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A TWO-PARAMETER EXTENSION OF URBANIK'S PRODUCT CONVOLUTION SEMIGROUP

BY

CHRISTIAN BERG (COPENHAGEN)

Abstract. We prove that $s_n(a,b) = \Gamma(an+b)/\Gamma(b), n = 0, 1, \ldots$, is an infinitely divisible Stieltjes moment sequence for arbitrary $a, b > 0$. Its powers $s_n(a,b)^c, c > 0$, are Stieltjes determinate if and only if $ac \leqslant 2$. The latter was conjectured in a paper by Lin (2019) in the case $b = 1$. We describe a product convolution semigroup $\tau_c(a,b)$, $c > 0$, of probability measures on the positive half-line with densities $e_c(a,b)$ and having the moments $s_n(a,b)^c$. We determine the asymptotic behavior of $e_c(a,b)(t)$ for $t \to 0$ and for $t \to \infty$, and the latter implies the Stieltjes indeterminacy when $ac > 2$. The results extend the previous work of the author and López (2015) and lead to a convolution semigroup of probability densities $\big(g_c(a,b)(x)\big)_{c>0}$ on the real line. The special case $\big(g_c(a,1)(x)\big)_{c>0}$ are the convolution roots of the Gumbel distribution with scale parameter $a > 0$. All the densities $g_c(a,b)(x)$ lead to determinate Hamburger moment problems.

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1. INTRODUCTION

A Stieltjes moment sequence is a sequence of non-negative numbers of the form

$$s_n = \int_0^\infty t^n \, d\mu(t), \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, \tag{1.1}$$

where μ is a positive measure on $[0, \infty)$ such that $t^n \in L^1(\mu)$ for all $n \in \mathbb{N}_0$. The sequence (s_n) is called *normalized* if $s_0 = \mu\big([0,\infty)\big) = 1$, and it is called *S-determinate* (resp. *S-indeterminate*) if (1.1) has exactly one solution (resp. several solutions) μ as positive measures on $[0, \infty)$. All these concepts go back to the fundamental memoir of Stieltjes [19].

A Stieltjes moment sequence (s_n) is called *infinitely divisible* if (s_n^c) is a Stieltjes moment sequence for any $c > 0$. These sequences were characterized in Tyan's Ph.D. thesis [21] and again in [5] without the knowledge of [21]. An important example of an infinitely divisible normalized Stieltjes moment sequence is $s_n = n!$, first established in Urbanik [22]. He proved that e_c in (1.2) is a probability density such that

$$(n!)^c = \int_0^\infty t^n e_c(t)\,dt, \quad e_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix-1}\Gamma(1-ix)^c\,dx, \quad c,t>0. \tag{1.2}$$

Here Γ is Euler's gamma function. The family $(\tau_c)_{c>0}$ with $d\tau_c(t) = e_c(t)dt$ is a convolution semigroup in the sense of [6] on the locally compact abelian group $G = (0,\infty)$ under multiplication. It is called the *Urbanik semigroup* in [7]. It turns out that the terminology "Urbanik semigroup" has been used in the literature for certain semigroups of operators on Banach spaces with the precise name "Urbanik decomposability semigroup", see Section 2 in [10]. We have therefore decided to use the more precise name "Urbanik's product convolution semigroup".

By Carleman's criterion for S-determinacy it is easy to prove that $(n!)^c$ is S-determinate for $c \leqslant 2$. That this estimate is sharp was first proved in [4], where it was established that $(n!)^c$ is S-indeterminate for $c > 2$ based on asymptotic results of Skorokhod [18] about stable distributions, see [23]. Another proof of the S-indeterminacy was given in [7] based on the asymptotic behavior of $e_c(t)$,

$$e_c(t) = \frac{(2\pi)^{(c-1)/2}}{\sqrt{c}}\frac{\exp(-ct^{1/c})}{t^{(c-1)/(2c)}}[1+\mathcal{O}(t^{-1/c})], \quad t\to\infty. \tag{1.3}$$

In the recent paper [11], Lin proposes the following conjecture:

CONJECTURE. *Let $a > 0$ be a real constant and let $s_n = \Gamma(na+1), n \in \mathbb{N}_0$. Then*:

(a) *(s_n) is an infinitely divisible Stieltjes moment sequence.*

(b) *For real $c > 0$ the sequence (s_n^c) is S-determinate if and only if $ac \leqslant 2$.*

(c) *For $0 < c \leqslant 2/a$ the unique probability measure μ_c corresponding to (s_n^c) has the Mellin transform*

$$\int_0^\infty t^s\,d\mu_c(t) = \Gamma(as+1)^c, \quad s \geqslant 0.$$

When $a = 1$, the conjecture is true because of the known results about Urbanik's product convolution semigroup, and for $a \in \mathbb{N}$, $a \geqslant 2$, the conjecture is true because of Theorems 4 and 7 in [11].

We shall prove that the conjecture is true, and it is a special case of similar results for the following more general normalized Stieltjes moment sequence

(1.4)
$$s_n(a,b) = \frac{\Gamma(an+b)}{\Gamma(b)} = \frac{1}{a\Gamma(b)} \int_0^\infty t^n t^{b/a-1} \exp(-t^{1/a})\, dt, \quad n = 0, 1, \ldots,$$

where $a, b > 0$ are arbitrary.

Defining

(1.5)
$$e_1(a,b)(t) = \frac{1}{a\Gamma(b)} t^{b/a-1} \exp(-t^{1/a}),$$

we get for $\operatorname{Re} z > -b/a$ and after a change of variable $t = s^a$

(1.6)
$$\int_0^\infty t^z e_1(a,b)(t)\, dt = \Gamma(az+b)/\Gamma(b).$$

This leads to our first main result.

