ASYMPTOTIC BEHAVIOR FOR QUADRATIC VARIATIONS OF NON-GAUSSIAN MULTIPARAMETER HERMITE RANDOM FIELDS

BY

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Abstract. Let $(Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t}\in[0,1]^d}$ denote a d-parameter Hermite random field of order $q \geqslant 1$ and self-similarity parameter $\mathbf{H} = (H_1,\ldots,H_d) \in \left(\frac{1}{2},1\right)^d$. This process is $\mathbf{H}$-self-similar, has stationary increments and exhibits long-range dependence. Particular examples include fractional Brownian motion ($q = 1$, $d = 1$), fractional Brownian sheet ($q = 1, d \geqslant 2$), the Rosenblatt process ($q = 2$, $d = 1$) as well as the Rosenblatt sheet ($q = 2, d \geqslant 2$). For any $q \geqslant 2, d \geqslant 1$ and $\mathbf{H} \in \left(\frac{1}{2},1\right)^d$ we show in this paper that a proper renormalization of the quadratic variation of $Z^{q,\mathbf{H}}$ converges in $L^2(\Omega)$ to a standard d-parameter Rosenblatt random variable with self-similarity index $\mathbf{H}'' = 1 + (2\mathbf{H} - 2)/q$.

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1. MOTIVATION AND MAIN RESULTS

In recent years, analyzing the asymptotic behavior of power variations of self-similar stochastic processes has attracted a lot of attention. This is because they play an important role in various aspects, both in probability and statistics. As far as quadratic variations are concerned, a classical application is to use them to construct efficient estimators for the self-similarity parameter (see e.g. [2], [15]). For a less conventional application, let us also mention the paper [5], where the authors have used weighted power variations of fractional Brownian motion to compute exact rates of convergence of some approximating schemes associated with one-dimensional fractional stochastic differential equations.

In this paper, we deal with the quadratic variation in the context of multi-parameter Hermite random fields. To be more specific, let $Z^{q,\mathbf{H}} = (Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t} \in [0,1]^d}$ stand for the d-parameter Hermite random field of order $q \geqslant 1$ and self-similarity parameter $\mathbf{H} = (H_1, \ldots, H_d) \in \left(\frac{1}{2}, 1\right)^d$ (see Definition 2.1 for the precise meaning), and consider a renormalized version of its quadratic variation, namely

$$(1.1) \qquad V_{\mathbf{N}} := \frac{1}{\mathbf{N}} \sum_{\mathbf{i}=0}^{\mathbf{N}-1} [\mathbf{N}^{2\mathbf{H}}(\Delta Z_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]}^{q,\mathbf{H}})^2 - 1],$$

where $\Delta Z_{[\mathbf{s},\mathbf{t}]}^{q,\mathbf{H}}$ are the increments of $Z^{q,\mathbf{H}}$ defined as

$$(1.2) \qquad \Delta Z_{[\mathbf{s},\mathbf{t}]}^{q,\mathbf{H}} = \sum_{\mathbf{r} \in \{0,1\}^d} (-1)^{d - \Sigma_i r_i} Z_{\mathbf{s}+\mathbf{r}\cdot(\mathbf{t}-\mathbf{s})}^{q,\mathbf{H}}$$

and where the bold notation is systematically used in presence of multi-indices (we refer to Section 2 for precise definitions). As illustrating examples, observe that (1.2) reduces to $\Delta Z_{[s,t]}^{q,H} = Z_t^{q,H} - Z_s^{q,H}$ when $d = 1$, and to $\Delta Z_{[\mathbf{s},\mathbf{t}]}^{q,H_1,H_2} = Z_{t_1,t_2}^{q,H_1,H_2} - Z_{t_1,s_2}^{q,H_1,H_2} - Z_{s_1,t_2}^{q,H_1,H_2} + Z_{s_1,s_2}^{q,H_1,H_2}$ when $d = 2$.

It is well known that each Hermite random field $Z^{q,\mathbf{H}}$ is $\mathbf{H}$-self-similar (that is, $(Z_{\mathbf{at}}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d} \overset{(d)}{=} (\mathbf{a}^{\mathbf{H}} Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d}$ for any $\mathbf{a} > 0$), has stationary increments (that is, $(\Delta Z_{[0,\mathbf{t}]}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d} \overset{(d)}{=} (\Delta Z_{[\mathbf{h},\mathbf{h}+\mathbf{t}]}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d}$ for all $\mathbf{h} \in \mathbb{R}^d$) and exhibits long-range dependence. Also, when $q = 1$, observe that $Z^{1,\mathbf{H}}$ is either the fractional Brownian motion (if $d = 1$) or the fractional Brownian sheet (if $d \geqslant 2$); in particular, among all the Hermite random fields $Z^{q,\mathbf{H}}$, it is the only one to be Gaussian. When $q = 2$, we use the usual terminology: the Rosenblatt process (if $d = 1$) or the Rosenblatt sheet (if $d \geqslant 2$).

Before describing our results, let us give a brief overview of the current state of the art. Firstly, let us consider the case $q = d = 1$, that is, the case where $Z^{1,H} = B^H$ is a fractional Brownian motion with Hurst parameter H. The behavior of the quadratic variation of B^H is well known since the 1980s, and dates back to the seminal works of Breuer and Major [1], Dobrushin and Major [3], Giraitis and Surgailis [4] or Taqqu [13]. We have, as $N \to \infty$:

- If $H < 3/4$, then

$$N^{-1/2} \sum_{j=1}^N \left(N^{2H}(B_{j/N}^H - B_{(j-1)/N}^H)^2 - 1\right) \overset{(d)}{\longrightarrow} \mathcal{N}(0, \sigma_H^2).$$

- If $H = 3/4$, then

$$(N \log N)^{-1/2} \sum_{j=1}^N \left(N^{3/2}(B_{j/N}^H - B_{(j-1)/N}^H)^2 - 1\right) \overset{(d)}{\longrightarrow} \mathcal{N}(0, \sigma_{3/4}^2).$$

- If $H > 3/4$, then

$$N^{1-2H} \sum_{j=1}^{N} \left(N^{2H} (B_{j/N}^H - B_{(j-1)/N}^H)^2 - 1 \right) \xrightarrow{L^2(\Omega)} \text{``Rosenblatt r.v.'',}$$

where "Rosenblatt r.v." denotes the random variable which is the value at time one of the Rosenblatt process.

