-zero probability, which contradicts the assumption that $E\Psi_\epsilon$ is continuous at 0. Thus, $\{a_n(\epsilon)/k_n, n \geqslant 1\}$ is tight and hence relatively compact. ∎

Proof of (iii). The proof is analogous to the one above. We prove that $\{S_n(\epsilon)/a_n(\epsilon), n \geqslant 1\}$ is tight. Assume the contrary; then there exists a subsequence $\{n_r, r \geqslant 1\}$ such that

$$P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < r\right] < 1 - 2\delta.$$

Since

$$P\left[\left|\frac{S_{n_r}(\epsilon)}{k_{n_r}}\right| > K_\delta\right] < \delta,$$

we have

$$1-\delta < P\left[\left|\frac{S_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta\right] = P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\,\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta\right]$$
$$< P\left[\left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta/r\right] + P\left[\left|\frac{S_{n_r}(\epsilon)}{a_{n_r}(\epsilon)}\right| < r\right]$$
$$< P\left[\left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| < K_\delta/r\right] + 1 - 2\delta,$$

and

$$P\left[\left|\frac{a_{n_r}(\epsilon)}{k_{n_r}}\right| > K_\delta/r\right] > \delta, \tag{2.4}$$

and since $\{a_{n_r}(\epsilon)/k_{n_r}, r \geqslant 1\}$ weakly tends to A, letting $r \to \infty$ in (2.4) we see that A is not continuous at 0, contrary to assumption. Therefore, $\{S_n(\epsilon)/a_n(\epsilon), n \geqslant 1\}$ is tight and so relatively compact. ∎

COROLLARY 2.1. (a) *Assume that the family $\mathcal{M}$ is decomposed into pairwise disjoint subfamilies $\{\mathcal{M}_j, 1 \leqslant j \leqslant k\}$. If for every $1 \leqslant j \leqslant k$ and every $\boldsymbol{\nu} \in \mathcal{M}_j$ we have*

$$\frac{S_n(\boldsymbol{\nu})}{a_n(\boldsymbol{\nu})} \xrightarrow{\mathcal{D}} \Psi_j \quad \text{as } n \to \infty,$$

and for some random sequence $\boldsymbol{\epsilon} \in \mathcal{M}$, a.s., such that $p_j = P[\boldsymbol{\epsilon} \in \mathcal{M}_j]$, $1 \leqslant j \leqslant k$, a sequence $\{k_n, n \geqslant 1\}$ diverging to infinity and a distribution function A we have

$$\frac{a_n(\boldsymbol{\epsilon})}{k_n} \xrightarrow{\mathcal{D}\ mixing} A \quad \text{as } n \to \infty,$$

where "$\mathcal{D}$ mixing" means mixing in the Rényi sense (for details see [6, Chapter VIII, §6, pp. 466–467]*), then*

$$\frac{S_n(\boldsymbol{\epsilon})}{k_n} \xrightarrow{\mathcal{D}} \Psi \quad \text{as } n \to \infty,$$

where

$$\int_{-\infty}^{\infty} e^{itx}\, \Psi(dx) = \sum_{j=1}^{k} p_j \int_0^{\infty} \psi_j(tx)\, A(dx) = \psi(t), \quad \text{say.}$$

If $k = 1$, then the assumption of mixing convergence may be omitted.

(b) *If for every $\boldsymbol{\nu} \in \mathcal{M}$ we have*

$$\frac{S_n(\boldsymbol{\nu})}{a_n(\boldsymbol{\nu})} \xrightarrow{\mathcal{D}} \Phi \quad \text{as } n \to \infty,$$

and for some $\boldsymbol{\epsilon}$ such that $P[\boldsymbol{\epsilon} \in \mathcal{M}] = 1$ and for some sequence $\{k_n, n \geqslant 1\}$ diverging to infinity we have

$$\frac{S_n(\boldsymbol{\epsilon})}{k_n} \xrightarrow{\mathcal{D}} \Psi \quad \text{as } n \to \infty,$$

with a distribution function Ψ continuous at 0, and for any distribution functions A_1, A_2 such that $A_1(0) = A_2(0) = 0$, we have

$$\forall_{t\in\mathbb{R}} \int_0^{\infty} \phi(tx)\, A_1(dx) = \int_0^{\infty} \phi(tx)\, A_2(dx) \implies A_1 \equiv A_2,$$

then

$$\frac{a_n(\boldsymbol{\epsilon})}{k_n} \xrightarrow{\mathcal{D}} A \quad \text{as } n \to \infty,$$

with

$$\int_0^{\infty} \phi(xt)\, A(dx) = \psi(t).$$

Proof of (a). Obviously, assumptions (a) and (e) of Theorem 2.1 hold. The mixing convergence of $\{a_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ means that for every event B with $P[B] > 0$ we have

$$P\left[\frac{a_n(\boldsymbol{\epsilon})}{k_n} < x \;\middle|\; B\right] \to A(x) \quad \text{as } n \to \infty,$$

for every x which is a point of continuity of A. Putting $B_j = \{\omega \in \Omega : \boldsymbol{\epsilon} \in \mathcal{M}_j\}$, $1 \leqslant j \leqslant k$, and taking into account that every characteristic function is continuous and bounded we have

$$E\left[\psi_j\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right) \;\middle|\; B_j\right] \to \int_0^\infty \psi_j(tx)\, A(dx) \quad \text{as } n \to \infty.$$