THEOREM 1.1. (i) $\big(s_n(a,b)\big)$ *is an infinitely divisible Stieltjes moment sequence.*

(ii) *There exists a uniquely determined convolution semigroup* $\big(\tau_c(a,b)\big)_{c>0}$ *of probability measures on the multiplicative group* $(0,\infty)$ *such that*

(1.7)
$$\int_0^\infty t^z \, d\tau_c(a,b)(t) = [\Gamma(az+b)/\Gamma(b)]^c, \quad \operatorname{Re} z > -b/a,$$

and, in particular, $\big(s_n(a,b)^c\big)$ *is the moment sequence of* $\tau_c(a,b)$.

(iii) $d\tau_c(a,b)(t) = e_c(a,b)(t)\, dt$ *on* $(0,\infty)$, *where*

(1.8)
$$e_c(a,b)(t) = \frac{1}{2\pi} \int_{-\infty}^\infty t^{ix-1} [\Gamma(b-iax)/\Gamma(b)]^c \, dx, \quad t > 0,$$

is a probability density belonging to $C^\infty(0,\infty)$.

(iv) $\big(s_n(a,b)^c\big)$ *is S-determinate if and only if* $ac \leqslant 2$, *hence independent of* $b > 0$.

Note that (1.4) is a special case of (1.6).

The measure $\tau_1(a,b)$ was considered in [20], where it was proved that the measure is S-indeterminate if $a > \max(2, 2b)$. This is a consequence of our result. Note that $\tau_1(a,1)$ is called the *Weibull distribution* with shape parameter $1/a$ and scale parameter one.

In (1.7) and (1.8) we use that $\Gamma(z)$ is a non-vanishing holomorphic function in the cut plane

(1.9)
$$\mathcal{A} = \mathbb{C} \setminus (-\infty, 0],$$

so we can define

$$\Gamma(z)^c := \exp\big(c\log\Gamma(z)\big), \quad z\in\mathcal{A},$$

using the holomorphic branch of $\log\Gamma$ which is zero for $z=1$. This branch is explicitly given in (3.1) below.

Let us recall a few facts about convolution semigroups of probability measures on LCA groups, see [6] for details.

The continuous characters of the multiplicative group $G=(0,\infty)$ can be given as $t\to t^{ix}$, where $x\in\mathbb{R}$ is arbitrary, and in this way the dual group $\widehat{G}$ of G can be identified with the additive group of real numbers. The convolution between measures μ and σ on $(0,\infty)$, called a *product convolution* and denoted by $\mu\diamond\sigma$, is defined as

$$\int_0^\infty f(t)\,d\mu\diamond\sigma(t) = \int_0^\infty\int_0^\infty f(ts)\,d\mu(t)\,d\sigma(s)$$

for suitable classes of continuous functions f on $(0,\infty)$, e.g. those of compact support.

A family $(\mu_c)_{c>0}$ of probability measures on the multiplicative group $G=(0,\infty)$ is called a *convolution semigroup* if

$$\mu_c\diamond\mu_d=\mu_{c+d},\ c,d>0, \quad \text{and} \quad \lim_{c\to 0}\mu_c=\varepsilon_1 \ \text{ vaguely.}$$

Here ε_1 is the Dirac measure with total mass one concentrated in the neutral element one of the group. Given a convolution semigroup $(\mu_c)_{c>0}$ on $(0,\infty)$, it is easy to see that if μ_1 has moments of order n, then all the measures μ_c have moments of order n and

$$\int_0^\infty t^n\,d\mu_c(t) = \Big(\int_0^\infty t^n\,d\mu_1(t)\Big)^c, \quad c>0.$$

By [6], Theorem 8.3, there is a one-to-one correspondence between convolution semigroups $(\mu_c)_{c>0}$ of probability measures on G and continuous negative definite functions $\rho:\mathbb{R}\to\mathbb{C}$ satisfying $\rho(0)=0$ such that

$$\int_0^\infty t^{-ix}\,d\mu_c(t) = \exp\big(-c\rho(x)\big), \quad c>0, x\in\mathbb{R}. \tag{1.10}$$

By the inversion theorem of Fourier analysis for LCA groups, if $\exp(-c\rho)$ is integrable on $\mathbb{R}$, then $d\mu_c(t)=f_c(t)\,dt$ for a continuous function $f_c(t)$ ($tf_c(t)$ is the density of μ_c with respect to Haar measure $(1/t)dt$ on $(0,\infty)$) given by

$$f_c(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix-1}\exp\big(-c\rho(x)\big)\,dx, \quad t>0. \tag{1.11}$$

(Note that the dual Haar measure of $(1/t)dt$ on $(0,\infty)$ is $1/(2\pi)\,dx$ on $\mathbb{R}$.)

PROPOSITION 1.1. *For* $a, b > 0$

$$\rho(x) := \log\Gamma(b) - \log\Gamma(b - iax), \quad x \in \mathbb{R}, \tag{1.12}$$

is a continuous negative definite function on $\mathbb{R}$ *satisfying* $\rho(0) = 0$.

Proposition 1.1 shows that there exists a uniquely determined product convolution semigroup $\big(\tau_c(a,b)\big)_{c>0}$ satisfying

$$\begin{aligned}\int_0^\infty t^{-ix}\, d\tau_c(a,b)(t) &= \exp\big[-c\big(\log\Gamma(b) - \log\Gamma(b-iax)\big)\big]\\ &= [\Gamma(b-iax)/\Gamma(b)]^c, \quad x \in \mathbb{R}.\end{aligned} \tag{1.13}$$

Putting $z = -ix$ in (1.6), we see by the uniqueness theorem for Fourier transforms that $d\tau_1(a,b)(t) = e_1(a,b)(t)\,dt$, and since $e_1(a,b)(t)$ has moments of any order by (1.6), we infer that all the measures $\tau_c(a,b)$ have moments of any order. This implies that the integral

$$\int_0^\infty t^z\, d\tau_c(a,b)(t), \quad \operatorname{Re} z \geqslant 0,$$

defines a continuous function of z in the half-plane $\operatorname{Re} z \geqslant 0$ and holomorphic in the interior $\operatorname{Re} z > 0$. By (1.13) this function equals $[\Gamma(b+az)/\Gamma(b)]^c$ on the imaginary axis and hence on $\operatorname{Re} z \geqslant 0$. As in the proof of [4], Lemma 2.1, it follows that this equality extends to the half-plane $\operatorname{Re} z > -b/a$, i.e. (1.7) holds.