Secondly, assume now that $q = 1$ and $d = 2$, that is, consider the case where $Z^{1,\mathbf{H}}$ is a two-parameter fractional Brownian sheet with Hurst parameter $\mathbf{H} = (H_1, H_2)$. According to Réveillac et al. [12] and with $\varphi(N, \mathbf{H})$ a suitable scaling factor, the quadratic variation of $Z^{1,\mathbf{H}}$ behaves as follows, as $N \to \infty$:

- If $\mathbf{H} \notin (3/4, 1)^2$, then

$$\varphi(N, \mathbf{H}) \sum_{i=1}^{N} \sum_{j=1}^{N} \left(N^{2H_1 + 2H_2} (\Delta Z_{[(\mathbf{i}-1)/\mathbf{N}, \mathbf{i}/\mathbf{N}]}^{1,\mathbf{H}})^2 - 1 \right) \xrightarrow{(d)} \mathcal{N}(0, \sigma_{\mathbf{H}}^2).$$

- If $\mathbf{H} \in (3/4, 1)^2$, then

$$\varphi(N, \mathbf{H}) \sum_{i=1}^{N} \sum_{j=1}^{N} \left(N^{2H_1 + 2H_2} (\Delta Z_{[(\mathbf{i}-1)/\mathbf{N}, \mathbf{i}/\mathbf{N}]}^{1,\mathbf{H}})^2 - 1 \right)$$

$$\xrightarrow{L^2(\Omega)} \text{``two-parameter Rosenblatt r.v.'',}$$

where "two-parameter Rosenblatt r.v." means the value at point $\mathbf{1} = (1, 1)$ of the two-parameter Rosenblatt sheet.

Here, we observe the following interesting phenomenon: the limit law in the mixture case (that is, when $H_1 \leqslant 3/4$ and $H_2 > 3/4$) is Gaussian. For simplicity of the exposition, we have only described above what happens when $d = 2$. But the asymptotic behavior for the quadratic variation of $Z^{1,\mathbf{H}}$ is actually known for any value of the dimension $d \geqslant 2$, and we refer to Pakkanen and Réveillac [9]–[11] for precise statements.

Let us finally review the existing literature on the quadratic variation of $Z^{q,\mathbf{H}}$ in the *non*-Gaussian case, that is, when $q \geqslant 2$. It is certainly because it is a more difficult case to deal with that only the case where $d = 1$ has been studied so far. Chronopoulou et al. have shown in [2] (see also [15], [14]) that, properly renormalized, the quadratic variation of $Z^{q,H}$ converges in $L^2(\Omega)$, for *any* $q \geqslant 2$ and *any* value of $H \in (1/2, 1)$, to the Rosenblatt random variable. A consequence of this finding is that fractional Brownian motion is the only Hermite process ($d = 1$) for which there exists a range of parameter such that its quadratic variation exhibits *normal* convergence; indeed, for all the other Hermite processes, it is shown in [2] that we have the convergence toward a non-Gaussian random variable belonging to the second Wiener chaos.

In the present paper, we study what happens in the remaining cases, that is, when q and d are both greater than or equal to 2. Thanks to our main result, Theorem 1.1, we have now a complete picture of the asymptotic behavior of the quadratic variation of *any* Hermite random field.

THEOREM 1.1. *Fix $q \geqslant 2, d \geqslant 1$ and $\mathbf{H} \in \left(\frac{1}{2}, 1\right)^d$. Let $Z^{q,\mathbf{H}}$ be a d-parameter Hermite random field of order q with self-similarity parameter $\mathbf{H}$ (see Definition 2.1). Then $c_{1,\mathbf{H}}^{-1/2} \mathbf{N}^{(2-2\mathbf{H})/q} (q!q)^{-1} V_{\mathbf{N}}$ converges, in $L^2(\Omega)$, to the standard d-parameter Rosenblatt sheet with self-similarity parameter $1 + (2\mathbf{H} - 2)/q$ evaluated at time $\mathbf{1}$, where $c_{1,\mathbf{H}}$ is given by (3.9).*

Our proof of Theorem 1.1 follows a strategy introduced by Tudor and Viens in [15], based on the use of chaotic expansion into multiple Wiener–Itô integrals. Let us sketch it. Since the Hermite random field $Z^{q,\mathbf{H}}$ is an element of the q-th Wiener chaos, we can firstly rely on the product formula for multiple integrals to see that the quadratic variation $V_{\mathbf{N}}$ can be decomposed into a sum of multiple integrals of even orders from 2 to $2q$. Secondly, by using the isometric property of multiple Wiener–Itô integrals and after checking the $L^2\left(([0,1]^d)^2\right)$ convergence of its kernel, we will prove that the projection onto the second Wiener chaos converges in $L^2(\Omega)$ to the d-parameter Rosenblatt random variable. Finally, we will check that all the remaining terms in the chaotic expansion are asymptotically negligible.

In conclusion, it is worth pointing out that, irrespective of the self-similarity parameter, the (properly normalized) quadratic variation of any *non-Gaussian* multiparameter Hermite random fields exhibits a convergence to a random variable belonging to the second Wiener chaos. It is in strong contrast with what happens in the Gaussian case ($q = 1$), where either central or non-central limit theorems may arise, depending on the value of the self-similarity parameter.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries and useful notation. The proof of our main result, Theorem 1.1, is then provided in Section 3.

2. PRELIMINARIES

This section describes the notation and the mathematical objects (together with their main properties) that are used throughout this paper.

2.1. Notation. Fix an integer $d \geqslant 1$. In what follows, we shall systematically use bold notation when dealing with multi-indexed quantities. We thus write:
- $\mathbf{a} = (a_1, a_2, \ldots, a_d)$.
- $\mathbf{ab} = (a_1 b_1, a_2 b_2, \ldots, a_d b_d)$ and $\mathbf{a}/\mathbf{b} = (a_1/b_1, a_2/b_2, \ldots, a_d/b_d)$.
- $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{d} [a_i, b_i], (\mathbf{a}, \mathbf{b}) = \prod_{i=1}^{d} (a_i, b_i)$.
- Summation is suppressed as follows: $\sum_{\mathbf{i}=1}^{\mathbf{N}} a_{\mathbf{i}} = \sum_{i_1=1}^{N_1} \cdots \sum_{i_d=1}^{N_d} a_{i_1, \ldots, i_d}$,

whereas for products we shall write $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^{d} a_i^{b_i}$. Finally, we shall write $\mathbf{a} < \mathbf{b}$

(resp. $\mathbf{a} \leqslant \mathbf{b}$) whenever $a_1 < b_1$, $a_2 < b_2$, ..., $a_d < b_d$ (resp. $a_1 \leqslant b_1$, $a_2 \leqslant b_2$, ..., $a_d \leqslant b_d$).