Thus

$$\begin{aligned} E\psi_{\boldsymbol{\epsilon}}\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right) &= \sum_{j=1}^k E\psi_j\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right) I[B_j] \\ &= \sum_{j=1}^k p_j E\left[\psi_j\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right) \;\middle|\; B_j\right] \\ &\to \sum_{j=1}^k p_j \int_0^\infty \psi_j(tx)\, A(dx) = \psi(t), \end{aligned}$$

and the assertion follows from Theorem 2.1(i). ∎

Proof of (b). From Theorem 2.1(ii) we see that $\{a_n(\boldsymbol{\epsilon})/k_n, n \geqslant 1\}$ is relatively compact. Assuming that for two subsequences n_k' and n_k'' and two distribution functions A_1 and A_2 we have

$$\lim_{k\to\infty} \frac{a_{n_k'}(\boldsymbol{\epsilon})}{k_{n_k'}} = A_1, \qquad \lim_{k\to\infty} \frac{a_{n_k''}(\boldsymbol{\epsilon})}{k_{n_k''}} = A_2,$$

by applying (a) we get

$$\psi(t) = \int_0^\infty \phi(tx)\, A_1(dx) = \int_0^\infty \phi(tx)\, A_2(dx),$$

which implies $A_1 = A_2$ and ends the proof. ∎

THEOREM 2.2. *Let $\{\psi_n, n \geqslant 1\}$ be a family of functions $\psi_n : \mathcal{M} \to \mathbb{R}$ and let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables independent of the sequence $\{\epsilon_n, n \geqslant 1\}$ of indicators. Assume that for some distribution function Ψ and some sequence $\{k_n, n \geqslant 1\}$ of positive reals,*

$$\sup_{\boldsymbol{\nu}\in\mathcal{M}} P\Big[\sum_{j=1}^n \nu_j X_j < x\psi_n(\nu_1, \ldots, \nu_n)\Big] \to \Phi(x) \quad \textit{as } n \to \infty,$$

$$\inf_{\boldsymbol{\nu}\in\mathcal{M}} P\Big[\sum_{j=1}^n \nu_j X_j < x\psi_n(\nu_1, \ldots, \nu_n)\Big] \to \Phi(x) \quad \textit{as } n \to \infty, \tag{2.5}$$

for every point x of continuity of Φ, and if additionally

$$\frac{\psi_n(\epsilon_1, \ldots, \epsilon_n)}{k_n} \xrightarrow{\mathcal{D}} A \quad \text{as } n \to \infty,$$

then

$$\frac{\sum_{j=1}^{n} \epsilon_j X_j}{k_n} \xrightarrow{\mathcal{D}} \Psi \quad \text{as } n \to \infty,$$

where $\psi(t) = \int_0^\infty \phi(tx)\, A(dx)$.

Proof. Condition (2.5) and the independence of $\boldsymbol{\epsilon}$ and $\{X_n, n \geqslant 1\}$ lead to

$$\frac{S_n(\boldsymbol{\epsilon})}{\psi_n(\boldsymbol{\epsilon})} \xrightarrow{\mathcal{D}} \Phi \quad \text{as } n \to \infty.$$

Similarly to the proof of Theorem 2.1(i) we have

$$\sup_{|t|<T} \left| Ee^{itS_n(\boldsymbol{\epsilon})/k_n} - E\phi\left(t\frac{a_n(\boldsymbol{\epsilon})}{k_n}\right)\right| \leqslant \int_{\boldsymbol{\nu}\in\mathcal{M}} \sup_{|t|<TK_\epsilon} \left| \prod_{j=1}^{n} \phi_j\left(\frac{t\nu_j}{a_n(\boldsymbol{\nu})}\right) - \Phi(t)\right| dP_{\boldsymbol{\epsilon}}(\boldsymbol{\nu}) + \epsilon,$$

which yields the conclusion. ■

When $\{X_n, n \geqslant 1\}$ are i.i.d., we obviously put $\psi_n(\nu_i, \ldots, \nu_n) = \sum_{i=1}^{n} \nu_i$ and obtain Theorem 1.2 as a corollary.