The function $\big(\Gamma(b-iax)/\Gamma(b)\big)^c$ is a Schwartz function on $\mathbb{R}$ and in particular integrable, so (1.8) follows from (1.7), and $e_c(a,b)$ is C^∞ on $(0,\infty)$.

In this way we have established (i)–(iii) of Theorem 1.1. The proof of the more difficult part (iv) as well as the proof of Proposition 1.1 will be given in Section 3.

By Riemann–Lebesgue's lemma we also see that $te_c(a,b)(t)$ tends to zero for t tending to zero and to infinity. Much more on the behavior near zero and infinity will be given in Section 2, where we extend the work of [7] leading to the asymptotic behavior of the densities $e_c(a,b)(t)$ for $t \to 0$ and $t \to \infty$. The behavior for $t \to \infty$ will lead to a proof of the S-indeterminacy for $ac > 2$ using the Krein criterion.

The fact that $\tau_c(a,b) \diamond \tau_d(a,b) = \tau_{c+d}(a,b)$ can be written as

$$e_{c+d}(a,b)(t) = \int_0^\infty e_c(a,b)(t/x) e_d(a,b)(x)\,\frac{dx}{x}, \quad c, d > 0. \tag{1.14}$$

In particular, for $c = d = 1$ and the explicit formula for $e_1(a,b)$ we get

$$\begin{aligned}e_2(a,b)(t) &= \frac{t^{b/a-1}}{[a\Gamma(b)]^2}\int_0^\infty \exp(-x^{-1/a}t^{1/a} - x^{1/a})\frac{dx}{x}\\ &= \frac{2t^{b/a-1}}{a\Gamma(b)^2}K_0(2t^{1/(2a)}),\end{aligned} \tag{1.15}$$

because the Macdonald function K_0 is given by

$$K_0(z) = \frac{1}{2}\int_0^\infty \exp\big(-(z/2)^2/y - y\big)\frac{dy}{y},$$

cf. [8], 8.432(6); [14], Chapter 10, Section 25.

Except for a scaling this result is the same as Lemma 1 in [12].

2. MAIN RESULTS

Our additional main results are the following:

THEOREM 2.1. *For $c > 0$ we have*

$$(2.1)\quad e_c(a,b)(t) = \frac{(2\pi)^{(c-1)/2}}{a\sqrt{c}\,\Gamma(b)^c}\frac{\exp(-ct^{1/(ac)})}{t^{1-(b-1/2+1/(2c))/a}}[1+\mathcal{O}(t^{-1/(ac)})], \quad t\to\infty.$$

THEOREM 2.2. *The measure $\tau_c(a,b)$ is S-indeterminate if and only if $ac > 2$.*

THEOREM 2.3. *For $c > 0$ and $0 < t < 1$ we have*

$$(2.2)\quad e_c(a,b)(t) = \frac{t^{b/a-1}}{[a\Gamma(b)]^c}\frac{[\log(1/t)]^{c-1}}{\Gamma(c)} + \mathcal{O}\big(t^{b/a-1}[\log(1/t)]^{c-2}\big), \quad t\to 0.$$

REMARK 2.1. Formula (2.2) shows that $e_c(a,b)(t)$ tends to zero for $t\to 0$ if $b/a > 1$, and to infinity if $b/a < 1$, independent of c. If $b/a = 1$, then $e_c(a,b)(t)$ tends to zero for $c < 1$ and to infinity as a power of $\log(1/t)$ when $c > 1$.

3. PROOFS

Proof of Proposition 1.1. From the Weierstrass product for the entire function $1/\Gamma(z)$ we get the following holomorphic branch in the cut plane $\mathcal{A}$, cf. (1.9):

$$(3.1)\quad -\log\Gamma(z) = \gamma z + \operatorname{Log} z + \sum_{k=1}^{\infty}\big(\operatorname{Log}(1+z/k) - z/k\big), \quad z\in\mathcal{A},$$

where Log denotes the principal logarithm, and γ is Euler's constant.

For $n\in\mathbb{N}$ and $z\in\mathcal{A}$ define

$$\rho_n(z) = \gamma z + \operatorname{Log} z + \sum_{k=1}^{n}\big(\operatorname{Log}(1+z/k) - z/k\big),$$

$$R_n(z) = \sum_{k=n+1}^{\infty}\big(\operatorname{Log}(1+z/k) - z/k\big),$$

so $\lim_{n\to\infty} \rho_n(z) = -\log\Gamma(z)$, uniformly on compact subsets of $\mathcal{A}$. Furthermore, we have

$$\log\Gamma(b) + \rho_n(b) + R_n(b) = 0,$$

and since $\log(1+x) < x$ for $x > 0$, we see that $R_n(b) < 0$ and hence $\log\Gamma(b) + \rho_n(b) > 0$.

We claim that $\log\Gamma(b) + \rho_n(b - iax)$ is a continuous negative definite function, and letting $n \to \infty$, we get the assertion of Proposition 1.1.