2.2. Multiple Wiener–Itô integrals. We will now briefly review the theory of multiple Wiener–Itô integrals with respect to Brownian sheet, as described e.g. in Nualart's book [8], Chapter 1, or in [9], Section 3. Let $f \in L^2\big((\mathbb{R}^d)^q\big)$ and let us denote by $I_q^W(f)$ the q-fold multiple Wiener–Itô integral of f with respect to the standard two-sided Brownian sheet $(W_{\mathbf{t}})_{\mathbf{t} \in \mathbb{R}^d}$. In symbols, such an integral is written

$$I_q^W(f) = \int\limits_{(\mathbb{R}^d)^q} f(\mathbf{u}_1, \ldots, \mathbf{u}_q) dW_{\mathbf{u}_1} \ldots dW_{\mathbf{u}_q}.$$

Moreover, one has $I_q^W(f) = I_q^W(\widetilde{f})$, where $\widetilde{f}$ is the symmetrization of f defined by the formula

$$(2.1) \qquad \widetilde{f}(\mathbf{u}_1, \ldots, \mathbf{u}_q) = \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} f(\mathbf{u}_{\sigma(1)}, \ldots, \mathbf{u}_{\sigma(q)}).$$

The set of random variables of the form $I_q^W(f)$, when f runs over $L^2\big((\mathbb{R}^d)^q\big)$, is called the *q-th Wiener chaos* of W. Furthermore, if $f \in L^2\big((\mathbb{R}^d)^p\big)$ and $g \in L^2\big((\mathbb{R}^d)^q\big)$ are two symmetric functions, then

$$(2.2) \qquad I_p^W(f)I_q^W(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^W(f \widetilde{\otimes}_r g),$$

where the contraction $f \otimes_r g$, which belongs to $L^2\big((\mathbb{R}^d)^{p+q-2r}\big)$ for every $r = 0, 1, \ldots, p \wedge q$, is given by

$$(2.3) \quad f \otimes_r g(\mathbf{u}_1, \ldots, \mathbf{u}_{p-r}, \mathbf{v}_1, \ldots, \mathbf{v}_{q-r})$$
$$= \int\limits_{(\mathbb{R}^d)^r} f(\mathbf{u}_1, \ldots, \mathbf{u}_{p-r}, \mathbf{a}_1, \ldots, \mathbf{a}_r) g(\mathbf{v}_1, \ldots, \mathbf{v}_{q-r}, \mathbf{a}_1, \ldots, \mathbf{a}_r) d\mathbf{a}_1 \ldots d\mathbf{a}_r.$$

For any $r = 0, \ldots, p \wedge q$, the Cauchy–Schwarz inequality yields

$$(2.4) \qquad \|f \widetilde{\otimes}_r g\|_{L^2((\mathbb{R}^d)^{p+q-2r})} \leqslant \|f \otimes_r g\|_{L^2((\mathbb{R}^d)^{p+q-2r})}$$
$$\leqslant \|f\|_{L^2((\mathbb{R}^d)^p)} \|g\|_{L^2((\mathbb{R}^d)^q)}.$$

Also, $f \otimes_p g = \langle f, g \rangle_{L^2((\mathbb{R}^d)^p)}$ when $q = p$. Furthermore, the multiple Wiener integrals satisfy the following isometry and orthogonality properties:

$$E[I_p^W(f)I_q^W(g)] = \begin{cases} p! \langle \widetilde{f}, \widetilde{g} \rangle_{L^2((\mathbb{R}^d)^p)} & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

2.3. Multiparameter Hermite random fields. Let us now introduce our main object of interest in this paper, the so-called multiparameter Hermite random field. We follow the definition given by Tudor in [14], Chapter 4.

DEFINITION 2.1. Let $q, d \geqslant 1$ be two integers and let $\mathbf{H} = (H_1, \ldots, H_d)$ be a vector belonging to $\left(\frac{1}{2}, 1\right)^d$. The *d-parameter Hermite random field of order q and self-similarity parameter* $\mathbf{H}$ is any random field of the form

$$(2.5) \quad Z^{q,\mathbf{H}}(\mathbf{t}) = c_{q,\mathbf{H}} \int\limits_{(\mathbb{R}^d)^q} dW_{u_{1,1},\ldots,u_{1,d}} \ldots dW_{u_{q,1},\ldots,u_{q,d}}$$

$$\times \Big(\int\limits_0^{t_1} da_1 \ldots \int\limits_0^{t_d} da_d \prod_{j=1}^q (a_1 - u_{j,1})_+^{-(1/2+(1-H_1)/q)} \ldots (a_d - u_{j,d})_+^{-(1/2+(1-H_d)/q)} \Big)$$

$$= c_{q,\mathbf{H}} \int\limits_{(\mathbb{R}^d)^q} dW_{\mathbf{u_1}} \ldots dW_{\mathbf{u_q}} \int\limits_0^{\mathbf{t}} d\mathbf{a} \prod_{j=1}^q (\mathbf{a} - \mathbf{u}_j)_+^{-(1/2+(1-\mathbf{H})/q)},$$

where $x_+ = \max(x, 0)$, W is a standard two-sided Brownian sheet, and $c(q, \mathbf{H})$ is a positive constant depending only on q and $\mathbf{H}$ chosen so that $E[Z^{q,\mathbf{H}}(\mathbf{1})^2] = 1$.

The above integral (2.5) represents a multiple Wiener–Itô integral of order q with respect to the standard two-sided Brownian sheet W.

In many cases (e.g. when one wants to simulate $Z^{q,\mathbf{H}}$ or when one looks for constructing a stochastic calculus with respect to it), the following finite-time interval representation for $Z^{q,\mathbf{H}}$ may be interested as well:

$$(2.6) \quad Z^{q,\mathbf{H}}(\mathbf{t}) \stackrel{(d)}{=} b_{q,\mathbf{H}} \int\limits_0^{t_1} \ldots \int\limits_0^{t_d} dW_{u_{1,1},\ldots,u_{1,d}} \ldots \int\limits_0^{t_1} \ldots \int\limits_0^{t_d} dW_{u_{q,1},\ldots,u_{q,d}}$$

$$\times \Big(\int\limits_{u_{1,1} \vee \ldots \vee u_{q,1}}^{t_1} da_1 \partial_1 K^{H_1'}(a_1, u_{1,1}) \ldots \partial_1 K^{H_1'}(a_1, u_{q,1}) \Big) \ldots$$

$$\times \Big(\int\limits_{u_{1,d} \vee \ldots \vee u_{q,d}}^{t_d} da_d \partial_1 K^{H_d'}(a_d, u_{1,d}) \ldots \partial_1 K^{H_d'}(a_d, u_{q,d}) \Big)$$

$$= b_{q,\mathbf{H}} \int\limits_{[0,\mathbf{t}]^q} dW_{\mathbf{u_1}} \ldots dW_{\mathbf{u_q}} \prod_{j=1}^d \int\limits_{u_{1,j} \vee \ldots \vee u_{q,j}}^{t_j} da \partial_1 K^{H_j'}(a, u_{1,j}) \ldots \partial_1 K^{H_j'}(a, u_{q,j}).$$