3. THE CASE WHEN ϵ IS DEPENDENT ON $\{X_n, n \geqslant 1\}$

EXAMPLE 3.1. Let $0 < p < 1/2$ be a real number and let x be defined by $p = G_{\alpha,0,\gamma,\lambda}(-x) = 1 - G_{\alpha,0,\gamma,\lambda}(x)$. Let $\{Y_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with symmetric law

$$P[Y_n \in A] = \frac{1}{2p} G_{\alpha,0,\gamma,\lambda}(A \cap [(-\infty, -x) \cup (x, \infty)]), \quad A \in \mathcal{B}(\mathbb{R}),\ n \geqslant 1,$$

and let $\{Z_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with uniform law on $[-x, x]$. Let $\{\epsilon_n, n \geqslant 1\}$ be an i.i.d. sequence with $P[\epsilon_i = 0] = 2p$ and $P[\epsilon_i = 1] = 1 - 2p$, and let $\{X_n, n \geqslant 1\}$ be an i.i.d. sequence of random variables defined by

$$X_n = Y_n I[\epsilon_n = 0] + Z_n I[\epsilon_n = 1], \quad n \geqslant 1.$$

Let $\mathcal{M}$ denote the family of infinite sequences $\boldsymbol{\nu}$ of 0s and 1s such that $\sum_{i=1}^{n} \nu_i \to \infty$ as $n \to \infty$. Then for every $\boldsymbol{\nu} \in \mathcal{M}$ we have

$$\frac{S_n(\boldsymbol{\nu})}{(2p\sum_{i=1}^{n} \nu_i)^{1/\alpha}} \xrightarrow{\mathcal{D}} G_{\alpha,0,\gamma,\lambda} \qquad \text{as } n \to \infty,$$
$$\frac{S_n(\boldsymbol{\epsilon})}{(x/\sqrt{3})\sqrt{(1-2p)n}} \xrightarrow{\mathcal{D}} N(0,1) \qquad \text{as } n \to \infty,$$

but for $\alpha < 2$ there is no distribution function A such that

$$\int_{-\infty}^{\infty} e^{itx}\, dG_{\alpha,0,\gamma,\lambda}(x) = \int_{0}^{\infty} e^{-t^2 y/2}\, dA(y).$$

Moreover, putting $a_n(\boldsymbol{\nu}) = (2p\sum_{i=1}^{n} \nu_i)^{1/\alpha}$ and $k_n = (x/\sqrt{3})\sqrt{(1-2p)n}$, and denoting by B_p^n the binomial random variable with chance of success equal to p and n trials, we have

$$\frac{a_n(\boldsymbol{\epsilon})}{k_n} = \frac{(2p\sum_{i=1}^{n} I[\epsilon_i = 1])^{1/\alpha}}{(x/\sqrt{3})\sqrt{(1-2p)n}} = \frac{(2pB_{1-2p}^n)^{1/\alpha}}{(x/\sqrt{3})\sqrt{(1-2p)n}}.$$

From the Central Limit Theorem we have

$$\left(\frac{B_{1-2p}^n - (1-2p)n}{\sqrt{n(1-2p)2p}}\right)^{1/\alpha} \xrightarrow{\mathcal{D}} (N(0,1))^{1/\alpha} \qquad \text{as } n \to \infty,$$

thus

$$P[B_{1-2p}^n > (1-2p)n] \geqslant \frac{1}{2} - \sup_x \left| P\left[\frac{B_{1-2p}^n - EB_{1-2p}^n}{\sqrt{\operatorname{Var}(B_{1-2p}^n)}} < x\right] - \Phi(x)\right|,$$

where $\Phi(x)$ is the standard normal distribution function. From the Berry–Esseen Theorem, putting $q_{x,p,\alpha} = \frac{\sqrt{3}(2p(1-2p))^{1/\alpha}}{x\sqrt{1-2p}}$ we have

$$P\left[\frac{a_n(\boldsymbol{\epsilon})}{k_n} > q_{x,p,\alpha} n^{1/\alpha - 1/2}\right] \geqslant \frac{1}{2} - 0.4748 \cdot \frac{4p-1}{\sqrt{2p(1-2p)}} n^{-1/2}.$$

As a consequence, the implication (ii) of Theorem 2.1 fails, as $\{a_n(\epsilon)/k_n, n \geqslant 1\}$ is not relatively compact when $\alpha < 2$.

It is easy to check that if we omit the assumption that $\{\epsilon_n, n \geqslant 1\}$ are the indicators under the small restriction ($P[\bigcup_{n\geqslant 1}[X_n = 0]] = 0$), we get convergence in probability to an arbitrary random variable X ($(\sum_{i=1}^{n} \frac{X}{X_i} X_i)/n = X$). For indicators we have not achieved this result in this case, but we only remark that since for every random variable X and indicator function ϵ we have

$$XI[X < 0] \leqslant X\epsilon \leqslant XI[X > 0],$$

it follows that if

$$\sum_{i=1}^{n} X_i \epsilon_i / k_n \xrightarrow{\mathcal{D}} F \quad \text{as } n \to \infty,$$

then

$$\liminf_{n\to\infty} P\Big[\sum_{i=1}^{n} X_i I[X_i > 0] < x k_n\Big] \leqslant F(x) \leqslant \limsup_{n\to\infty} P\Big[\sum_{i=1}^{n} X_i I[X_i < 0] < x k_n\Big].$$

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