To see the claim, we write

$$\begin{aligned}\log\Gamma(b) + \rho_n(b-iax) &= \log\Gamma(b) + (b-iax)\Big(\gamma - \sum_{k=1}^{n}\frac{1}{k}\Big)\\ &\quad + \operatorname{Log}(b-iax) + \sum_{k=1}^{n}\operatorname{Log}\Big(1+\frac{b-iax}{k}\Big)\\ &= \log\Gamma(b) + \rho_n(b) - iax\Big(\gamma - \sum_{k=1}^{n}\frac{1}{k}\Big) + \sum_{k=0}^{n}\operatorname{Log}\Big(1 - i\frac{ax}{b+k}\Big),\end{aligned}$$

and the assertion follows since $\alpha + i\beta x$ and $\operatorname{Log}(1+i\beta x)$ are negative definite functions when $\alpha \geqslant 0, \beta \in \mathbb{R}$, see [6], [16]. ■

P r o o f o f T h e o r e m 2.1. We modify the proof given in [7] and start by applying Cauchy's integral theorem to move the integration in (1.8) to a horizontal line

$$H_\delta := \{z = x + i\delta \ : \ x \in \mathbb{R}\}, \quad \delta > -b/a. \tag{3.2}$$

LEMMA 3.1. *With H_δ as in* (3.2) *we have*

$$e_c(a,b)(t) = \frac{1}{2\pi}\int_{H_\delta} t^{iz-1}[\Gamma(b-iaz)/\Gamma(b)]^c\,dz, \quad t > 0. \tag{3.3}$$

P r o o f. For $t, c > 0$ fixed, $f(z) = t^{iz-1}[\Gamma(b-iaz)/\Gamma(b)]^c$ is holomorphic in the simply connected domain $\mathbb{C}\setminus i(-\infty,-b/a]$, so (3.3) follows from Cauchy's integral theorem provided the integral

$$\int_0^\delta f(x+iy)\,dy$$

tends to zero for $x \to \pm\infty$. We have

$$|f(x+iy)| = t^{-y-1}|\Gamma(b+y-iax)/\Gamma(b)|^c,$$

and the result follows since

$$|\Gamma(u+iv)| \sim \sqrt{2\pi}e^{-\pi|v|/2}|v|^{u-1/2}, \quad |v| \to \infty, \text{ uniformly for bounded real } u,$$

cf. [1], p. 141, equation 5.11.9; [8], 8.328(1). ■

In the following we will use Lemma 3.1 with the line of integration H_δ, where $\delta = (t^{1/(ac)} - b)/a$. Therefore,

$$e_c(a,b)(t) = t^{(b-t^{1/(ac)})/a-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} t^{ix}[\Gamma(t^{1/(ac)} - iax)/\Gamma(b)]^c\,dx,$$

and after the change of variable $x = a^{-1}t^{1/(ac)}u$ and putting $A := (1/c+b-a)/a$ we get

(3.4)
$$e_c(a,b)(t) = t^{A-a^{-1}t^{1/(ac)}}\frac{1}{2\pi a}\int_{-\infty}^{\infty} t^{iua^{-1}t^{1/(ac)}}\big[\Gamma\big(t^{1/(ac)}(1-iu)\big)/\Gamma(b)\big]^c\,du.$$

Binet's formula for Γ is ([8], 8.341(1))

$$\Gamma(z) = \sqrt{2\pi}z^{z-1/2}e^{-z+\mu(z)}, \quad \mathrm{Re}\,(z) > 0, \tag{3.5}$$

where

$$\mu(z) = \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}\,dt, \quad \mathrm{Re}\,(z) > 0. \tag{3.6}$$

Notice that $\mu(z)$ is the Laplace transform of a positive function, so we have the estimates for $z = r + is, r > 0$, that is,

$$|\mu(z)| \leqslant \mu(r) \leqslant \frac{1}{12r}, \tag{3.7}$$

where the last inequality is a classical version of Stirling's formula, thus showing that the estimate is uniform in $s \in \mathbb{R}$.

Inserting this in (3.4), we get after some simplification

(3.8)
$$e_c(a,b)(t) = \frac{(2\pi)^{c/2-1}}{a\Gamma(b)^c}t^{A-1/(2a)}e^{-ct^{1/(ac)}}\int_{-\infty}^{\infty} e^{ct^{1/(ac)}f(u)}g_c(u)M(u,t)\,du,$$

where

$$f(u) := iu + (1-iu)\,\mathrm{Log}(1-iu), \quad g_c(u) := (1-iu)^{-c/2} \tag{3.9}$$

and

$$M(u,t) := \exp\big[c\mu\big(t^{1/(ac)}(1-iu)\big)\big]. \tag{3.10}$$

From (3.7) we get $M(u,t) = 1 + \mathcal{O}(t^{-1/(ac)})$ for $t \to \infty$, uniformly in u. We shall therefore consider the behavior for large x of the integral

$$\int_{-\infty}^{\infty} e^{xf(u)}g_c(u)\,du, \quad x = ct^{1/(ac)}. \tag{3.11}$$

This is the same integral as was treated in [7], equation (28), leading to

$$\int_{-\infty}^{\infty} e^{xf(u)} g_c(u)\, du = (2\pi/x)^{1/2}[1 + \mathcal{O}(x^{-1})]$$

by methods from [13].

For $x = ct^{1/(ac)}$ we find

$$\int_{-\infty}^{\infty} \exp\big(ct^{1/(ac)} f(u)\big) g_c(u) du = \frac{\sqrt{2\pi}}{\sqrt{c}t^{1/(2ac)}}[1 + \mathcal{O}(t^{-1/(ac)})],$$

hence

$$e_c(a,b)(t) = \frac{(2\pi)^{(c-1)/2}}{a\sqrt{c}\Gamma(b)^c} \frac{\exp(-ct^{1/(ac)})}{t^{1-(b-1/2+1/(2c))/a}}[1 + \mathcal{O}(t^{-1/(ac)})]. \ \blacksquare$$

P r o o f o f T h e o r e m 2.2. We first prove that $\big(s_n(a,b)^c\big)$ is S-determinate for $ac \leqslant 2$ by Carleman's criterion, cf. [17], p. 20. In fact, from Stirling's formula we have

$$s_n(a,b)^{c/(2n)} = \big(\Gamma(na+b)/\Gamma(b)\big)^{c/(2n)} \sim (na/e)^{ac/2}, \quad n \to \infty,$$

so $\sum s_n(a,b)^{-c/(2n)} = \infty$ if and only if $ac \leqslant 2$.