In (2.6), K^H stands for the usual kernel appearing in the classical expression of the fractional Brownian motion B^H as a Volterra integral with respect to Brownian motion (see e.g. [6], [7]), that is, $B_t^H = \int_0^t K^H(t,s)dB_s$, whereas
(2.7)
$$b_{q,\mathbf{H}} := \frac{\big(\mathbf{H}(2\mathbf{H}-1)\big)^{1/2}}{\big(q!\big(\mathbf{H}'(2\mathbf{H}'-1)\big)^q\big)^{1/2}} = (\sqrt{q!})^{d-1} \prod_{j=1}^d \frac{\big(H_j(2H_j-1)\big)^{1/2}}{\big(q!\big(H_j'(2H_j'-1)\big)^q\big)^{1/2}}$$

is the unique positive constant ensuring that $E[Z^{q,\mathbf{H}}(\mathbf{1})^2] = 1$, where

$$(2.8) \qquad \mathbf{H}' := 1 + \frac{\mathbf{H} - 1}{q} \qquad (\text{iff } (2\mathbf{H}' - 2)q = 2\mathbf{H} - 2).$$

For a proof of (2.6) when $d = 2$, we refer to [14], Chapter 4. The extension to any value of d as presented here is straightforward.

3. PROOF OF THEOREM 1.1

We are now in a position to give the proof of our Theorem 1.1. It is divided into three steps.

3.1. Expanding into Wiener chaos. In preparation for analyzing the quadratic variation (1.1), let us find an explicit expression for the chaos decomposition of $V_{\mathbf{N}}$. Using (2.6) and proceeding by induction on the dimension d, we can write $\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]}$ as a q-th Wiener–Itô integral with respect to the standard two-sided Brownian sheet $(W_{\mathbf{t}})_{\mathbf{t} \in \mathbb{R}^d}$ as follows: for every $0 \leqslant \mathbf{i} \leqslant \mathbf{N} - 1$, one has

$$(3.1) \qquad \Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]} = I_q(f_{\mathbf{i},\mathbf{N}}),$$

where

$$(3.2) \qquad f_{\mathbf{i},\mathbf{N}}(\mathbf{x}_1, \ldots, \mathbf{x}_q) = b_{q,\mathbf{H}} \prod_{j=1}^{d} f_{i_j, N_j}(x_{1,j}, \ldots, x_{q,j}),$$

with $f_{i,N}(x_1, \ldots, x_q)$ denoting the expression

$$(3.3) \quad \mathbf{1}_{[0,(i+1)/N]}(x_1 \vee \ldots \vee x_q) \int\limits_{x_1 \vee \ldots \vee x_q}^{(i+1)/N} du \, \partial_1 K^{H'}(u, x_1) \ldots \partial_1 K^{H'}(u, x_q)$$

$$- \mathbf{1}_{[0,i/N]}(x_1 \vee \ldots \vee x_q) \int\limits_{x_1 \vee \ldots \vee x_q}^{i/N} du \, \partial_1 K^{H'}(u, x_1) \ldots \partial_1 K^{H'}(u, x_q)$$

and with $b_{q,\mathbf{H}}$ and $\mathbf{H}'$ given by (2.7) and (2.8), respectively. Indeed, for $d = 1$, see [2], Section 3, p. 8, it reduces to

$$\Delta Z^{q,H}_{[i/N,(i+1)/N]} = Z^{q,H}_{(i+1)/N} - Z^{q,H}_{i/N} = b_{q,H} I_q(f_{i,N}),$$

while for $d = 2$ it is easy to verify that

$$\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]} = Z^{q,H_1,H_2}_{(i+1)/N,(j+1)/M} - Z^{q,H_1,H_2}_{i/N,(j+1)/M} - Z^{q,H_1,H_2}_{(i+1)/N,j/M} + Z^{q,H_1,H_2}_{i/N,j/M}$$
$$= I_q(f_{i,j,N,M}),$$

where

$$f_{i,j,N,M}(x_1, y_1, \ldots, x_q, y_q)$$

$$= b_{q,H_1,H_2} \mathbf{1}_{[0,(i+1)/N]}(x_1 \vee \ldots \vee x_q) \int_{x_1 \vee \ldots \vee x_q}^{(i+1)/N} du \partial_1 K^{H_1'}(u, x_1) \ldots \partial_1 K^{H_1'}(u, x_q)$$

$$\times \mathbf{1}_{[0,(j+1)/M]}(y_1 \vee \ldots \vee y_q) \int_{y_1 \vee \ldots \vee y_q}^{(j+1)/M} dv \partial_1 K^{H_2'}(v, y_1) \ldots \partial_1 K^{H_2'}(v, y_q)$$

$$- b_{q,H_1,H_2} \mathbf{1}_{[0,(i+1)/N]}(x_1 \vee \ldots \vee x_q) \int_{x_1 \vee \ldots \vee x_q}^{(i+1)/N} du \partial_1 K^{H_1'}(u, x_1) \ldots \partial_1 K^{H_1'}(u, x_q)$$

$$\times \mathbf{1}_{[0,j/M]}(y_1 \vee \ldots \vee y_q) \int_{y_1 \vee \ldots \vee y_q}^{j/M} dv \partial_1 K^{H_2'}(v, y_1) \ldots \partial_1 K^{H_2'}(v, y_q)$$

$$- b_{q,H_1,H_2} \mathbf{1}_{[0,i/N]}(x_1 \vee \ldots \vee x_q) \int_{x_1 \vee \ldots \vee x_q}^{i/N} du \partial_1 K^{H_1'}(u, x_1) \ldots \partial_1 K^{H_1'}(u, x_q)$$

$$\times \mathbf{1}_{[0,(j+1)/M]}(y_1 \vee \ldots \vee y_q) \int_{y_1 \vee \ldots \vee y_q}^{(j+1)/M} dv \partial_1 K^{H_2'}(v, y_1) \ldots \partial_1 K^{H_2'}(v, y_q)$$

$$+ b_{q,H_1,H_2} \mathbf{1}_{[0,i/N]}(x_1 \vee \ldots \vee x_q) \int_{x_1 \vee \ldots \vee x_q}^{i/N} du \partial_1 K^{H_1'}(u, x_1) \ldots \partial_1 K^{H_1'}(u, x_q)$$

$$\times \mathbf{1}_{[0,j/M]}(y_1 \vee \ldots \vee y_q) \int_{y_1 \vee \ldots \vee y_q}^{j/M} dv \partial_1 K^{H_2'}(v, y_1) \ldots \partial_1 K^{H_2'}(v, y_q)$$

$$= b_{q,H_1,H_2} f_{i,N}(x_1, \ldots, x_q) f_{j,M}(y_1, \ldots, y_q).$$

The last equality above is obtained by grouping each term of $f_{i,j,N,M}$ together. Suppose that the expressions (3.1) and (3.2) are true for d, that is, the kernel of $\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]}$ is equal to