Since Carleman's criterion is only a sufficient condition for S-determinacy, we need to prove that $e_c(a,b)$ is S-indeterminate for $ac > 2$. We apply the Krein criterion for S-indeterminacy of probability densities concentrated on the half-line, using a version due to H. L. Pedersen given in [9], Theorem 4. It states that if

$$\int_{K}^{\infty} \frac{\log e_c(a,b)(t^2)\, dt}{1+t^2} > -\infty \tag{3.12}$$

for some $K \geqslant 0$, then $e_c(a,b)$ is S-indeterminate. This version of the Krein criterion is a simplification of a stronger version given in [15]. We shall see that (3.12) holds for $ac > 2$.

From Theorem 2.1 we see that (3.12) holds for sufficiently large $K > 0$ if and only if

$$\int_{K}^{\infty} \frac{-ct^{2/(ac)}}{1+t^2}\, dt > -\infty,$$

and the latter holds precisely for $ac > 2$. This shows that $\tau_c(a,b)$ is S-indeterminate for $ac > 2$. $\blacksquare$

P r o o f o f T h e o r e m 2.3. The proof uses the same ideas as in [7], but since the proof is quite technical, we give the full proof with the necessary modifications. Since we are studying the behavior for $t \to 0$, we assume that $0 < t < 1$ so that $\Lambda := \log(1/t) > 0$.

We will need integration along vertical lines

$$V_\alpha := \{\alpha + iy \mid y = -\infty \dots \infty\}, \quad \alpha \in \mathbb{R}, \tag{3.13}$$

and we can therefore express (1.8) as

$$e_c(a,b)(t) = \frac{t^{b/a-1}}{2\pi i a\Gamma(b)^c} \int_{V_{-b}} t^{z/a}\Gamma(-z)^c dz. \tag{3.14}$$

By the functional equation for Γ we get

$$e_c(a,b)(t) = \frac{t^{b/a-1}}{2\pi i a\Gamma(b)^c} \int_{V_{-b}} g(z)\varphi(z) dz, \tag{3.15}$$

where we have defined

$$\varphi(z) := t^{z/a}\Gamma(1-z)^c, \quad g(z) := (-z)^{-c} = \exp\big(-c\,\mathrm{Log}(-z)\big).$$

Note that φ is holomorphic in $\mathbb{C} \setminus [1,\infty)$, while g is holomorphic in $\mathbb{C} \setminus [0,\infty)$.

For $x > 0$ we define

$$g_\pm(x) := \lim_{\varepsilon \to 0+} g(x \pm i\varepsilon) = x^{-c}e^{\pm i\pi c}.$$

Case 1. Assume $0 < c < 1$.

We fix $0 < s < 1$, choose $0 < \varepsilon < \min(s,b)$ and integrate $g(z)\varphi(z)$ over the contour

$$\begin{aligned}\mathcal{C} := \{-b + iy \mid y = \infty \dots 0\} &\cup [-b,-\varepsilon] \cup \{\varepsilon e^{i\theta} \mid \theta = \pi \dots 0\} \\ &\cup [\varepsilon, s] \cup \{s + iy \mid y = 0 \dots \infty\}\end{aligned}$$

and get zero by the integral theorem of Cauchy. On the interval $[\varepsilon, s]$ we will use $g = g_+$.

Similarly, we get zero by integrating $g(z)\varphi(z)$ over the complex conjugate contour $\overline{\mathcal{C}}$, and now we use $g = g_-$ on the interval $[\varepsilon, s]$.

Subtracting the second contour integral from the first leads to

$$\int_{V_s} - \int_{V_{-b}} - \int_{|z|=\varepsilon} g(z)\varphi(z)\,dz + \int_\varepsilon^s \varphi(x)\big(g_+(x) - g_-(x)\big)\,dx = 0,$$

where the integral over the circle is with positive orientation. Note that the two integrals over $[-b,-\varepsilon]$ cancel. Since $0 < c < 1$, it is easy to see that the just mentioned integral converges to zero for $\varepsilon \to 0$, and we finally get for $\varepsilon \to 0$

$$\begin{aligned}e_c(a,b)(t) &= \frac{t^{b/a-1}}{2\pi i a\Gamma(b)^c} \int_{V_s} g(z)\varphi(z)\,dz + \frac{t^{b/a-1}\sin(\pi c)}{\pi a\Gamma(b)^c} \int_0^s x^{-c}\varphi(x)\,dx \\ &:= I_1 + I_2.\end{aligned}$$

We claim that I_1 is $o(t^{(s+b)/a-1})$ for $t \to 0$. To see this, we insert the parametrization of V_s and get

$$\begin{aligned} I_1 &= \frac{t^{b/a-1}}{2\pi a\Gamma(b)^c} \int\limits_{-\infty}^{\infty} (-s-iy)^{-c} t^{(s+iy)/a} \Gamma(1-s-iy)^c \, dy \\ &= \frac{t^{(s+b)/a-1}}{2\pi a\Gamma(b)^c} \int\limits_{-\infty}^{\infty} e^{-iy\Lambda/a} (-s-iy)^{-c} \Gamma(1-s-iy)^c \, dy, \end{aligned}$$

and the integral is $o(1)$ for $t \to 0$ by Riemann–Lebesgue's lemma because $\Lambda := \log(1/t) \to \infty$.

The substitution $u = x\Lambda$ in the integral in the term I_2 leads to

$$I_2 = \frac{t^{b/a-1}\sin(\pi c)}{\pi a\Gamma(b)^c} \Lambda^{c-1} \int\limits_0^{s\Lambda} u^{-c} e^{-u/a} \Gamma(1-u/\Lambda)^c \, du. \tag{3.16}$$

We split the integral in (3.16) as

$$\int\limits_0^{s\Lambda} u^{-c} e^{-u/a} \left[\Gamma(1-u/\Lambda)^c - 1\right] du + \int\limits_0^{\infty} u^{-c} e^{-u/a} \, du - \int\limits_{s\Lambda}^{\infty} u^{-c} e^{-u/a} \, du. \tag{3.17}$$

Calling the three terms J_1, J_2, J_3, we have $J_2 = a^{1-c}\Gamma(1-c)$ and

$$J_3 = -a^{1-c}\Gamma(1-c, s\Lambda/a),$$

where $\Gamma(\alpha, x)$ is the incomplete gamma function with the asymptotics

$$\Gamma(\alpha, x) = \int\limits_x^{\infty} u^{\alpha-1} e^{-u} \, du \sim x^{\alpha-1} e^{-x}, \quad x \to \infty$$

(cf. [8], 8.357), hence

$$J_3 = \mathcal{O}(t^{s/a}\Lambda^{-c}), \quad t \to 0.$$

Using the digamma function $\Psi = \Gamma'/\Gamma$, we get by the mean-value theorem

$$\Gamma(1-u/\Lambda)^c - 1 = -\frac{u}{\Lambda} c\Gamma(1-\theta u/\Lambda)^c \Psi(1-\theta u/\Lambda)$$

for some $0 < \theta < 1$, but this implies that

$$|\Gamma(1-u/\Lambda)^c - 1| \leqslant \frac{cu}{\Lambda} M(s), \quad 0 < u < s\Lambda,$$

where

$$M(s) := \max\{\Gamma(x)^c |\Psi(x)| \mid 1-s \leqslant x \leqslant 1\},$$

so $J_1 = \mathcal{O}(\Lambda^{-1})$ for $t \to 0$.

This gives

$$\begin{aligned} I_2 &= \frac{t^{b/a-1}\sin(\pi c)}{\pi a\Gamma(b)^c}\Lambda^{c-1}\big(\mathcal{O}(\Lambda^{-1}) + a^{1-c}\Gamma(1-c) + \mathcal{O}(t^{s/a}\Lambda^{-c})\big) \\ &= \frac{t^{b/a-1}\Lambda^{c-1}}{\big(a\Gamma(b)\big)^c\Gamma(c)} + \mathcal{O}(t^{b/a-1}\Lambda^{c-2}), \end{aligned}$$

where we have used Euler's reflection formula for Γ. Since finally

$$I_1 = o(t^{(s+b)/a-1}) = \mathcal{O}(t^{b/a-1}\Lambda^{c-2}),$$

we see that (2.2) holds.

Case 2. Assume $1 < c < 2$.

The gamma function decays so rapidly on vertical lines $z = \alpha + iy, y \to \pm\infty$, that we can integrate by parts in (3.15) to get

$$e_c(a,b)(t) = -\frac{t^{b/a-1}}{2\pi i a\Gamma(b)^c}\int\limits_{V_{-1}} \frac{(-z)^{-(c-1)}}{c-1}\frac{d}{dz}\big(t^{z/a}\Gamma(1-z)^c\big)dz. \tag{3.18}$$

Defining

$$\varphi_1(z) := \frac{d}{dz}\big(t^{z/a}\Gamma(1-z)^c\big) = t^{z/a}\Gamma(1-z)^c\big((1/a)\log t - c\Psi(1-z)\big)$$

and using the same contour technique as in Case 1 to the integral in (3.18), where now $0 < c-1 < 1$, we get for $0 < s < 1$ fixed the equality

$$e_c(a,b)(t) = -\frac{t^{b/a-1}}{a\Gamma(b)^c}(\tilde{I}_1 + \tilde{I}_2),$$

where

$$\tilde{I}_1 = \frac{1}{2\pi i(c-1)}\int\limits_{V_s}(-z)^{-(c-1)}\varphi_1(z)\,dz, \quad \tilde{I}_2 = \frac{\sin\big(\pi(c-1)\big)}{\pi(c-1)}\int\limits_0^s x^{-(c-1)}\varphi_1(x)\,dx.$$

We have $\tilde{I}_1 = o(t^{s/a}\Lambda)$ for $t \to 0$ by Riemann–Lebesgue's lemma, and the substitution $u = x\Lambda$ in the second integral leads to

$$\begin{aligned} &\int\limits_0^s x^{-(c-1)}\varphi_1(x)\,dx \\ &= \Lambda^{c-2}\int\limits_0^{s\Lambda} u^{-(c-1)}\varphi_1(u/\Lambda)\,du \\ &= -(1/a)\Lambda^{c-1}\Big(\int\limits_0^{s\Lambda} u^{-(c-1)}e^{-u/a}\,du + \int\limits_0^{s\Lambda} u^{-(c-1)}e^{-u/a}\big(\Gamma(1-u/\Lambda)^c - 1\big)\,du\Big) \end{aligned}$$

$$
\begin{aligned}
&- c\Lambda^{c-2}\int_0^{s\Lambda} u^{-(c-1)}e^{-u/a}\Gamma(1-u/\Lambda)^c\Psi(1-u/\Lambda)\,du\\
&= -a^{1-c}\Lambda^{c-1}\Gamma(2-c)+\mathcal{O}(\Lambda^{c-2}).
\end{aligned}
$$

Using the equality

$$
\frac{\sin\big(\pi(c-1)\big)}{(c-1)\pi}\big(-a^{1-c}\Lambda^{c-1}\Gamma(2-c)\big) = -a^{1-c}\frac{\Lambda^{c-1}}{\Gamma(c)}
$$

obtained by Euler's reflection formula, we see that (2.2) holds.