$$b_{q,\mathbf{H}} \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^d r_i} \prod_{j=1}^d \mathbf{1}_{[0,(i_j+r_j)/N_j]}(x_{1,j} \vee \ldots \vee x_{q,j})$$

$$\times \int_{x_{1,j} \vee \ldots \vee x_{q,j}}^{(i_j+r_j)/N_j} du \partial_1 K^{H_j'}(u, x_{1,j}) \ldots \partial_1 K^{H_j'}(u, x_{q,j})$$

$$= b_{q,\mathbf{H}} \prod_{j=1}^d f_{i_j,N_j}(x_{1,j}, \ldots, x_{q,j}).$$

Then, for the case $d + 1$ we have

$$
\begin{aligned}
\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]} &= \sum_{\mathbf{r}\in\{0,1\}^{d+1}} (-1)^{d+1-\sum_{i=1}^{d+1} r_i} Z^{q,\mathbf{H}}_{(\mathbf{i}+\mathbf{r})/\mathbf{N}} \\
&= \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^{d} r_i} Z^{q,\mathbf{H}}_{((i_1+r_1)/N_1,\ldots,(i_d+r_d)/N_d,(i_{d+1}+1)/N_{d+1})} \\
&\quad + \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} (-1)^{d+1-\sum_{i=1}^{d} r_i} Z^{q,\mathbf{H}}_{((i_1+r_1)/N_1,\ldots,(i_d+r_d)/N_d,i_{d+1}/N_{d+1})} \\
&= \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^{d} r_i} Z^{q,\mathbf{H}}_{((i_1+r_1)/N_1,\ldots,(i_d+r_d)/N_d,(i_{d+1}+1)/N_{d+1})} \\
&\quad - \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} (-1)^{d-\sum_{i=1}^{d} r_i} Z^{q,\mathbf{H}}_{((i_1+r_1)/N_1,\ldots,(i_d+r_d)/N_d,i_{d+1}/N_{d+1})}.
\end{aligned}
$$

It belongs to the q-th Wiener chaos with the kernel $f_{\mathbf{i},\mathbf{N}}$ given by

$$
\begin{aligned}
f_{\mathbf{i},\mathbf{N}} = b_{q,\mathbf{H}} \sum_{(r_1,\ldots,r_d)\in\{0,1\}^d} &(-1)^{d-\sum_{i=1}^{d} r_i} \prod_{j=1}^{d} \mathbf{1}_{[0,(i_j+r_j)/N_j]}(x_{1,j} \vee \ldots \vee x_{q,j}) \\
&\times \int_{x_{1,j}\vee\ldots\vee x_{q,j}}^{(i_j+r_j)/N_j} du\, \partial_1 K^{H'_j}(u,x_{1,j})\ldots\partial_1 K^{H'_j}(u,x_{q,j}) \\
&\times \Big(\int_{x_{1,d+1}\vee\ldots\vee x_{q,d+1}}^{(i_{d+1}+1)/N_{d+1}} du'\, \partial_1 K^{H'_{d+1}}(u',x_{1,d+1})\ldots\partial_1 K^{H'_{d+1}}(u',x_{q,d+1}) \\
&\quad - \int_{x_{1,d+1}\vee\ldots\vee x_{q,d+1}}^{i_{d+1}/N_{d+1}} du'\, \partial_1 K^{H'_{d+1}}(u',x_{1,d+1})\ldots\partial_1 K^{H'_{d+1}}(u',x_{q,d+1})\Big).
\end{aligned}
$$

By induction, one gets $f_{\mathbf{i},\mathbf{N}} = b_{q,\mathbf{H}} \prod_{j=1}^{d+1} f_{i_j,N_j}(x_{1,j},\ldots,x_{q,j})$, which is our desired expression.

Next, by applying the product formula (2.2), we can write
(3.4)
$$
(\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]})^2 - E[(\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]})^2] = \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}(f_{\mathbf{i},\mathbf{N}} \widetilde{\otimes}_r f_{\mathbf{i},\mathbf{N}}).
$$

Let us compute the contractions appearing on the right-hand side of (3.4). For every $0 \leqslant r \leqslant q-1$, we have

(3.5)
$$
\begin{aligned}
(f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}})(\mathbf{x}_1,\ldots,\mathbf{x}_{2q-2r}) \\
= \int_{([0,1]^d)^r} d\mathbf{a}_1 \ldots d\mathbf{a}_r f_{\mathbf{i},\mathbf{N}}(\mathbf{x}_1,\ldots,\mathbf{x}_{q-r},\mathbf{a}_1,\ldots,\mathbf{a}_r) \\
\times f_{\mathbf{i},\mathbf{N}}(\mathbf{x}_{q-r+1},\ldots,\mathbf{x}_{2q-2r},\mathbf{a}_1,\ldots,\mathbf{a}_r)
\end{aligned}
$$

$$= b_{q,\mathbf{H}}^2 \int\limits_{([0,1]^d)^r} d\mathbf{a}_1 \dots d\mathbf{a}_r \prod_{j=1}^d f_{i_j,N_j}(x_{1,j}, \dots, x_{q-r,j}, a_{1,j}, \dots, a_{r,j})$$

$$\times \prod_{j=1}^d f_{i_j,N_j}(x_{q-r+1,j}, \dots, x_{2q-2r,j}, a_{1,j}, \dots, a_{r,j})$$

$$= b_{q,\mathbf{H}}^2 \prod_{j=1}^d (f_{i_j,N_j} \otimes_r f_{i_j,N_j})(x_{1,j}, \dots, x_{2q-2r,j}),$$

where

$$(3.6) \quad (f_{i,N} \otimes_r f_{i,N})(x_1, \dots, x_{2q-2r}) = \big(H'(2H'-1)\big)^r$$

$$\times \Big\{ \mathbf{1}_{[0,(i+1)/N]}(x_1 \vee \dots \vee x_{q-r}) \int\limits_{x_1 \vee \dots \vee x_{q-r}}^{(i+1)/N} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r})$$

$$\times \mathbf{1}_{[0,(i+1)/N]}(x_{q-r+1} \vee \dots \vee x_{2q-2r}) \int\limits_{x_{q-r+1} \vee \dots \vee x_{2q-2r}}^{(i+1)/N} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots$$

$$\dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r}$$

$$- \mathbf{1}_{[0,(i+1)/N]}(x_1 \vee \dots \vee x_{q-r}) \int\limits_{x_1 \vee \dots \vee x_{q-r}}^{(i+1)/N} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r})$$

$$\times \mathbf{1}_{[0,i/N]}(x_{q-r+1} \vee \dots \vee x_{2q-2r}) \int\limits_{x_{q-r+1} \vee \dots \vee x_{2q-2r}}^{i/N} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots$$

$$\dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r}$$

$$- \mathbf{1}_{[0,i/N]}(x_1 \vee \dots \vee x_{q-r}) \int\limits_{x_1 \vee \dots \vee x_{q-r}}^{i/N} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r})$$

$$\times \mathbf{1}_{[0,(i+1)/N]}(x_{q-r+1} \vee \dots \vee x_{2q-2r}) \int\limits_{x_{q-r+1} \vee \dots \vee x_{2q-2r}}^{(i+1)/N} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots$$

$$\dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r}$$

$$+ \mathbf{1}_{[0,i/N]}(x_1 \vee \dots \vee x_{q-r}) \int\limits_{x_1 \vee \dots \vee x_{q-r}}^{i/N} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r})$$

$$\times \mathbf{1}_{[0,i/N]}(x_{q-r+1} \vee \dots \vee x_{2q-2r}) \int\limits_{x_{q-r+1} \vee \dots \vee x_{2q-2r}}^{i/N} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots$$

$$\dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r} \Big\}.$$

(See [2], p. 10, for a detailed computation of the expression (3.6).) Moreover, since