Case 3. Assume $c > 2$.

We perform the change of variable $w = (1/a)\Lambda z$ in (3.15) and assume that $\Lambda > a$. This gives

$$
e_c(a,b)(t) = \frac{t^{b/a-1}\Lambda^{c-1}}{[a\Gamma(b)]^c}\frac{1}{2\pi i}\int_{V_{-(b/a)\Lambda}} (-w)^{-c}e^{-w}\Gamma(1-aw/\Lambda)^c\,dw.
$$

Using Cauchy's integral theorem, we can shift the contour $V_{-(b/a)\Lambda}$ to V_{-1} as the integrand is holomorphic in the vertical strip between both paths and exponentially small at both extremes of that vertical strip. For the holomorphic function $h(z) = \Gamma(1-z)^c$ in the domain $G = \mathbb{C}\setminus[1,\infty)$, which is star-shaped with respect to zero, we have

$$
h(z) = h(0) + z\int_0^1 h'(uz)\,du, \quad z\in G,
$$

hence

$$
\Gamma(1-aw/\Lambda)^c = 1 - \frac{caw}{\Lambda}\int_0^1 \Gamma(1-uaw/\Lambda)^c\Psi(1-uaw/\Lambda)\,du. \tag{3.19}
$$

Defining

$$
R(w) = \int_0^1 \Gamma(1-uaw/\Lambda)^c\Psi(1-uaw/\Lambda)\,du,
$$

we get

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{V_{-1}} (-w)^{-c}e^{-w}\Gamma(1-aw/\Lambda)^c\,dw\\
&\qquad = \frac{1}{2\pi i}\int_{V_{-1}} (-w)^{-c}e^{-w}dw + \frac{ac/\Lambda}{2\pi i}\int_{V_{-1}} (-w)^{1-c}e^{-w}R(w)dw.
\end{aligned}
$$

For any $w\in V_{-1}, 0\leqslant u\leqslant 1$ and for $\Lambda\geqslant a$ it follows that $1-uaw/\Lambda$ belongs to the closed vertical strip located between the vertical lines V_1 and V_2. Because

$\Gamma(z)^c\Psi(z)$ is continuous and bounded in this strip, $R(w)$ is bounded for $w \in V_{-1}$ by a constant independent of $\Lambda \geqslant a$. Furthermore, $(-w)^{1-c}e^{-w}$ is integrable over V_{-1} because $c > 2$.

On the other hand, in the integral

$$\frac{1}{2\pi i}\int_{V_{-1}} (-w)^{-c}e^{-w}dw$$

the contour V_{-1} may be deformed to a Hankel contour

$$\mathcal{H} := \{x - i \mid x = \infty \ldots 0\} \cup \{e^{i\theta} \mid \theta = -\pi/2 \ldots -3\pi/2\}$$
$$\cup \{x + i \mid x = 0 \ldots \infty\}$$

surrounding $[0, \infty)$, and the integral over $\mathcal{H}$ is Hankel's integral representation of the reciprocal gamma function:

$$\frac{1}{2\pi i}\int_{\mathcal{H}} (-w)^{-c}e^{-w}dw = \frac{1}{\Gamma(c)}.$$

Therefore, when we join everything, we obtain for $c > 2$:

$$e_c(a,b)(t) = \frac{t^{b/a-1}}{[a\Gamma(b)]^c}\frac{[\log(1/t)]^{c-1}}{\Gamma(c)} + \mathcal{O}\big(t^{b/a-1}[\log(1/t)]^{c-2}\big), \quad t \to 0.$$

Case 4. $c = 1, c = 2$.

These cases are easy since $e_1(a,b)(t)$ is explicitly given by (1.5) and $e_2(a,b)(t)$ by (1.15). The asymptotics of K_0 is known:

$$K_0(t) = \log(2/t) + \mathcal{O}(1), \quad t \to 0. \quad \blacksquare$$

REMARK 3.1. The behavior of $e_c(a,b)(t)$ for $t \to 0$ can be obtained from (3.14) by using the residue theorem when c is a natural number. In fact, in this case $\Gamma(-z)^c$ has a pole of order c at $z = 0$, and a shift of the contour V_{-1} to V_s, where $0 < s < 1$, has to be compensated by a residue, which will give the behavior for $t \to 0$.

When c is a natural number, one can actually express $e_c(a,b)(t)$ in terms of Meijer's G-function:

$$e_c(a,b)(t) = \frac{t^{b/a-1}}{a\Gamma(b)^c}G^{c,0}_{0,c}\left(t^{1/a} \mid \begin{matrix} - & \cdots & - \\ 0 & \cdots & 0 \end{matrix}\right),$$

cf. Section 9.3 in [8].

4. ONE-PARAMETER EXTENSION OF THE GUMBEL DISTRIBUTIONS

The group isomorphism $x = \log(1/t)$ of the multiplicative group $(0, \infty)$ onto the additive group $\mathbb{R}$ transforms the convolution semigroup $\big(\tau_c(a,b)\big)_{c>0}$ into an ordinary convolution semigroup $\big(G_c(a,b)\big)_{c>0}$ of probability measures on $\mathbb{R}$ with densities given by

$$g_c(a,b)(x) = e^{-x} e_c(a,b)(e^{-x}), \quad x \in \mathbb{R}, \tag{4.1}$$

and $a, b, c > 0$ are arbitrary. For $c = 1$ we have

$$g_1(a,b)(x) = \frac{1}{a\Gamma(b)} \exp(-bx/a - e^{-x/a}), \quad x \in \mathbb{R}. \tag{4.2}$$

This density is infinitely divisible and the uniquely determined convolution roots are given by (4.1).