$Z^{q,\mathbf{H}}$ is $\mathbf{H}$-self-similar and has stationary increments, we obtain

$$\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]} \overset{(d)}{=} \mathbf{N}^{-\mathbf{H}} \Delta Z^{q,\mathbf{H}}_{[\mathbf{i},\mathbf{i}+1]} \overset{(d)}{=} \mathbf{N}^{-\mathbf{H}} Z^{q,\mathbf{H}}_{[\mathbf{0},\mathbf{1}]}.$$

It follows that

$$E[\mathbf{N}^{2\mathbf{H}}(\Delta Z^{q,\mathbf{H}}_{[\mathbf{i}/\mathbf{N},(\mathbf{i}+1)/\mathbf{N}]})^2] = E[Z^{q,\mathbf{H}}(\mathbf{1})^2] = 1.$$

As a consequence, we have

(3.7) $$V_{\mathbf{N}} = F_{2q,\mathbf{N}} + c_{2q-2} F_{2q-2,\mathbf{N}} + \ldots + c_4 F_{4,\mathbf{N}} + c_2 F_{2,\mathbf{N}},$$

where $c_{2q-2r} = r!\binom{q}{r}^2$, $r = 0, \ldots, q-1$, are the combinational constants coming from the product formula, and

(3.8) $$F_{2q-2r,\mathbf{N}} := \mathbf{N}^{2\mathbf{H}-1} I_{2q-2r}\Big(\sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \widetilde{\otimes}_r f_{\mathbf{i},\mathbf{N}} \Big)$$

for the kernels $f_{\mathbf{i},\mathbf{N}} \widetilde{\otimes}_r f_{\mathbf{i},\mathbf{N}}$ computed in (3.5).

3.2. Evaluating the $L^2(\Omega)$-norm. Set

(3.9)

$$
\begin{aligned}
c_{1,\mathbf{H}} &= \frac{2! 2^d b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q}}{(4\mathbf{H}'-3)(4\mathbf{H}'-2)[(2\mathbf{H}'-2)(q-1)+1]^2[(\mathbf{H}'-1)(q-1)+1]^2} \\
&= \frac{(q!)^{2(d-1)}}{2^{d-1}} \prod_{j=1}^{d} \frac{4\big(H_j(2H_j-1)\big)^2}{(q!)^2(4H_j'-3)(4H_j'-2)} \\
&\quad \times \prod_{j=1}^{d} \frac{1}{[(2H_j'-2)(q-1)+1]^2[(H_j'-1)(q-1)+1]^2}.
\end{aligned}
$$

We claim that

(3.10) $$\lim_{\mathbf{N}\to\infty} E[c_{1,\mathbf{H}}^{-1} \mathbf{N}^{2(2-2\mathbf{H}')} c_2^{-2} V_{\mathbf{N}}^2] = 1.$$

Let us prove (3.10). Due to the orthogonality property for Wiener chaos of different orders, it is sufficient to evaluate the $L^2(\Omega)$-norm of each multiple Wiener–Itô integral appearing in the chaotic decomposition (3.7) of $V_{\mathbf{N}}$. Let us start with the double integral:

$$F_{2,\mathbf{N}} = \mathbf{N}^{2\mathbf{H}-1} I_2\Big(\sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}} \Big).$$

Since the kernel $\sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}$ is symmetric, we have

$$E[F_{2,\mathbf{N}}^2] = 2N^{4\mathbf{H}-2} \Big\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}} \Big\|_{L^2(([0,1]^d)^2)}^2$$

$$= 2\mathbf{N}^{4\mathbf{H}-2} \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \langle f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_{q-1} f_{\mathbf{k},\mathbf{N}} \rangle_{L^2(([0,1]^d)^2)} .$$

Let us now compute the scalar products in the above expression. By using (3.5), we get

$$\langle f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_{q-1} f_{\mathbf{k},\mathbf{N}} \rangle_{L^2(([0,1]^d)^2)}$$

$$= b_{q,\mathbf{H}}^4 \prod_{j=1}^d \langle f_{i_j,N_j} \otimes_{q-1} f_{i_j,N_j}, f_{k_j,N_j} \otimes_{q-1} f_{k_j,N_j} \rangle_{L^2([0,1]^d)}$$

$$= b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q} \prod_{j=1}^d \int_{i_j/N_j}^{(i_j+1)/N_j} du_j \int_{i_j/N_j}^{(i_j+1)/N_j} dv_j \int_{k_j/N_j}^{(k_j+1)/N_j} du_j' \int_{k_j/N_j}^{(k_j+1)/N_j} dv_j'$$

$$\times |u_j - v_j|^{(2H_j'-2)(q-1)} |u_j' - v_j'|^{(2H_j'-2)(q-1)} |u_j - u_j'|^{2H_j'-2} |v_j - v_j'|^{2H_j'-2}.$$

The change of variables $u' = (u - i/N)N$ for each u_j, u_j', v_j, v_j' with j from 1 to d yields

$$E[F_{2,\mathbf{N}}^2] = 2b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q} \mathbf{N}^{4\mathbf{H}-2} \mathbf{N}^{-4} \mathbf{N}^{-(2\mathbf{H}'-2)2q}$$

$$\times \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \prod_{j=1}^d \int_0^1 du_j \int_0^1 dv_j \int_0^1 du_j' \int_0^1 dv_j' |u_j - v_j|^{(2H_j'-2)(q-1)} |u_j' - v_j'|^{(2H_j'-2)(q-1)}$$

$$\times |u_j - u_j' + i_j - k_j|^{2H_j'-2} |v_j - v_j' + i_j - k_j|^{2H_j'-2}.$$

We split the sum $\sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1}$ appearing in $E[F_{2,\mathbf{N}}^2]$ just above into

$$\sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} = \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ \exists 1 \leqslant j \leqslant d: i_j = k_j}}^{\mathbf{N}-1} + \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ \forall j: i_j \neq k_j}}^{\mathbf{N}-1} .$$

For the first term, without loss of generality, we can assume that $i_1 = k_1$ and $i_j \neq k_j$ for all $j \neq 1$. Then,

$$\mathbf{N}^{-2} \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ i_1=k_1}}^{\mathbf{N}-1} \prod_{j=1}^{d} \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)}$$

$$\times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2}$$

$$= N_1^{-1} \int_{[0,1]^4} du_1 dv_1 du'_1 dv'_1 |u_1 - v_1|^{(2H'_1-2)(q-1)} |u'_1 - v'_1|^{(2H'_1-2)(q-1)}$$

$$\times |u_1 - u'_1|^{2H'_1-2} |v_1 - v'_1|^{2H'_1-2}$$

$$\times \prod_{j=2}^{d} \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)}$$

$$\times 2N_j^{-2} \sum_{\substack{i_j,k_j=0 \\ i_j>k_j}}^{N_j-1} |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2}.$$

Following [2] or [15], we see that

$$N^{-2} \sum_{\substack{i,k=0 \\ i>k}}^{N-1} |u - u' + i - k|^{2H'-2} |v - v' + i - k|^{2H'-2}$$

$$= N^{2(2H'-2)} \frac{1}{N} \sum_{n=1}^{N} \left(1 - \frac{n}{N}\right) \left|\frac{u-u'}{N} + \frac{n}{N}\right|^{2H'-2} \left|\frac{v-v'}{N} + \frac{n}{N}\right|^{2H'-2}$$

is asymptotically equivalent to

$$N^{2(2H'-2)} \int_0^1 (1-x) x^{4H'-4} dx = N^{2(2H'-2)} \frac{1}{(4H'-3)(4H'-2)}.$$

It follows that

$$\mathbf{N}^{-2} \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ i_1=k_1}}^{\mathbf{N}-1} \prod_{j=1}^{d} \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)}$$

$$\times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2}$$

$$\approx N_1^{-1} \int_{[0,1]^4} du_1 dv_1 du'_1 dv'_1 |u_1 - v_1|^{(2H'_1-2)(q-1)} |u'_1 - v'_1|^{(2H'_1-2)(q-1)}$$

$$\times |u_1 - u'_1|^{2H'_1-2} |v_1 - v'_1|^{2H'_1-2} \prod_{j=2}^{d} 2N_j^{2(2H'_j-2)}$$

$$\times \frac{1}{(4H'_j - 3)(4H'_j - 2)[(2H'_j - 2)(q-1) + 1]^2 [(H'_j - 1)(q-1) + 1]^2}.$$

Similarly for the second term, where $i_j \neq k_j$ for all $1 \leqslant j \leqslant d$, we have

$$
\mathbf{N}^{-2} \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ \forall j: i_j \neq k_j}}^{\mathbf{N}-1} \prod_{j=1}^{d} \int_{[0,1]^4} du_j dv_j du_j' dv_j' |u_j - v_j|^{(2H_j'-2)(q-1)} |u_j' - v_j'|^{(2H_j'-2)(q-1)}
$$

$$
\times |u_j - u_j' + i_j - k_j|^{2H_j'-2} |v_j - v_j' + i_j - k_j|^{2H_j'-2}
$$

$$
\approx \prod_{j=1}^{d} 2N_j^{2(2H_j'-2)}
$$

$$
\times \frac{1}{(4H_j' - 3)(4H_j' - 2)[(2H_j' - 2)(q-1) + 1]^2[(H_j' - 1)(q-1) + 1]^2}.
$$

Putting everything together, we conclude that

$$
(3.11) \qquad \lim_{\mathbf{N}\to\infty} E[c_{1,\mathbf{H}}^{-1} \mathbf{N}^{2(2-2\mathbf{H}')} F_{2,\mathbf{N}}^2] = 1.
$$

Let us now consider the remaining terms $F_{4,\mathbf{N}}, \dots, F_{2q,\mathbf{N}}$ in the chaos decomposition (3.7). Since $\|\tilde{g}\|_{L^2} \leqslant \|g\|_{L^2}$ for any square integrable function g, one can write, for every $0 \leqslant r \leqslant q - 2$,

$$
E[F_{2q-2r,\mathbf{N}}^2] = \mathbf{N}^{4\mathbf{H}-2}(2q - 2r)! \Big\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \widetilde{\otimes}_r f_{\mathbf{i},\mathbf{N}} \Big\|_{L^2(([0,1]^d)^{2q-2r})}^2
$$

$$
\leqslant \mathbf{N}^{4\mathbf{H}-2}(2q - 2r)! \Big\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}} \Big\|_{L^2(([0,1]^d)^{2q-2r})}^2
$$

$$
= (2q - 2r)! \mathbf{N}^{4\mathbf{H}-2} \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \langle f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_r f_{\mathbf{k},\mathbf{N}} \rangle_{L^2(([0,1]^d)^{2q-2r})}.
$$

Proceeding as above, we obtain

$$
\langle f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_r f_{\mathbf{k},\mathbf{N}} \rangle_{L^2([0,1]^{d\cdot(2q-2r)})}
$$

$$
= b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}' - 1)\big)^{2q} \prod_{j=1}^{d} \int_{i_j/N_j}^{(i_j+1)/N_j} du_j \int_{i_j/N_j}^{(i_j+1)/N_j} dv_j \int_{k_j/N_j}^{(k_j+1)/N_j} du_j' \int_{k_j/N_j}^{(k_j+1)/N_j} dv_j'
$$

$$
\times |u_j - v_j|^{(2H_j'-2)r} |u_j' - v_j'|^{(2H_j'-2)r} |u_j - u_j'|^{(2H_j'-2)(q-r)} |v_j - v_j'|^{(2H_j'-2)(q-r)}.
$$

Using the change of variables $u' = (u - i/N)N$ for each u_j, u_j', v_j, v_j' with $j = 1, \dots, d$, we get

$$E[F_{2q-2r,\mathbf{N}}^2]$$

$$\leqslant (2q-2r)! b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q} \mathbf{N}^{4\mathbf{H}-2} \mathbf{N}^{-4} \mathbf{N}^{-(2\mathbf{H}'-2)2q}$$

$$\times \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \prod_{j=1}^{d} \int_0^1 du_j \int_0^1 dv_j \int_0^1 du'_j \int_0^1 dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r}$$

$$\times |u_j - u'_j + i_j - k_j|^{(2H'_j-2)(q-r)} |v_j - v'_j + i_j - k_j|^{(2H'_j-2)(q-r)}$$

$$= (2q-2r)! b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q} \mathbf{N}^{-2}$$

$$\times \prod_{j=1}^{d} \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r}$$

$$\times \sum_{i_j,k_j=0}^{N_j-1} |u_j - u'_j + i_j - k_j|^{(2H'_j-2)(q-r)} |v_j - v'_j + i_j - k_j|^{(2H'_j-2)(q-r)}$$

$$= (2q-2r)! b_{q,\mathbf{H}}^4 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{2q} \mathbf{N}^{2(2\mathbf{H}'-2)(q-r)}$$

$$\times \prod_{j=1}^{d} \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r}$$

$$\times \frac{2}{N_j} \sum_{n_j=1}^{N_j} \left(1 - \frac{n_j}{N_j}\right) \left|\frac{u_j - u'_j}{N_j} + \frac{n_j}{N_j}\right|^{(2H'_j-2)(q-r)} \left|\frac{v_j - v'_j}{N_j} + \frac{n_j}{N_j}\right|^{(2H'_j-2)(q-r)}.$$