The special density $g_1(a,1)(x)$ is the Gumbel density with scale parameter $a > 0$, and the basic case $a = 1$ is discussed in [7]. From the asymptotic behavior of $e_c(a,b)$ in Theorems 2.1 and 2.3 we can obtain the asymptotic behavior of the convolution roots $g_c(a,b)$:

(4.3)
$$g_c(a,b)(x) = \frac{(2\pi)^{(c-1)/2}}{a\sqrt{c}\,\Gamma(b)^c} \frac{\exp(-ce^{-x/(ac)})}{\exp\big(x\big(b - 1/2 + 1/(2c)\big)/a\big)} \Big[1 + \mathcal{O}\Big(\exp\big(x/(ac)\big)\Big)\Big]$$

for $x \to -\infty$, and

$$g_c(a,b)(x) = \frac{\exp(-bx/a)x^{c-1}}{[a\Gamma(b)]^c \Gamma(c)} + \mathcal{O}\left(\exp(-bx/a)x^{c-2}\right), \quad x \to \infty. \tag{4.4}$$

THEOREM 4.1. *All densities $g_c(a,b)$ belong to determinate Hamburger moment problems.*

P r o o f. We first prove that $g_1(a,b)$ is determinate, and for this it suffices to verify that the moments

$$s_n = \int_{-\infty}^{\infty} x^n g_1(a,b)(x)\, dx \tag{4.5}$$

satisfy Carleman's condition $\sum_{n=0}^{\infty} s_{2n}^{-1/(2n)} = \infty$ (cf. [17], p. 19). From (4.5) we get

$$\begin{aligned} s_{2n} &= \frac{1}{a\Gamma(b)} \int_0^{\infty} (\log t)^{2n} t^{b/a-1} \exp(-t^{1/a})\, dt = \frac{1}{\Gamma(b)} \int_0^{\infty} (a \log s)^{2n} s^{b-1} e^{-s}\, ds \\ &< \frac{a^{2n}}{\Gamma(b)} \Big(\int_0^1 (\log s)^{2n} s^{b-1}\, ds + \int_1^{\infty} s^{2n+b-1} e^{-s}\, ds \Big). \end{aligned}$$

By changing variables we see that

$$\int_0^1 (\log s)^{2n} s^{b-1}\, ds = \frac{(2n)!}{b^{2n+1}},$$

and

$$\int_1^\infty s^{2n+b-1} e^{-s}\, ds < \Gamma(2n+b),$$

hence

$$s_{2n}^{1/(2n)} < \frac{a}{\Gamma(b)^{1/(2n)}}\left[\left(\frac{(2n)!}{b^{2n+1}}\right)^{1/(2n)} + \Gamma(2n+b)^{1/(2n)}\right],$$

and the Carleman condition follows from Stirling's formula, which shows that the right-hand side is bounded by Kn for sufficiently large $K > 0$. We next use Corollary 3.3 in [2] to infer that the Carleman condition also holds for all convolution roots $g_c(a,b)$. ■

Concerning the moments

$$s_n(c) = \int_{-\infty}^{\infty} x^n g_c(a,b)(x)\, dx, \qquad n \in \mathbb{N}_0, \tag{4.6}$$

of the convolution roots we have the following result:

THEOREM 4.2. *The moment* $s_n(c)$ *of* (4.6) *is a polynomial*

$$s_n(c) = \sum_{k=1}^{n} a_{n,k} c^k, \quad n \geqslant 1, \tag{4.7}$$

of degree at most n *in the variable* c*. The coefficients* $a_{n,k}$ *are given below.*

Proof. From (1.7) we get

$$\int_{-\infty}^{\infty} e^{-ixy}\, dG_c(a,b)(x) = \int_0^\infty t^{iy} e_c(a,b)(t)\, dt = [\Gamma(b+iay)/\Gamma(b)]^c,$$

which shows that the negative definite function ρ corresponding to the convolution semigroup $\big(G_c(a,b)\big)_{c>0}$ is

$$\rho(y) = \log\Gamma(b) - \log\Gamma(b+iay), \quad y \in \mathbb{R}.$$

The derivatives of ρ can be expressed in terms of the digamma function Ψ, namely

$$\rho^{(n+1)}(y) = -(ia)^{n+1}\Psi^{(n)}(b+iay), \quad n \in \mathbb{N}_0,$$

so if for $n \in \mathbb{N}_0$ we define (cf. [3], equation (2.7))

$$\sigma_n := -i^{n+1}\rho^{(n+1)}(0) = (-a)^{n+1}\Psi^{(n)}(b),$$

we find

$$\sigma_0 = a\gamma + \frac{a}{b} - ab\sum_{k=1}^{\infty}\frac{1}{k(b+k)},$$

$$\sigma_n = a^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(b+k)^{n+1}} = a^{n+1}n!\zeta(n+1,b), \quad n \in \mathbb{N},$$

where $\zeta(z,q)$ is Hurwitz' zeta function (cf. [8], 9.521).

According to [3] we have $s_1(c) = \sigma_0 c, s_2(c) = \sigma_1 c + \sigma_0^2 c^2$ and in general $s_n(c)$ is given by (4.7), where the coefficients $a_{n,k}$ are determined by the recursion

$$a_{n+1,k+1} = \sum_{j=k}^{n} a_{j,k}\binom{n}{j}\sigma_{n-j}, \quad n \geqslant k \geqslant 0.$$

It is easy to see that

$$a_{n,1} = \sigma_{n-1}, \quad a_{n,n-1} = \binom{n}{2}\sigma_0^{n-2}\sigma_1, \quad a_{n,n} = \sigma_0^n. \quad \blacksquare$$

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Christian Berg
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen Ø, Denmark
E-mail: berg@math.ku.dk

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