Apart from the diagonal terms, the term $(u - u')/N$ is dominated by n/N for n, N large enough. Using a Riemann sum approximation, one has for $0 \leqslant r \leqslant q - 2$ and as $\mathbf{N} \to \infty$,

$$(3.12) \qquad E[\mathbf{N}^{(2-2\mathbf{H}')(2q-2r)} F_{2q-2r,\mathbf{N}}^2] = O(1).$$

We deduce that

$$(3.13) \qquad \lim_{\mathbf{N}\to\infty} E[\mathbf{N}^{2(2-2\mathbf{H}')} F_{2q-2r,\mathbf{N}}^2] = 0,$$

and the proof of (3.10) follows.

3.3. Concluding the proof of Theorem 1.1. Thanks to (3.10), to understand the asymptotic behavior of the renormalized sequence of $V_{\mathbf{N}}$, it is enough to analyze the convergence of the term

$$(3.14) \qquad I_2\big(\mathbf{N}^{2\mathbf{H}-1} \mathbf{N}^{2-2\mathbf{H}'} \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}\big)$$

with

$$f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}(\mathbf{x}_1, \mathbf{x}_2) = b_{q,\mathbf{H}}^2 \prod_{j=1}^{d} (f_{i_j,N_j} \otimes_{q-1} f_{i_j,N_j})(x_{1,j}, x_{2,j})$$

$$= b_{q,\mathbf{H}}^2 \big(\mathbf{H}'(2\mathbf{H}'-1)\big)^{q-1}$$

$$\times \prod_{j=1}^d \bigg(\mathbf{1}_{[0,i_j/N_j]}(x_{1,j})\mathbf{1}_{[0,i_j/N_j]}(x_{2,j}) \int_{i_j/N_j}^{(i_j+1)/N_j} du \int_{i_j/N_j}^{(i_j+1)/N_j} du' \partial_1 K^{H_j'}(u,x_{1,j})$$

$$\times \partial_1 K^{H_j'}(u',x_{2,j})|u-u'|^{(2H_j'-2)(q-1)}$$

$$+ \mathbf{1}_{[0,i_j/N_j]}(x_{1,j})\mathbf{1}_{[i_j/N_j,(i_j+1)/N_j]}(x_{2,j}) \int_{i_j/N_j}^{(i_j+1)/N_j} du \int_{x_{2,j}}^{(i_j+1)/N_j} du' \partial_1 K^{H_j'}(u,x_{1,j})$$

$$\times \partial_1 K^{H_j'}(u',x_{2,j})|u-u'|^{(2H_j'-2)(q-1)}$$

$$+ \mathbf{1}_{[i_j/N_j,(i_j+1)/N_j]}(x_{1,j})\mathbf{1}_{[0,(i_j+1)/N_j]}(x_{2,j}) \int_{x_{1,j}}^{(i_j+1)/N_j} du$$

$$\times \int_{i_j/N_j}^{(i_j+1)/N_j} du' \partial_1 K^{H_j'}(u,x_{1,j})\partial_1 K^{H_j'}(u',x_{2,j})|u-u'|^{(2H_j'-2)(q-1)}$$

$$+ \mathbf{1}_{[i_j/N_j,i_j/N_j]}(x_{1,j})\mathbf{1}_{[i_j/N_j,(i_j+1)/N_j]}(x_{2,j}) \int_{x_{1,j}}^{(i_j+1)/N_j} du$$

$$\times \int_{x_{2,j}}^{(i_j+1)/N_j} du' \partial_1 K^{H_j'}(u,x_{1,j})\partial_1 K^{H_j'}(u',x_{2,j})|u-u'|^{(2H_j'-2)(q-1)}\bigg).$$

Among the four terms of each factor on the right-hand side of the above expression, only the first one is not asymptotically negligible in $L^2(\Omega)$, see [2], pp. 14 and 15, or follow the lines of [15] for details. Furthermore, by the isometry property for multiple Wiener–Itô integrals, evaluating the $L^2(\Omega)$-limit of a sequence belonging to the second Wiener chaos is equivalent to evaluating the $L^2\big(([0,1]^d)^2\big)$-limit of the sequence of their corresponding symmetric kernels. Therefore, we are left to find the limit of $f_2^{\mathbf{N}}$ in $L^2\big(([0,1]^d)^2\big)$, where

$$f_2^{\mathbf{N}}(\mathbf{x}_1,\mathbf{x}_2) := (q!)^{d-1} \prod_{j=1}^d \frac{H_j(2H_j-1)}{q!\big(H_j'(2H_j'-1)\big)^q} \big(H_j'(2H_j'-1)\big)^{q-1} N_j^{2H_j-1} N_j^{2-2H_j'}$$

$$\times \sum_{1_j=0}^{N_j-1} \mathbf{1}_{[0,i_j/N_j]}(x_{1,j})\mathbf{1}_{[0,i_j/N_j]}(x_{2,j}) \int_{i_j/N_j}^{(i_j+1)/N_j} du \int_{i_j/N_j}^{(i_j+1)/N_j} du' \partial_1 K^{H_j'}(u,x_{1,j})$$

$$\times \partial_1 K^{H_j'}(u',x_{2,j})|u-u'|^{(2H_j'-2)(q-1)}.$$

According to [2], Theorem 3.2, by using integral approximation and the Lebesgue dominated convergence theorem, each term under the product for j from 1 to d converges in $L^2([0,1]^2)$ to the constant a_j times the kernel of a standard Rosenblatt

random variable $Z^{2,2H'_j-1}(1)$ as $N_j \to \infty$ with

$$a_j = \frac{2H_j(2H_j-1)}{q!(4H'_j-3)^{1/2}(4H'_j-2)^{1/2}[(2H'_j-2)(q-1)+1][(H'_j-1)(q-1)+1]}.$$

Therefore, $f_2^{\mathbf{N}}$ converges in $L^2\big(([0,1]^d)^2\big)$ to the constant $2^{(d-1)/2}c_{1,\mathbf{H}}^{1/2}$ times the kernel of a standard Rosenblatt sheet at time $\mathbf{1}$ as $\mathbf{N} \to \infty$, which leads to our desired conclusion.

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